QUEEN'S MATHEMATICAL COMMUNICATOR

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DEPARTMENTAL

PHOTO

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OCTOBER 1983

Cover Picture – On Wednesday March 16, 1983 at 3:30 pm the entire Queen's Mathematics and Statistics Department assembled on the steps outside Jeffery Hall for a first (in recent memory) group photograph. Various Math undergraduates assembled to jeer, and an extremely aimiable chap from Flair Photographic darted energetically back and forth and kept us, as you can see, in fine form throughout the ordeal (mathematicians are very shy). Missing from the company, with manifold regrets, were Paulo Ribenboim and Leslie Roberts. Copies of the picture were presented to Hu Ellis and John Coleman on their retirement AND a copy awaits you, dear reader, on the back page of this issue. May it encourage you to drop us an occasional line. (See next page).

An Invitation to Readers of the Communicator

We would like to enhance the role of the <u>Communicator</u> as a medium of communication among alumn(ae)(i) of the Queen's Department of Mathematics and Statistics. The purpose of this message is to urge our readers to write to us.

News and views are what we want, especially views. What are your interests, your priorities, your thoughts on society, its problems and possibilities, local and global? The readers of the Communicator constitute a rather special community, diverse, but with one strong common bond. In such a community, the sharing of views is of great interest to its members. News is important too, but is of greatest interest when it serves to provide a context for a point of view.

Imagine we had a gathering of all graduates. There would be much reviewing of old acquaintances, exchanging of news, and frank discussion of ideas, dilemmas and hopes. What we are seeking in your response to this message is an extract from the conversations you would have at such a reunion. Since it will be written it may be somewhat more polished and coherent than if it were recorded at such an imaginary gathering. Or it may not; spontaneity has its virtue too.

Please take our invitation seriously. Sit down for an evening or two and tear a page from that fascinating book of life you are no doubt busily writing. We are all curious to know what you are up to.

The Mathematical Communicator is a magazine for the alumni(ae) of the Department of Mathematics and Statistics at Queen's. We welcome comments, problems, solutions, news letters, and even articles, from this constituency. Please address all correspondence to:

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Please let us know if you change your address.

        Peter Taylor
        Editor

A Transcendental Number
Ken Braithwaite

[Ken graduated from Queen's in Mathematics in 1980. Since then he has been working part-time on an M.Sc. in Set Theory and Logic with Doug Hoover at Queen's and teaching high school in Georgia.]

Real numbers are sometimes classified as either algebraic or transcendental. A real number is <u>algebraic</u> if it is the root of some polynomial with integer coefficients. For example $\sqrt{3/5}$ is algebraic (take $f(x) = 5x^2-3$) and so is the cosine of $15°$ (can you find an f?). If p is algebraic we say that an integral polynomial $f(x)$ for which $f(p) = 0$ is a <u>minimal polynomial</u> of p if f has minimal degree in the set of all integral polynomials which have p as a root. As an exercise show that the minimal polynomial of an algebraic number is unique up to a constant multiple.

A number is <u>transcendental</u> if it is not algebraic. The numbers π, e and $\log 2$ are all transcendental, but this is not easy to prove. In a very real sense most numbers are transcendental: there are uncountably many transcendental numbers, but only countably many algebraic ones. [Can you prove this? Count the integer polynomials.] Yet to prove that any given number is transcendental is in general very difficult. It is even difficult to produce a number we are sure is transcendental; nearly everything we can write down explicitly is algebraic. But there is one number, discovered by Liouville, which is easy to prove transcendental.

Let α be a decimal with 1 in the $K!$ place for $K = 1,2,3,\ldots$ and zero elsewhere, $\alpha = 0.1100010000\ldots$, or $\alpha = \sum_{m=1}^{\infty} 10^{-m!}$. Assume α is algebraic and let $f(x) = \sum_{i=0}^{n} C_i x^i$ be a minimal polynomial of α of degree n .

