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Moduli spaces of vortex knots for an exact fluid flow

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The moduli spaces $\mathcal{S}(\mathcal{D})$ of non-isotopic vortex knots are introduced for the ideal fluid flows in invariant domains $\mathcal{D}$. The analogous moduli spaces of the magnetic fields $\mathbf{B}$ knots are defined. We derive and investigate new exact fluid flows (and analogous plasma equilibria) satisfying the Beltrami equation which have nested invariant balls $\mathbb{B}_k^3$ with radii $R_k \approx (k+1)\pi$, $k \longrightarrow \infty$. The first flow is z-axisymmetric; the other ones do not possess any rotational symmetries. The axisymmetric flow has an invariant plane $z = 0$. Due to an involutive symmetry of the flow, its vortex knots in the invariant half-spaces $z > 0$ and $z < 0$ are equivalent. It is demonstrated that the moduli space $\mathcal{S}(\mathbb{R}^3)$ for the derived fluid flow in $\mathbb{R}^3$ is naturally isomorphic to the set of all rational numbers p/q in the interval $J_1 : 0.25 < \mathbf{q} < \tilde{M}_1 \approx 0.5847$, where $\mathbf{q}$ is the safety factor. For the fluid flow in the first invariant ball $\mathbb{B}_1^3$, it is shown that all values of the safety factor $\mathbf{q}$ belong to a small interval of length $\ell \approx 0.1261$. It is established that only torus knots $K_{p,q}$ with $0.25 < p/q < 0.5847$ are realized as vortex knots for the constructed flow in $\mathbb{R}^3$. Each torus knot $K_{p,q}$ with $0.25 < p/q < 0.5$ is realized on countably many invariant tori $\mathbb{T}_k^2$ located between the invariant spheres $\mathbb{S}_k^2$ and $\mathbb{S}_{k+1}^2$, while torus knots with $0.5 < p/q < \tilde{M}_1$ are realized only on finitely many invariant tori. The moduli spaces $\mathcal{S}_m(\mathbb{B}_a^3)(m = 1, 2, \ldots)$ of vortex knots are constructed for some axisymmetric steady fluid flows that are solutions to the boundary eigenvalue problem for the curl operator on a ball $\mathbb{B}_a^3$. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4973802]

I. INTRODUCTION

I. Equations of magnetohydrodynamics[1] in case of constant density ρ and vanishing resistivity have the form

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \operatorname{grad})\mathbf{V} = -\frac{1}{\rho}\operatorname{grad}\tilde{p} + \frac{1}{\rho\mu}\operatorname{curl}\mathbf{B} \times \mathbf{B} + \nu\Delta\mathbf{V}, \tag{1.1}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{V} \times \mathbf{B}), \quad \operatorname{div}\mathbf{V} = 0, \quad \operatorname{div}\mathbf{B} = 0,$$

where $\mathbf{B}$ is the magnetic field, $\mathbf{V}$ is the fluid velocity, $\tilde{p}$ is the pressure, μ is the magnetic permeability, and ν is the kinematic viscosity. Equations (1.1) imply[2] that the magnetic field $\mathbf{B}$ lines are transformed in time by the flow diffeomorphisms (or are "frozen in the flow").

Definition 1. Moduli space $\mathcal{S}(\mathcal{D})$ of magnetic field $\mathbf{B}$ *knots for a solution to Equations (1.1) in an invariant with respect to the vector field* $\mathbf{B}$ *domain $\mathcal{D}$ is a set that is in a one-to-one correspondence with all classes of isotopy equivalence of knots $\mathcal{K} \subset \mathcal{D}$ formed by the closed magnetic field* $\mathbf{B}$ *lines for the considered solution at a given time t.*

The frozenness of the magnetic field lines in the magnetohydrodynamic flow yields that the moduli space $\mathcal{S}(\mathcal{D})$ does not depend on time t and hence is an invariant of solutions to Equations (1.1).

II. Euler equations of dynamics of an ideal incompressible fluid with a constant density ρ have the form

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \operatorname{grad})\mathbf{V} = -\frac{1}{\rho}\operatorname{grad} p, \qquad \operatorname{div}\mathbf{V} = 0. \tag{1.2}$$

Here $\mathbf{V}(t,\mathbf{x})$ is the fluid velocity and $p(t,\mathbf{x})$ is the pressure.

As is well-known,[3,4] the vortex field curl $\mathbf{V}$ is transformed in time by the flow diffeomorphisms (or "is frozen in the flow"). This yields that any knot formed by a closed vortex line at a time t is transformed by the flow into an isotopic knot.

Definition 2. Moduli space $\mathcal{S}(\mathcal{D})$ of vortex knots for a solution to Equations (1.2) in an invariant domain $\mathcal{D}$ is a set that is in a one-to-one correspondence with all classes of isotopy equivalence of vortex knots $\mathcal{K} \subset \mathcal{D}$ existing for the considered solution at a given time t.

The frozenness of the vortex field curl $\mathbf{V}$ implies that the moduli space $\mathcal{S}(\mathcal{D})$ does not depend on time t and hence is an invariant of the fluid flows.