Temporarily fix a positive integer j , and define $\beta = \sum_{m=1}^{j} 10^{-m!}$. Then

$$\alpha - \beta = \sum_{m=j+1}^{\infty} 10^{-m!} < 2 \cdot 10^{-(j+1)!} \quad \text{(Not hard to show)}.$$

<u>Lemma 1</u>: $f(\beta) \neq 0$.

<u>Proof</u>: If $f(\beta) = 0$ then $(x-\beta)$ divides $f(x)$ and $f(x) = (x-\beta)q(x)$ for some $q(x)$ with rational coefficients and degree $n-1$ (strictly less than degree f). Since $f(\alpha) = 0$ and $(\alpha-\beta) \neq 0$, we must have $q(\alpha) = 0$. Let D be the common denominator of the coefficients of $q(x)$. Then $Dq(x)$ is an integer

polynomial of degree less than that of f with $Dq(\alpha) = 0$. This contradicts the minimality of f , so $f(\beta) \neq 0$. [Exercise: prove q has rational coefficients. Hint: suppose not and multiply out $(x-\beta)q(x)$.]

Lemma 2: $|f(\alpha) - f(\beta)| \; 10^{n \cdot j!} \geq 1$ for any j .

Proof: As $f(\alpha) = 0$, $|f(\alpha)-f(\beta)| = |f(\beta)|$. Now we can express $\beta = \sum\limits_{m=1}^{j} 10^{-m!}$ as a fraction $\dfrac{t}{10^{j!}}$ where t is a positive integer whose exact value is of no interest. Then

$$f(\beta) = \frac{C_n t^n}{10^{n \cdot j!}} + \frac{C_{n-1} t^{n-1}}{10^{(n-1) \cdot j!}} + \ldots + \frac{C_1 t}{10^{j!}} + C_0$$

so

$$f(\beta) \cdot (10^{n \cdot j!}) = C_n t^n + 10^{j!} \cdot C_{n-1} t^{n-1} + \ldots + 10^{(n-1) \cdot j!} C_1 t^1 + 10^{n \cdot j!} C_0$$

which is an integer, and non-zero by Lemma 1. Taking absolute values gives Lemma 2.

Lemma 3: There is an $N > 0$, independent of j , such that $|f(\alpha)-f(\beta)| < N(\alpha-\beta)$.

Proof: $f(x)$ has a continuous derivative so if $|x-\alpha| < 1$ say, we can find an integer $N > 0$ such that $f'(x) < N$. Now by the Mean Value theorem there is a $c \; \epsilon \; [\beta,\alpha]$ with $f(\alpha) - f(\beta) = (\alpha-\beta)f'(c)$. As $|\alpha-\beta| < 1$, we have $|\alpha-c| < 1$ so, taking absolute values yields Lemma 3.

Lemma 4: For some value of j , $|f(\alpha)-f(\beta)| \cdot 10^{n \cdot j!} < 1$

Proof: $|f(\alpha)-f(\beta)| \cdot 10^{n \cdot j!} < N \cdot (\alpha-\beta) \cdot 10^{n \cdot j!}$ by Lemma 3

$$< N \cdot \frac{2}{10^{(j+1)!}} \cdot 10^{n \cdot j!} = \frac{2N}{10^{(j+1)!-n \cdot j!}}$$

$$= \frac{2N}{10^{(j+1-n) \cdot j!}}$$ which we can make small by taking j large.

Theorem: α is transcendental.

Proof: Under the assumption that α was algebraic we derived the contradictory Lemmas 4 and 2.

For a more generalized result than this one, see John Oxtaby, Measure and Category, pp. 6-8.

Coleman-Ellis Retirement

On March 26, 1983 a banquet was held at Queen's University to mark the retirements of A. J. Coleman and H. W. Ellis. Many students and colleagues, past and present, were on hand to honour John and Hu. After some witty remarks by the Chairman, Lorne Campbell and the Principal, Ron Watts, Hu Ellis told the story of his life (which itself was worth the price of admission) and John Coleman indulged in some uncharacteristically brief reminiscences. At this point presentation was made of a photograph of the entire Department taken on the steps outside Jeffery Hall. The evening ended with assorted songs and poems from some of the junior staff, one of which is reproduced at the end.