Remark 1. The moduli space $\mathcal{S}(\mathcal{D})$ evidently does exist for any hydrodynamic flow (1.2) and always is either countable or finite. Indeed, this follows from the fact that there is only a countable number of isotopy classes of knots in $\mathbb{R}^3$.[5,6]

We study the following well-known problem that was around since Kelvin's works.[7–9]

To classify all vortex knots $\mathcal{K} \subset \mathbb{R}^3$ (up to the isotopy equivalence) for concrete solutions to Euler equations (1.1).

In this paper, we develop a method to resolve this problem for axisymmetric fluid flows by constructing the corresponding moduli spaces $\mathcal{S}(\mathbb{R}^3)$ of vortex knots. As an application of the method we describe the moduli space $\mathcal{S}(\mathbb{R}^3)$ for the simplest of the derived fluid flows.

For the ideal fluid flows, the problem of finding the moduli space $\mathcal{S}(\mathcal{D})$ is closely connected to Kelvin's papers.[7–9] According to the historical studies by Epple,[12] the works by Helmholtz,[3] Kelvin,[7–9] and Tait[10,11] on vortex knots published in the 1850s–1880s had laid the foundation of what are now called the topological methods of hydrodynamics. Figure 1 represents the knots and links envisioned by Kelvin.

Remark 2. In Ref. 13 the existence theorem 1.1 is proven that states that for *any* link $L \subset \mathbb{R}^3$ and any $\lambda \neq 0$ "*one can transform L by a C^1 diffeomorphism Φ of $\mathbb{R}^3$ arbitrarily close to the identity in any C^r norm, so that $\Phi(L)$ is a set of stream lines of a Beltrami field u, which satisfies* curl $u = \lambda u$ *in* $\mathbb{R}^3$." Thus the authors of Ref. 13 have claimed that for any knot K (or link L) there exists a Beltrami flow such that an isotopic to K knot (or link) is realized by some trajectory of the flow. We study a completely different problem: for a *concrete* fluid flow to classify *all* isotopy types of knots which are realized by its vortex lines. This problem was not discussed in Ref. 13 and it is not the inverse problem for the one considered in Ref. 13.

III. In this paper, we present the moduli spaces $\mathcal{S}(\mathbb{R}^3)$ and $\mathcal{S}(\mathbb{B}^3_a)$ of vortex knots for an exact steady solution to Equation (1.2) in the Euclidean space $\mathbb{R}^3$ and in a ball $\mathbb{B}^3_a$ of radius a and the moduli spaces $\mathcal{S}(\mathbb{R}^3)$ and $\mathcal{S}(\mathbb{B}^3_a)$ of the magnetic field $\mathbf{B}$ knots for the analogous steady solution to the MHD Equations (1.1).

The steady ideal MHD equations ($\nu = 0$) have the form

$$\mathbf{V} \times \operatorname{curl} \mathbf{V} - \frac{1}{\rho\mu}\mathbf{B} \times \operatorname{curl} \mathbf{B} = \operatorname{grad}\left(\frac{\tilde{p}}{\rho} + \frac{1}{2}|\mathbf{V}|^2\right), \tag{1.3}$$

$$\operatorname{curl}(\mathbf{V} \times \mathbf{B}) = 0, \quad \operatorname{div} \mathbf{V} = 0, \quad \operatorname{div} \mathbf{B} = 0.$$

For the collinear vector fields $\mathbf{V}(\mathbf{x}) = \alpha\mathbf{B}(\mathbf{x})$, Equations (1.3) reduce to

$$(\alpha^2\rho\mu - 1)\mathbf{B} \times \operatorname{curl} \mathbf{B} = \operatorname{grad}\left(\mu\tilde{p} + \alpha^2\rho\mu|\mathbf{B}|^2/2\right), \quad \operatorname{div} \mathbf{B} = 0. \tag{1.4}$$

The simplest case of Equations (1.4) for $\alpha = 0, \mathbf{V} = 0$ describes the plasma equilibria

$$\mathbf{B} \times \operatorname{curl} \mathbf{B} = \operatorname{grad}(-\mu\tilde{p}), \quad \operatorname{div} \mathbf{B} = 0. \tag{1.5}$$

It is evident that for any constant $\alpha \neq \pm 1/\sqrt{\rho\mu}$, Equations (1.4) are equivalent to the plasma equilibrium Equations (1.5).