Many (indeed most) of the readers of the Communicator will have passed through (and no doubt lingered in) the classrooms of one or both of these men and so we cannot resist the opportunity to highlight their careers.

John Coleman graduated in mathematics from the University of Toronto in 1939. He was a member of "the famous Putnam trio": Kaplansky, Mendelsohn and Coleman, who won the first Putnam Math Competition in 1938 for the University of Toronto. He continued with an M.Sc. at Princeton and in '43 received his Ph.D. from Toronto in Relativistic Quantum Mechanics under Leopold Infeld. From '43 - '45 he was on staff at Queen's under the headship of Ralph Jeffery. He left in 1945 for a four year stint as travelling secretary for the World Student Christian Federation in Geneva. On his way home, in '49, a rare distinction befell him: he was admitted to the secret château where the famous French enigma Nicholas Bourbaki did his work. He was thus one of the first people (and probably the second North American) to discover that Bourbaki was actually the nom de plume of a group of the best young French mathematicians of the day.

From '49 - '60 he was on staff at the University of Toronto and in 1960 came to Queen's to succeed Jeffery as Head of the Mathematics Department. He thereby became only the fifth Head of the Department since its founding in 1842. (That's 30 years per Head!) With such a tradition of longevity behind him, it is little wonder that he retained this position for 20 years, stepping down in 1980. During this period he presided over extraordinary growth and change in the Department: from 9 staff in Carruthers Hall in 1960 to 48 in Jeffery Hall in 1980.

The career of Hu Ellis is best chronicled by the man himself. Let us reproduce the text of his banquet address:

"On one's 65th birthday it is natural for one to think he'll soon be getting old. Since it is assumed that an old gaffer will be garrulous and I've always been a conformist, prepare yourselves while I talk about myself.

I was born in Port Maitland, a fishing village in Nova Scotia where the Bay of Fundy meets the Atlantic - about 8 miles from Jeffery's birthplace. My ancestors came to Nova Scotia from Massachussets before the revolution.

I'm a product of the Port Maitland school system, not a one room school, mind you, but a four roomer with four teachers to take from Grade 1 through 11. Six of us took Grade 12 with some help out of school hours by the principal. Doug Snow of Acadia and I passed the provincial exams.

We graduated in 1934 in the middle of the great depression. I became a fisherman (in Jeffery's footsteps). He often said, facitiously I think, that he wasn't sure he'd made the right choice in leaving fishing for an academic career. Since I sometimes got seasick it was no contest for me.

Leaving shore at 6 a.m. on a cold February morning to pound 15 miles off shore to haul lobster traps on the Lucher shoals was challenging, especially when you knew it would take three hours to run to safety if the wind breezed up too much. It was nearly as challenging as meeting a class of second year engineers at 8:30 a.m.

In 1937, as a retired fisherman at age 19, I enrolled at Acadia thanks to a generous scholarship of $150 (about one third of a year's expenses). As far as I know only one of my ancestors had gone to university. He graduated from Harvard in the Class of '68, 1668 that is. He was my great[6] grandfather Zecheriah Whitman, whose father had come to Weymouth, Massachussets in 1638.

After the first year I financed myself with scholarships, waiting on tables and summer bell-hopping at the C.P.R. Digby Pines Hotel.

I met Dorothy in my second year. Her freshette placard during initiation read, "Saint Petersburg, Florida, the Sunshine City".

I graduated in 1940 and continued on to take a Master's degree under Jeffery. In February 1942 I went to Halifax to join the naval branch of the N.R.C., concerned with degaussing protection of ships against magnetic mines. A year later the Naval Research Establishment was formed and we were given commissions in the navy. Dorothy and I were married that year.

In '45 I was posted to Vancouver and at the end of the war I got out of the navy in time to register for graduate work at the University of Toronto in the Fall.

In September '47 I came to Queen's for a salary of $2700 and, except for sabbaticals, have been here since. I was tempted to accept an offer at the U. of Florida, Gainesville after my first sabbatical which was at the Institute for Advanced Study. I didn't go. Queen's was a good choice. I like the Kingston area in the summer, autumn and spring in that order. I've found the university and department to be most congenial. A big plus was the Summer Institute that met here for more than fifteen years. Above all the students have been inspiring.

There have always been a lot of "Yes, but" students - you go through a long explanation and as its sits there crystal clear in your mind the student says "Yes, but". However, we've had a lot of "Yes, and" students over the years. I can think of a lot of "world class" students: Hale Trotter, Jim Woods, Norm Rice, John Holbrook, Dave Gregory, Peter Taylor, Jim Verner. I'm pleased that my first Ph.D. student, Bill Eames, is here.

I'll close by plagiarizing words of Dr. MacNeil, who was Treasurer of Queen's during the great depression. "I'm not a Queen's man born, nor a Queen's man bred, but when I die there'll be a Queen's man dead."

Thanks for coming."

Hu Ellis, March 1983

Coloured Chalk

Dedicated to Hu and John, March 26, 1983.

Hu Ellis asked John Coleman,
And John Coleman asked Eileen:
Could we have some coloured chalk
For classroom three-nineteen?
AJC asked Mrs. Wight and
Mrs. Wight said:
That's alright
I'll go and ask the janitor
Before he starts to clean.

Mrs. Wight she curtsied,
And went and asked the janitor:
Don't forget the coloured chalk
For classroom three-nineteen.
The janitor said sleepily:
You'd better tell
Herr Doktor
That many people nowadays
Use overhead
And screen.

Mrs. Wight
Said, Fancy!
And went to see
John Coleman,
She curtsied to her Head, and
She turned a little green:
Excuse me,
Dr. Coleman,
For taking of
The liberty,
But overhead projectors are the latest thing
And we've just bought six of them
And you can get ten colours
And they even sometimes work if you
Don't stand
In between.

John Coleman said:
Oh!
And went to
Hu Ellis:
Talking of the coloured chalk
For classroom three nineteen,
Many people
Think that
Overheads
Are nicer.
Would you like to try an
Overhead
And screen?

Hu Ellis said:
Bother!
And then he said:
Oh, dear me!
Hu Ellis sobbed:
Oh, deary me!
And went to see the Dean.
Nobody,
He whimpered,
Could call me
A fussy man;
I only want
Some coloured chalk
For classroom
Three-nineteen.

The Dean said:
There, there!
And went to
Mrs. Wight.
Mrs. Wight said:
There, there!
And went downstairs.
The janitor said:
There, there!
I didn't really
Mean it;
Here's a brand new box
Of coloured chalk
With brown and blue and green.

John Coleman took
The coloured chalk
And brought it to
His colleague;
Hu Ellis said:
By cracky!
Get those blackboards clean.
Nobody, he said,
As he raced down the corridor.
Nobody, he said,
As he commandeered the elevator,
Nobody,
Except possibly my Math 120 students
Could call me
A fussy man,
BUT
**I do like a little coloured chalk
in three-nineteen!**

Mathematics and Engineering News
Bill Woodside

The Mathematics and Engineering program (formerly known as course J) has now produced about 240 engineers since the first class graduated in 1967. The program, still the only one of its kind in Canada (and perhaps North America) has just recently been re-accredited for a further five year period by the Canadian Accreditation Board of the Canadian Council of Professional Engineers. There are currently options in the following areas of engineering: control and communications, computer science, applied mechanics, thermosciences, process control, and structural engineering.