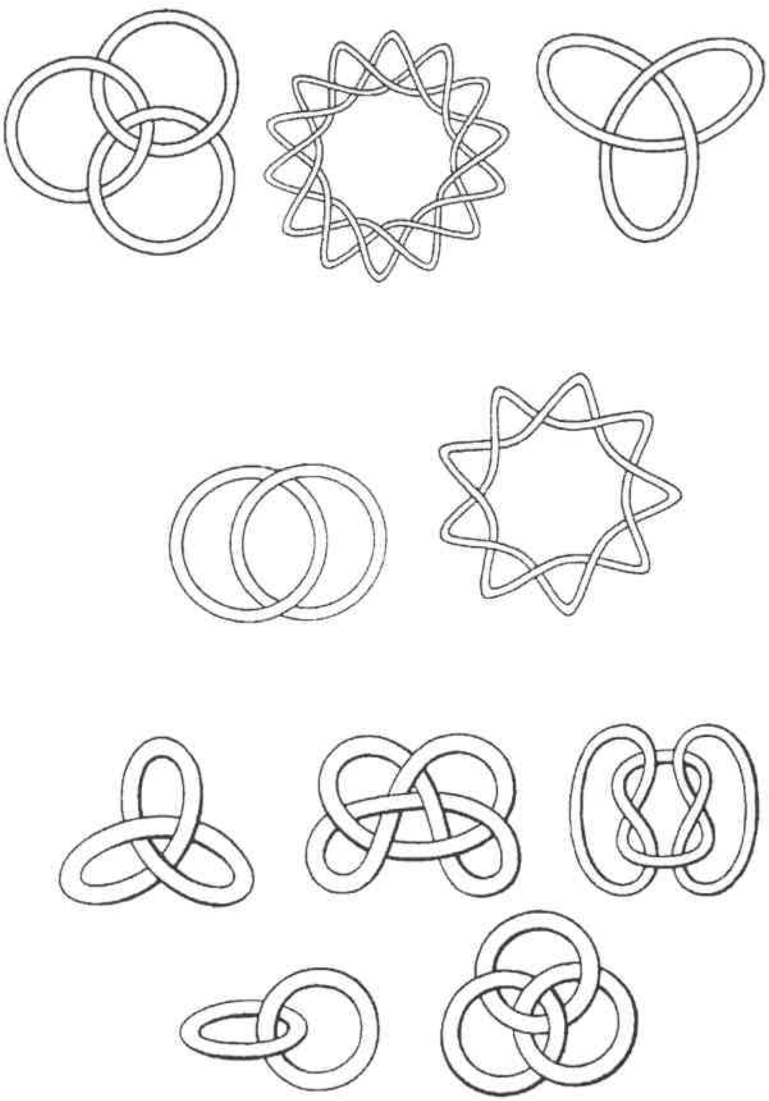

FIG. 1. Figures of knots and links obtained from Refs. 8 and 9.

The steady hydrodynamics equations in the Bernoulli form are

$$\mathbf{V} \times \operatorname{curl} \mathbf{V} = \operatorname{grad}\left(p/\rho + |\mathbf{V}|^2/2\right), \quad \operatorname{div} \mathbf{V} = 0. \tag{1.6}$$

Since plasma equilibrium Equations (1.5) and the steady hydrodynamics Equations (1.6) are equivalent,[14] any result proven for one of them is equally applicable to another.

In a pioneering paper in 1958,[15] Kruskal and Kulsrud proved for Equations (1.5) that surfaces $\tilde{p}(\mathbf{x}) = \text{const}$ *"by* $\mathbf{B} \cdot \nabla p = 0$ *are "magnetic surfaces," in the sense that they are made up of lines of magnetic force, and simultaneously by* $\mathbf{j} \cdot \nabla p = 0$ *they are "current surfaces." If such a surface lies in a bounded volume of space and has no edges and either* $\mathbf{B}$ *or* $\mathbf{j}$ *nowhere vanishes on it then by a well-known theorem*[16] *it must be a toroid (by which we mean a topological torus) or a Klein bottle. The latter, however is not realizable in physical space."*

In a paper in 1959,[17] Newcomb stated that *"It is easy to verify that the lines of force on a pressure surface are closed if and only if* $\mathbf{i}(P)/2\pi$ *is rational; if it is irrational, the lines of force cover the surface ergodically."* Here $\mathbf{i}(P)$ is the rotational transform connected with the safety factor $\mathbf{q}(P)$[18] by the relation $\mathbf{q}(P) = 2\pi/\mathbf{i}(P)$.

In 1965, the analogous results for the equivalent Equations (1.6) were published by Arnold in Ref. 19 and in Ref. 20 where he added to Refs. 15 and 17 a statement that if a Bernoulli's surface M intersects the boundary of the invariant domain D then M has *"co-ordinates of the ring"* and *"all streamlines on M are closed."*

The results of Refs. 15 and 17 yield that for the plasma equilibrium Equations (1.5) (and hence for the equivalent hydrodynamics Equations (1.6)) all magnetic field **B** knots (and correspondently all vortex curl **V** knots) are torus knots $K_{p,q}$ defined by the rational values p/q of the safety factor $\mathbf{q}(P)$.[40] Therefore, to classify the magnetic knots it is necessary to know the range of the safety factor $\mathbf{q}(P)$.

IV. We study in this paper the vector fields $\mathbf{B}(\mathbf{x})$ obeying the Beltrami equation

$$\operatorname{curl}\mathbf{B} = \lambda\mathbf{B} \tag{1.7}$$

and therefore satisfying plasma equilibrium Equations (1.5) with $\tilde{p} \equiv \text{const}$. The equivalent hydrodynamics equilibria (1.6) are defined by equations

$$\operatorname{curl}\mathbf{V} = \lambda\mathbf{V}, \quad p = C - \rho|\mathbf{V}|^2/2. \tag{1.8}$$

Chandrasekhar,[21] Chandrasekhar and Kendall,[22] and Woltjer[23] presented an infinite basis of solutions to the Beltrami equation (1.7) in terms of the Bessel and Legendre functions.

In Refs. 24 and 25, we derived an integral representation of Beltrami fields (1.7) which depends on an arbitrary vector field $\mathbf{T}(\mathbf{x})$ tangent to the unit sphere $\mathbb{S}^2$.

The spectrum and the eigenvector fields for the boundary eigenvalue problems for the operator curl on different domains in $\mathbb{R}^3$ were studied in Refs. 26–30 which use the Bessel and Legendre functions representation of Refs. 21–23.

V. In Sections II and III we present exact solutions to Beltrami equation (1.7) and (1.8) in terms of elementary functions. The constructed fluid flows have nested invariant balls $\mathbb{B}^3_k$. Some of the flows are axisymmetric; the other ones have no rotational symmetries and allow only discrete symmetries.

A method of construction of the moduli spaces of vortex knots $\mathcal{S}(\mathcal{D})$ for the axisymmetric fluid flows is presented in Section IV. The method is based on the investigation of functions of periods $\tau_k(H)$ *(in a special time variable τ)* of closed trajectories of a certain 2-dimensional dynamical system in invariant domains $\mathcal{D}_{k,\pm} \subset \mathbb{R}^2$. The system is obtained by *(1)* reduction of the main axisymmetric dynamical system in $\mathbb{R}^3$,

$$\frac{d\mathbf{x}}{dt} = \operatorname{curl}\mathbf{V}(\mathbf{x}), \tag{1.9}$$

in the cylindrical coordinates r, z, φ to a 2-dimensional system on the plane $\mathbb{R}^2$ with coordinates r, z and *(2)* a special choice of the time variable τ satisfying equation $d\tau/dt = H(r,z)/(2\pi r^2)$ which becomes singular at the boundaries $H(r,z) = 0$ of the invariant domains $\mathcal{D}_{k,\pm}$. The function $H(r,z)$ is a first integral of the dynamical system (1.9) and coincides with the Stokes stream function $\psi(r,z)$ of the axisymmetric flow $\mathbf{V}(\mathbf{x})$. The function of periods $\tau(H)$ of the closed trajectories C_H ($H(r,z) = \text{const}$) is connected with the safety factor $\mathbf{q}(H)$ and the pitch $p(H)$ of the corresponding helical vortex lines on the invariant tori $\mathbb{T}^2_H$ by the relations

$$\tau(H) = \mathbf{q}(H) = \frac{1}{2\pi}p(H). \tag{1.10}$$

Therefore, for the analogous magnetic fields $\mathbf{B}(\mathbf{x})$, the function of periods $\tau(H)$ coincides with the safety factor $\mathbf{q}(\psi)$.[18,31]

All vortex knots for the derived exact axisymmetric fluid flows are torus knots $K_{p,q}$ that correspond to the rational values of a function of periods $\tau(H) = p/q$. Therefore, the moduli space of vortex knots $\mathcal{S}(\mathcal{D})$ is the set of all rational numbers in the range of function $\tau(H)$ that coincides with the safety factor $\mathbf{q}(H)$.

Using two different limiting procedures, we derive the exact lower and upper bounds for the ranges of the continuous functions $\tau_k(H)$ for each invariant domain $\mathcal{D}_{k\pm}$. Both the bounds are finite positive numbers that are presented by exact formulae. Our construction of the moduli spaces

$\mathcal{S}(\mathcal{D}_{k\pm})$ is based on the derived exact lower and upper bounds for all functions of periods $\tau_k(H)$ for $k = 1, 2, \ldots$.

The ranges and domains of functions $\tau_k(H)$ and pitch functions $p_k(H)$ (1.10) are obtained in exact form in Sections V and VI.

In Section II we derive an exact axisymmetric Beltrami flow $\mathbf{V}_2(\mathbf{x})$ (2.21) for which the moduli space $\mathcal{S}(\mathbb{R}^3)$ of vortex knots (as we demonstrate in Section VIII) is naturally isomorphic to the set of all rational numbers p/q in the interval

$$J_1: \quad 0.25 < \tau < \tilde{M}_1 = \frac{1}{2\sqrt{1 - 6\tilde{R}_1^{-2}}} \approx 0.5847, \tag{1.11}$$

where $\tilde{R}_1 \approx 3.8702$ is the smallest positive root of equation $\tan r = r(r^2 - 6)/(3r^2 - 6)$. The interval J_1 (1.11) is the complete range of function of periods $\tau(H)$ in the whole space $\mathbb{R}^3$. The complete range of the pitch function $p(H)$ (1.10) is $2\pi \cdot J_1$.

The same moduli space $\mathcal{S}(\mathbb{R}^3)$ exists for the magnetic field $\mathbf{B}$ knots for the equivalent exact solution to Equations (1.5) and for the analogous MHD equilibrium solutions to Equations (1.4) with $\alpha \neq \pm 1/\sqrt{\rho\mu}$.

VI. In Ref. 32, Moffatt considered the general steady axisymmetric fluid flows corresponding to arbitrary smooth functions $F(\psi)$ and $G(\psi)$ in the Grad - Shafranov equation[4] for the stream function $\psi(r, z)$,

$$\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} = r^2\frac{\mathrm{d}F}{\mathrm{d}\psi} - G\frac{\mathrm{d}G}{\mathrm{d}\psi}. \tag{1.12}$$

Equation (1.12) is equivalent to the steady axisymmetric Equations (1.6). Moffatt formulated in Ref. 32, p. 29, the statement.

"The streamlines within these vortices are topologically similar to those of the special case when $F(\psi)$ and $G(\psi)$ are linear in ψ, i.e., they are helices wrapped on the family of nested tori $\psi = \mathrm{cst}(0 < \psi < \psi_{\max})$, the pitch of the helix varying continuously from zero ... to infinity ..."