Three members of the 1983 graduating class, **Stewart Morris, Glenn Swanson** and **Amanda Hubbard,** won NSERC Postgraduate Scholarships. Stewart, who won the University Medal in Mathematics and Engineering, accepted a position with Miller Communications in Ottawa. Glenn has already embarked on a Master's program in Electrical Engineering at Queen's and Amanda has been accepted as a doctoral student in Plasma Physics at Imperial College of Science and Technology in London. All three, together with **Felix Lee,** graduated with first-class honours.

Even in the very difficult job market which existed last spring, roughly half of the graduating class accepted permanent positions with the other half proceeding to graduate studies in a wide variety of fields including mathematics, physics, computing science, electrical and mechanical engineering.

The program, although just as difficult and demanding as ever, is attracting increasing interest. Last year the second year enrolment was 31; this September we may be welcoming as many as 36 students.

A successful first annual Mathematics and Engineering Banquet, organized by the Applied Mathematics Club, was held in the General Wolfe Hotel on Wolfe Island on the evening of Saturday, January 22.

An Applied Science Microcomputer Study Committee including Professor **Jon Davis** was formed recently to investigate the potential use of microcomputers in engineering programs at Queen's. The Committee is considering the desirability of requiring every engineering student to have his (her) own personal microcomputer (something comparable to the IBM PC) beginning perhaps with the first year class entering Queen's in September 1984 or 1985.

NEWS

From Jim Verner

Queen's - Rabat Exchange

For two years our Department has been co-operating with the Department of Mathematics of L'Université Mohammed V, Rabat, Morocco. Some faculty from each Department have the opportunity to visit the other to undertake short periods of research and discussion. This stimulus is of particular benefit to the Department of Rabat.

Until four years ago, Rabat was the only university in Morocco. At the time, there were schools of Letters, Science and Engineering which have in the interim been amalgamated. Now there are five universities, and the Ministry of Education plans for a total of nine in the next few years.

Because Rabat had the most advanced Department of Mathematics, it is presently the only one offering a graduate program. It will be graduates of this program that will become faculty in the developing Departments of Mathematics. With the proposed rapid development, the demand for faculty is considerable. Accordingly, the Department at Rabat has a considerable responsibility in producing graduates. Moreover, any assistance provided by Queen's will be very welcome.

This past May, I spent twelve days at Rabat. I gave lectures (primarily in English) on both instructional and research material in Numerical Analysis. My hosts responded with both questions and problems for discussions (primarily in French). (Because the large fraction of mathematical literature is in English, most of the faculty have a good reading knowledge in English and many can understand the spoken language.)

Their undergraduate honours Mathematics program is at least as difficult if not more so than our own. The Moroccan program puts a greater emphasis on tutorial classes to supplement core lectures with as much as one and one-half hours of tutorial for each hour of lecture. In spite of the lower standard of teaching facilities, the students seem to enjoy and work diligently at their program of study.

From Terry Smith

At the 143rd Annual Meeting of the American Statistical Association, held in Toronto in August, the honour of a Fellowship in the Association was conferred upon Dr. **Agnes M. Herzberg** (Hons. Math '62). The citation was as follows:

> AGNES M. HERZBERG, Lecturer, Imperial College of Science and Technology, University of London; for outstanding contributions to the theory of design of experiments; for excellence in teaching; for significant editorial activities; and for unstinting service to the profession.

Dr. Herzberg had her first serious encounter with statistics in a course given by Emeritus Professor G. L. **Edgett**.

At the same meeting several members of the department presented papers. Professor **C. R. Blyth** spoke on "Maximum Probability Estimation" at a special invited session in memory of **Jacob Wolfowitz**. At a session entitled <u>Statistical Education for the 90's</u>, Professor D. G. **Watts** presented an invited paper by Professors Watts, Griffin and Smith entitled "Using Objectives to Structure a Statistics Program". And at a special contributed session organized by the Canadian Statistical Society, Professor **T. W. F. Stroud** spoke on "Two-Stage Sample Design Based on Knowledge of the Total Variance".