In this paper we show that constructed in Section II exact axisymmetric flow $\mathbf{V}_2(\mathbf{x})$ (2.21) provides a counterexample to the above statement.[32] Indeed, from formulae (1.10) and (1.11) we get that the pitch function $p(\psi)$ is changing in the limits

$$0.5\pi < p(\psi) < 1.1694\pi \tag{1.13}$$

for all possible values of the function $\psi(r, z) = H(r, z)$ in the whole space $\mathbb{R}^3$. Moffatt claims in Ref. 32 that the pitch function $p(\psi)$ is "varying continuously from zero to infinity." The inequalities (1.13) prove that the quoted statement of Ref. 32 does not correspond to the facts.

For the exact flow $\mathbf{V}_2(\mathbf{x})$ (2.21), Equation (1.11) yields that only those torus knots $K_{p,q}$ are realized as vortex knots for which $0.25 < p/q < 0.5847$ and not for any p/q as it would follow from Moffatt's statement.[32]

VII. In Section VII we study the pitch functions $p(\psi) = 2\pi\mathbf{q}(\psi)$ for the general axisymmetric fluid flows. We prove that at any stable vortex axis defined by $\psi(r, z) = \psi_m$ (where ψ_m is a local maximum or minimum of function $\psi(r, z)$) the pitch function $p(\psi)$ has a finite non-zero limit $\lim_{\psi \longrightarrow \psi_m} p(\psi) = p(\psi_m) < \infty$. The number $p(\psi_m) \neq 0$ is one of the two exact bounds for the range of function $p(\psi)$. The latter therefore cannot change continuously from zero to infinity. This result holds for any axisymmetric fluid flow and provides counterexamples to one of the statements of Moffatt's highly quoted paper Ref. 33, p. 129 about the pitch function $p(\psi)$ for some concrete solutions to Equation (1.12):

"This quantity clearly increases continuously from zero to infinity as ψ increases from zero (on $R = a$) to $\psi_{\max}$ (on the vortex axis)."

Analogously we get counterexamples to Moffatt's statement of Ref. 34, pp. 30 - 31 concerning the spheromak field $\mathbf{B}$ satisfying Beltrami equation (1.7):

"Each $\mathbf{B}$-line is a helix and the pitch of the helices decreases continuously from infinity on the magnetic axis to zero on the sphere $r = R$ as we move outwards across the family of toroidal surfaces."

For any steady axisymmetric fluid flow in an invariant domain $\mathcal{D}$ the moduli space of vortex knots $\mathcal{S}(\mathcal{D})$ is the set of all rational numbers in the range of the safety factor $\mathbf{q}(\psi) = p(\psi)/(2\pi)$. Therefore, results of Section VII imply that different steady fluid flows have different moduli spaces of vortex knots. The moduli spaces of vortex knots for the spheromak fluid flow in invariant balls $\mathbb{B}_a^3$ and in the whole Euclidean space $\mathbb{R}^3$ will be published in Ref. 35.

VIII. In Section IX we construct the moduli spaces $\mathcal{S}_m(\mathbb{B}_a^3)$ $(m = 1,2,\ldots)$ of vortex knots for the axisymmetric fluid flows $\mathbf{V}_{2m}(\mathbf{x})$ inside a ball $\mathbb{B}_a^3$ of radius a which are tangent to the boundary sphere $\mathbb{S}_a^2$. Here integer $m \geq 1$ enumerates a special series of eigenvalues λ_m and the corresponding axisymmetric eigenvector fields $\mathbf{V}_{2m}(\mathbf{x})$ for the boundary eigenvalue problem for the operator curl on the ball $\mathbb{B}_a^3$. The spectrum of this eigenvalue problem was studied in Refs. 26–30. We show that the moduli spaces $\mathcal{S}_m(\mathbb{B}_a^3)$ and $\mathcal{S}_\ell(\mathbb{B}_a^3)$ are different for $m \neq \ell$, do not depend on the radius a, and that $\mathcal{S}_m(\mathbb{B}_a^3)$ tends to $\mathcal{S}(\mathbb{R}^3)$ when $m \longrightarrow \infty$.

II. EXACT STEADY FLUID FLOWS

I. Let us construct exact axisymmetric solutions to the steady Euler equation (1.6) which simultaneously are solutions to the Beltrami equation

$$\operatorname{curl} \mathbf{V}(\mathbf{x}) = \mathbf{V}(\mathbf{x}). \tag{2.1}$$

The change of variables $\tilde{x}_i = \lambda x_i$ yields the equivalence of eigenvector fields (1.7) and (1.8) for the curl operator with eigenvector fields (2.1) corresponding to the eigenvalue $\lambda = 1$. Chandrasekhar in Ref. 21 and Chandrasekhar and Kendall in Ref. 22 introduced a representation of Beltrami fields in the form

$$\mathbf{V}(\mathbf{x}) = \operatorname{curl} \mathbf{S}(\mathbf{x}) + \operatorname{curl}\operatorname{curl} \mathbf{S}(\mathbf{x}), \tag{2.2}$$

where $\mathbf{S}(\mathbf{x})$ satisfies the vector Helmholtz equation

$$\Delta\mathbf{S}(\mathbf{x}) = -\mathbf{S}(\mathbf{x}). \tag{2.3}$$

Applying the identity $\operatorname{curl}\operatorname{curl} \mathbf{S}(\mathbf{x}) = \operatorname{grad}(\operatorname{div}\mathbf{S}(\mathbf{x})) - \Delta\mathbf{S}(\mathbf{x})$ to (2.2) and using Equation (2.3), we find

$$\mathbf{V}(\mathbf{x}) = \mathbf{S}(\mathbf{x}) + \operatorname{curl} \mathbf{S}(\mathbf{x}) + \operatorname{grad}(\operatorname{div}\mathbf{S}(\mathbf{x})). \tag{2.4}$$

In what follows, we use the representation (2.4) for the Beltrami fields.