From Bruce Kirby

I have received a letter from **Alan Poplove** (Maths. and Eng. '79, M.Sc. with Lorne Campbell) who plans to pursue a Ph.D. He writes

> I have been working at the Communications Security Establishment of the Department of National Defence since December, 1980. The work involves the design, development, and evaluation of communications security equipment for DND and other government agencies.
>
> Since our department at work is currently understaffed, my responsibilities are mostly associated with the engineering of cryptographic algorithms. Thus I have decided to pursue a Ph.D. in combinatorics at Waterloo University.

From **Bill Woodside**

    Charlie Whitfield (Math-Eng. '68, M.Sc., Case Western Reserve 1970) Chief Supervisor, Process Control, Algoma Steel Corporation, Sault Ste. Marie has accepted an invitation to join the Queen's Engineering Advisory Council's Sub-Committee for Mathematics and Engineering. The Council visits the university each February for an intense two-day program and provides a vital external perspective on the curriculum. He is the first graduate of the program so honoured. **Ron Dimock** (Math-Eng. '71, L.L.B. Queen's 1973) is now a partner in the patent and trade-mark law firm of Sims, Hughes in Toronto. **Hugh Cameron** (Math-Eng. '73, Ph.D. Cambridge, under Sir James Lighthill) is with Bell Northern Research in Montreal and leaves shortly for a one year leave in Paris with his wife, **Heather Hume** (Math '72, M.D. '78). His brother **Alan Cameron** (Math-Eng. '76) will shortly complete a Ph.D. in Chemistry at Queen's. **John Cartledge** (Math-Eng. '74, Ph.D. Queen's 1979 under L. L. Campbell) who was with Bell Northern Research is now an Assistant Professor in Electrical Engineering at Queen's. **Alan Poplove** (Math-Eng. '79, M.Sc. 1980), formerly working on cryptographic devices with the Communications Security Establishment of the Department of National Defence in Ottawa, has now begun a Ph.D. program in combinatorics at the University of Toronto. **Paul Tseng** (Math-Eng. '81) has almost completed his Ph.D. studies in operations research at M.I.T.

Too Short Poems

(i)
(e)yes

(ii)
(d)anger

- Manontroppo

("Manontroppo" is an alter ego of a being who is among other things an Associate Professor in the Department of Mathematics and Statistics at Queen's.)

PROBLEM: The Game of Wythoff

Wythoff is a number-theoretic game which deserves to be better known. The rules and objective of the game are easy enough to state, but discovering how to play it is another matter. Recently (last June) the 15 national winners of this year's Canadian Math Olympiad spent a week at Queen's University learning a bit of mathematics, playing with some new problems, and "besporting" themselves. In one of the sessions I had with them, I introduced them to Wythoff and challenged them to figure out how to play. Let me summarize what I told them.

Wythoff is a 2 person game. At any moment the _game position_ is a pair (x,y) of numbers (positive integers). Players move alternately, each time reducing the game position (x,y) to any new one of one the following three types:

(1) $(x-k,y)$ for some $k>0$

(2) $(x,y-k)$ for some $k>0$

(3) $(x-k,y-k)$ for some $k>0$.

Any such position is called a _descendent_ of (x,y) . A player wins by moving to the position $(0,0)$.

A position is called _W (winning)_ if the player who has that position at the _start_ of his move has a strategy which will guarantee a win no matter what the other player does. A position is called _L (losing)_ if the player who leaves that position at the _end_ of his move has a winning strategy no matter what the other player does. Clearly a position is L if and only if all its descendents are W , and a position is W if and only if it has at least one descendent which is L .

Which positions are W and which are L ? We can find examples of each kind by starting at $(0,0)$ and working up. Clearly $(0,0)$ is L (a player who starts with this has already lost). Thus $(0,1)$ is W and indeed any $(0,k)$ is W . Also $(1,1)$ is W [move to $(0,0)$] but $(1,2)$ is L [all descendents are W]. Thus $(1,k)$ is W for all $k>2$ [move to $(1,2)$]. We see we can continue to work up and assign either W or L to every position. Thus every position is either a W or L . A list of the first few L positions appears below. We will index these by n (starting at n=0) and call the nth L position (x_n,y_n) with $x_n \leq y_n$.

n	(x_n, y_n)	n	(x_n, y_n)	n	(x_n, y_n)
0	(0,0)	6	(9,15)	12	(19,31)
1	(1,2)	7	(11,18)	13	(21,34)
2	(3,5)	8	(12,20)	14	(22,36)
3	(4,7)	9	(14,23)	15	(24,39)
4	(6,10)	10	(16,26)	16	(25,41)
5	(8,13)	11	(17,28)	17	(27,44)

It becomes clear that the pattern is determined by the following rules:

(1) At each step x_n is the least integer which has not yet occurred and

(2) $y_n = x_n + n$.

I made the boys (they were alas all male!) generate this table themselves. Take the time to satisfy yourself that this is really the pattern of the losing positions.

Now how do we play the game? Well, if we have a copy of the L table, it's really quite simple. Suppose I give you (24,34) and it's your move. Observe that $34 - 24 = 10$ and move to the 10th losing pair (16,26). Or you could subtract 3 from 24 and move to (21,34) . But if I give you (24,47) , you have no choice; you must move to (24,39) . Of course if I give you (24,39) , then you have been given a losing pair and you will have no choice but to move to some winning pair and hope I make a mistake.

What's the general rule? As an excellent exercise write out a precise set of instructions (a computer program if you like) for moving from any W position, assuming the L table is known.

Now here's the crunch. What happens if I give you (6721,8214) and say it's your move. What do you do? Of course you could generate some more of the table, but that takes time, even for a computer. It would be better to have an explicit formula for x_n and y_n in terms of n . Can you find one?

That's in fact the challenge I gave "the boys". I thought it would take them at least an hour. It didn't. Would you believe 5 minutes?

Solution to the Problem: How to exceed g   (March '83)

In terms of response this was one of our most successful problems. We received many letters: some with great humour and tact, and some with a comprehensive analysis of the problem. (And some with some of each!)

The problem concerned a cone-shaped cup of water with the bottom snipped off at height ε . Some equations were presented for the height h of the water in the cup as it runs out, which predict that the acceleration d^2h/dt^2 would exceed g (downwards) as soon as h fell below $4\sqrt{3}\varepsilon$. How could this be?