We choose solutions to Equation (2.3) in the form $\mathbf{S}(\mathbf{x}) = f(\mathbf{x})\hat{\mathbf{e}}_z$, where $f(\mathbf{x})$ obeys the scalar Helmholtz equation

$$\Delta f(\mathbf{x}) = -f(\mathbf{x}) \tag{2.5}$$

and $\hat{\mathbf{e}}_z$ is the unit ort in the Cartesian coordinates x, y, z. We will use cylindrical coordinates r, φ, z defined by relations $r = \sqrt{x^2 + y^2}$, $x = r\cos\varphi$, $y = r\sin\varphi$. The corresponding orthogonal unit orts are

$$\hat{\mathbf{e}}_z, \quad \hat{\mathbf{e}}_r = \cos\varphi\hat{\mathbf{e}}_x + \sin\varphi\hat{\mathbf{e}}_y, \quad \hat{\mathbf{e}}_\varphi = \hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_r = -\sin\varphi\hat{\mathbf{e}}_x + \cos\varphi\hat{\mathbf{e}}_y. \tag{2.6}$$

For the Cartesian coordinates x, y, z, we have

$$\frac{dx}{dt} = \frac{dr}{dt}\cos\varphi - \frac{d\varphi}{dt} r\sin\varphi, \qquad \frac{dy}{dt} = \frac{dr}{dt}\sin\varphi + \frac{d\varphi}{dt} r\cos\varphi.$$

Hence using formulae (2.6) we find

$$\frac{d}{dt}\left(x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z\right) = \frac{dr}{dt}\hat{\mathbf{e}}_r + r\frac{d\varphi}{dt}\hat{\mathbf{e}}_\varphi + \frac{dz}{dt}\hat{\mathbf{e}}_z. \tag{2.7}$$

II. As known, for the axisymmetric functions $f(\mathbf{x}) = f(r,z)$ Equation (2.5) takes the form

$$\Delta f(r,z) = f_{rr} + \frac{1}{r} f_r + f_{zz} = -f, \tag{2.8}$$

where we denote partial derivatives as f_r, f_{rr}, f_{zz}. Vector field $\mathbf{V}(\mathbf{x})$ (2.4) for $\mathbf{S}(\mathbf{x}) = f(r,z)\hat{\mathbf{e}}_z$ takes the form

$$\mathbf{V}(\mathbf{x}) = f\hat{\mathbf{e}}_z + (\mathrm{grad} f) \times \hat{\mathbf{e}}_z + \mathrm{grad} f_z = f\hat{\mathbf{e}}_z + f_r\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_z + f_{zr}\hat{\mathbf{e}}_r + f_{zz}\hat{\mathbf{e}}_z,$$

where we used the identity $\mathrm{curl}\,(f\hat{\mathbf{e}}_z) = (\mathrm{grad} f) \times \hat{\mathbf{e}}_z$. Substituting here $f_{zz} + f = -f_{rr} - \frac{1}{r} f_r$ from Equation (2.8) and formula (2.6) for vector $\hat{\mathbf{e}}_\varphi$, we find

$$\mathbf{V}(\mathbf{x}) = -f_r\hat{\mathbf{e}}_\varphi + \frac{1}{r}(rf_r)_z\hat{\mathbf{e}}_r - \frac{1}{r}(rf_r)_r\hat{\mathbf{e}}_z. \tag{2.9}$$

Hence, the function $-rf_r(r,z)$ coincides with the stream function $\psi(r,z)^4$ of the axisymmetric flow $\mathbf{V}(\mathbf{x})$.

Dynamical system (1.9) for the Beltrami field $\mathbf{V}(\mathbf{x}) = \mathrm{curl}\,\mathbf{V}(\mathbf{x})$ (2.9) takes the form

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{1}{r}\frac{\partial H}{\partial z}, \quad \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{1}{r}\frac{\partial H}{\partial r}, \quad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{1}{r^2}H. \tag{2.10}$$

Here

$$H(r,z) = -rf_r(r,z) = \psi(r,z) \tag{2.11}$$

and we used formulae (2.7). System (2.10) evidently has first integral $H(r,z)$ (2.11). Hence, the axisymmetric submanifolds $H(r,z) = \mathrm{const}$, $0 \le \varphi \le 2\pi$ are invariant under the flow (2.10).

The Beltrami equation (2.1) and the Helmholtz equation (2.8) are invariant with respect to the z-differentiation. Therefore, vector field $\mathbf{V}_z(\mathbf{x}) = \partial\mathbf{V}(\mathbf{x})/\partial z$ is also a Beltrami field. From Equation (2.9) we find $\mathbf{V}_z(\mathbf{x}) = -f_{rz}\hat{\mathbf{e}}_\varphi + \frac{1}{r}(rf_{rz})_z\hat{\mathbf{e}}_r - \frac{1}{r}(rf_{rz})_r\hat{\mathbf{e}}_z$. The corresponding dynamical system (2.10) is obtained by the z-differentiation, where $H_z = -rf_{rz}$,

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{1}{r}\frac{\partial H_z}{\partial z}, \quad \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{1}{r}\frac{\partial H_z}{\partial r}, \quad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{1}{r^2}H_z. \tag{2.12}$$

III. Let $R = \sqrt{r^2 + z^2}$ be the radius in the Cartesian coordinates x, y, z. Following our paper,[24] we consider the spherically symmetric exact solution to the Helmholtz equation (2.8),

$$f(r,z) = G_1(R) = \frac{\sin R}{R}. \tag{2.13}$$

Evidently we have $\partial G_1/\partial r = rG_1'(R)/R$, $\partial G_1/\partial z = zG_1'(R)/R$. We denote

$$G_2(R) = \frac{1}{R}\frac{\mathrm{d}G_1(R)}{\mathrm{d}R} = \frac{1}{R^2}\left(\cos R - \frac{\sin R}{R}\right), \tag{2.14}$$

$$G_3(R) = \frac{1}{R}\frac{\mathrm{d}G_2(R)}{\mathrm{d}R} = \frac{1}{R^4}\left((3 - R^2)\frac{\sin R}{R} - 3\cos R\right), \tag{2.15}$$

$$G_4(R) = \frac{1}{R}\frac{\mathrm{d}G_3(R)}{\mathrm{d}R} = \frac{1}{R^6}\left((6R^2 - 15)\frac{\sin R}{R} - (R^2 - 15)\cos R\right). \tag{2.16}$$

Using Watson formula for the Bessel functions $J_{n+1/2}(R)$ Ref. 37, p. 56, we find $G_2(R) = -\sqrt{\pi/2}R^{-3/2}J_{3/2}(R)$. Functions $G_k(R)$ are analytic everywhere and have the following values at $R = 0$:

$$G_1(0) = 1, \quad G_2(0) = -1/3, \quad G_3(0) = 1/15, \quad G_4(0) = -1/105. \tag{2.17}$$

Functions $G_k(R)$ satisfy the easily verifiable identities

$$G_1(R) + 3G_2(R) + R^2G_3(R) = 0, \tag{2.18}$$

$$G_2(R) + 5G_3(R) + R^2G_4(R) = 0. \tag{2.19}$$

Using formulae (2.14)–(2.6), we present vector field (2.9) with $f_r = rG_2$ in the Cartesian coordinates x, y, z,

$$\mathbf{V}_1(x,y,z) = (yG_2 + xzG_3)\hat{\mathbf{e}}_x + (-xG_2 + yzG_3)\hat{\mathbf{e}}_y + (G_1 + G_2 + z^2G_3)\hat{\mathbf{e}}_z. \tag{2.20}$$

Remark 3. Exact solution (2.20) coincides with the spheromak magnetic field **B** derived by Chandrasekhar and Kendall[22] and Woltjer[23] in terms of the Bessel and Legendre functions. The term *"spheromak"* for a plasma equilibrium inside a ball was first introduced by Rosenbluth and Bussac.[36]

Dynamical system (2.10), corresponding to the spheromak flow (2.20), has first integral $H(r,z) = -rG_{1r}(R) = -r^2 G_2(R)$. For this dynamical system, we have $\mathrm{d}R/\mathrm{d}t = -2z R^{-1}G_2 + zR^{-1}F_2 = -2zR^{-1}G_2(R)$, where $F_2 = G_1 + 3G_2 + R^2G_3 = 0$ due to identity (2.18). Hence system (2.10) has invariant balls $\mathbb{B}^3_m$ bounded by invariant spheres $\mathbb{S}^2_m$ defined by equations $R = R_m$ where $G_2(R_m) = 0$ that is equivalent to $\tan R_m = R_m$.

IV. Differentiating the spheromak vector field $\mathbf{V}_1(x,y,z)$ (2.20) with respect to the z-variable we get a new z-axisymmetric Beltrami field

$$\mathbf{V}_2(x,y,z) = ((x+yz)G_3 + xz^2G_4)\hat{\mathbf{e}}_x + ((y-xz)G_3 + yz^2G_4)\hat{\mathbf{e}}_y + z(G_2 + 3G_3 + z^2G_4)\hat{\mathbf{e}}_z. \qquad (2.21)$$

Remark 4. A detailed analysis of the force free magnetic fields **B** and Beltrami flows is presented in Refs. 26 and 27. These works extensively use the Chandrasekhar, Kendall, and Woltjer[21–23] general solution of Beltrami equation in terms of Bessel and Legendre functions. The exact solution (2.21) is given in terms of elementary functions and was not presented in Refs. 26 and 27.