It turns out that there are two flaws in our equations - only one of which we were aware of when we set the problem. The first is that a more careful definition of h will lead to slightly different conservation of energy equation than $v^2 = 2gh$. Letting h be, not the depth of the water, but its height above the apex of the cone, the equation becomes $\frac{dh}{dt} = -2g(h-\varepsilon)$ which solves to give an acceleration $d^2h/dt^2 = -g(3x-4)/x^5$ where x = h/ε . This acceleration never even comes close to g but attains a maximum of $(0.6)^5g$ (downwards) at h = 5ε/3 .
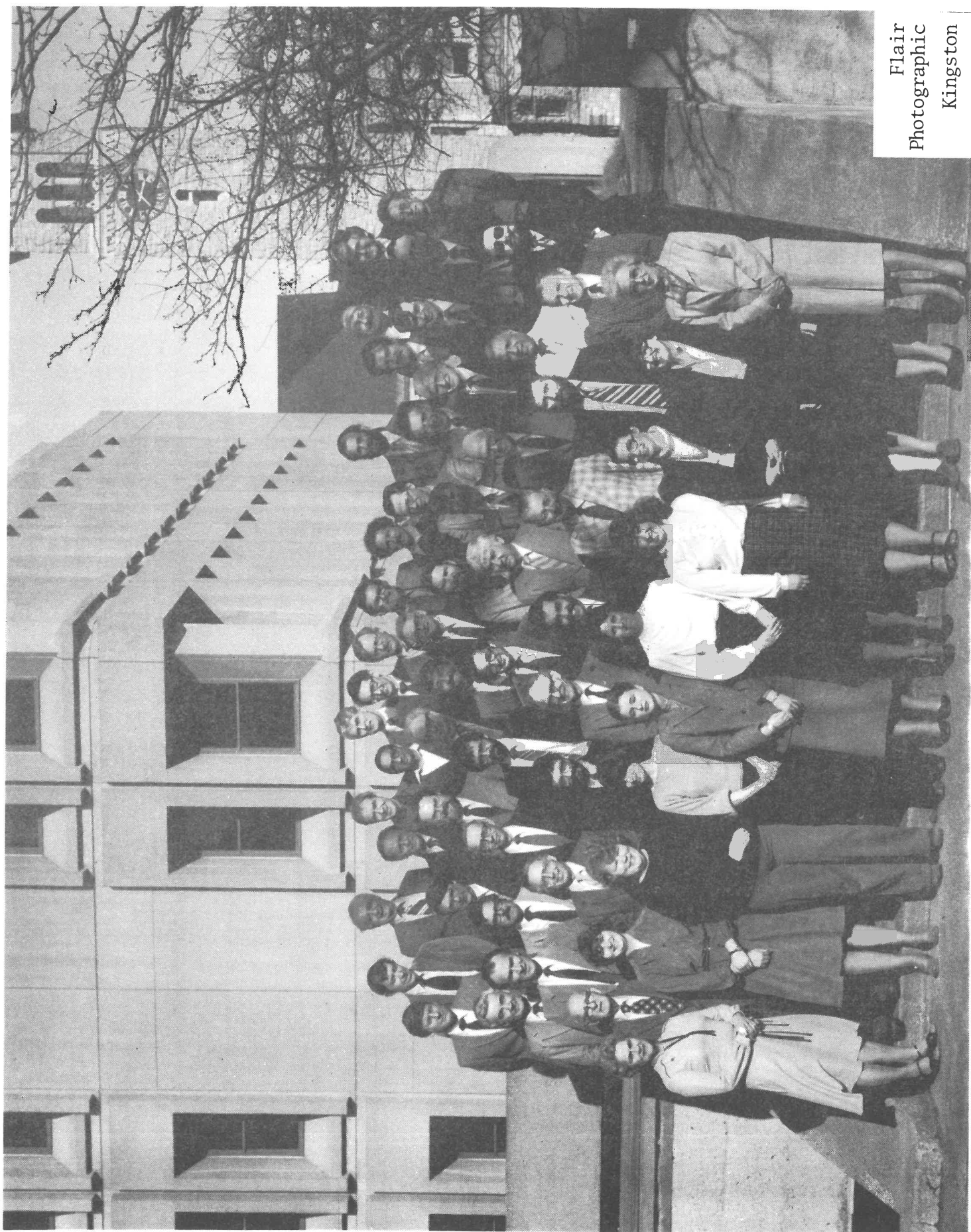
So far so good. This much we non-engineering types at the Communicator office had figured out, and so did several readers. But a second look at the expression for d^2h/dt^2 indicates that the acceleration will become zero when h drops to 4ε/3 and will point up for h < 4ε/3 !   This must be some cup that manages an upward force on its contents!

In fact, our conservation of energy equation is still not quite right. When balancing the kinetic energy of the escaping water with the loss of potential energy of the water in the cup we must also include the kinetic energy of the water in the cup, which is slowly moving down. The correct equation, which derives from Bernoulli's Law is

$$dh/dt = 2g(h-\varepsilon) + (dh/dt)^2 .$$

In most cases the last term is small and can be ignored, but when h gets close to ε , it becomes relatively important. This equation is not easy to solve and we will say no more about it. I invite any reader to solve it numerically and compare the results with those obtained from the omission of the last term.

The first flaw was noticed by Robin Giles, Ross Ethier, Alan Poplove, Santo D'Agostino (to whom we owe an apology for our cavalier treatment of physicists) and Kok-Chung Quan, and the second flaw was noted by the first two above and by Jon Davis and Mac Smith. Our thanks to all contributers.