Corresponding to (2.21) dynamical system, (2.12) has the following explicit form in the Cartesian coordinates:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (x+yz)G_3 + xz^2G_4,\quad \frac{\mathrm{d}y}{\mathrm{d}t} = (y-xz)G_3 + yz^2G_4,\quad \frac{\mathrm{d}z}{\mathrm{d}t} = z(G_2 + 3G_3 + z^2G_4) \qquad (2.22)$$

and possesses a first integral

$$H_2(r,z) = \frac{\partial H(r,z)}{\partial z} = -r^2\frac{\partial G_2(R)}{\partial z} = -zr^2G_3(R). \qquad (2.23)$$

Using identity (2.19) we derive from Equations (2.22)

$$\nabla_{\mathbf{V}_2}R = \frac{\mathrm{d}R}{\mathrm{d}t} = \frac{1}{R}(R^2 - 3z^2)G_3(R). \qquad (2.24)$$

Therefore system (2.22) has invariant balls $\mathbb{B}^3_k$ where $R \le R_k$, $G_3(R_k) = 0$. Formula (2.15) yields that R_k are the roots of equation

$$\tan R = \frac{3R}{3 - R^2} \qquad (2.25)$$

that has infinitely many solutions with the asymptotics

$$R_k \approx (k+1)\pi, \quad k \longrightarrow \infty. \qquad (2.26)$$

The first four solutions to Equation (2.25) are

$$R_1 \approx 5.7635, \quad R_2 \approx 9.0950, \quad R_3 \approx 12.3229, \quad R_4 \approx 15.5146. \qquad (2.27)$$

Fluid flow (2.21) has the following properties: *(1)* Dynamical system (2.22) has invariant plane $z = 0$ and therefore invariant hemispheres $\mathbb{H}_{k+}$ and $\mathbb{H}_{k-}$ defined by conditions $R \le R_k$, $z \ge 0$ and $R \le R_k$, $z \le 0$, respectively, where $G_3(R_k) = 0$.

(2) Dynamical system (2.22) is invariant under the automorphisms S and T,

$$S(x) = -x,\ S(y) = -y,\ S(z) = z, \quad T(x) = x,\ T(y) = -y,\ T(z) = -z, \qquad (2.28)$$

that generate a group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

V. The z-translational invariance of Equation (2.8) and its linearity imply that together with solution $f_2(r,z) = \partial G_1(R)/\partial z = zG_2(R)$ an arbitrary linear combinations $\hat{f}(r,z)$ of its z-translations

$$\hat{f}(r,z) = \sum_{k=1}^{n} \frac{a_k(z - z_k)}{r^2 + (z - z_k)^2}\left(\cos\sqrt{r^2 + (z - z_k)^2} - \frac{\sin\sqrt{r^2 + (z - z_k)^2}}{\sqrt{r^2 + (z - z_k)^2}}\right) \qquad (2.29)$$

also are exact solutions to Equation (2.8). Here a_k, z_k are arbitrary parameters. The corresponding Beltrami fields $\mathbf{V}_{\hat{f}}(\mathbf{x})$ are defined by formula (2.9).

VI. Consider the vector field $\tilde{\mathbf{V}}_2(\mathbf{x})$ obtained from the z-axisymmetric field $\mathbf{V}_2(\mathbf{x})$ (2.21) by the cyclic permutation of variables $x \longrightarrow y$, $y \longrightarrow z$, $z \longrightarrow x$,

$$\tilde{\mathbf{V}}_2(\mathbf{x}) = x(G_2 + 3G_3 + x^2G_4)\hat{\mathbf{e}}_x + ((y + xz)G_3 + x^2yG_4)\hat{\mathbf{e}}_y + ((z - xy)G_3 + x^2zG_4)\hat{\mathbf{e}}_z.$$

Vector field $\tilde{\mathbf{V}}_2(\mathbf{x})$ is x-axisymmetric and satisfies Beltrami equation (2.1). Analogously to Equation (2.24), we have

$$\nabla_{\tilde{\mathbf{V}}_2} R = R^{-1}(R^2 - 3x^2)G_3(R). \tag{2.30}$$

We define vector fields

$$\mathbf{W}_\alpha(\mathbf{x}) = \mathbf{V}_2(\mathbf{x}) + \alpha\tilde{\mathbf{V}}_2(\mathbf{x}), \tag{2.31}$$

that depend on an arbitrary parameter α. Equations (2.24) and (2.30) yield

$$\nabla_{\mathbf{W}_\alpha} R = R^{-1}\left[(1 + \alpha)R^2 - 3(\alpha x^2 + z^2)\right] G_3(R). \tag{2.32}$$

Equation (2.32) implies that fluid flows (2.31) have nested invariant balls $\mathbb{B}^3_k$ defined by inequalities $R \leq R_k$ where $G_3(R_k) = 0$. The fluid flows $\mathbf{W}_\alpha(\mathbf{x})$ (2.31) satisfy Beltrami equation (2.1), possess the group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ generated by the discrete automorphisms (2.28), and have no rotational symmetries for any $\alpha \neq 0, 1$ because Equation (2.32) evidently does not allow one.

III. DYNAMICAL SYSTEM IN $\mathbb{R}^2$, ITS INVARIANT DOMAINS AND EQUILIBRIUM POINTS

I. Partial derivatives of the function $H_2(r, z) = -zr^2G_3(R)$ have the form

$$\frac{\partial H_2}{\partial r} = -rz(2G_3 + r^2G_4), \quad \frac{\partial H_2}{\partial z} = -r^2(G_3 + z^2G_4). \tag{3.1}$$

Substituting formulae (3.1) into dynamical system (2.10) with $H(r, z) = H_2(r, z) = -zr^2G_3(R)$ we obtain its explicit form

$$\frac{\mathrm{d}r}{\mathrm{d}t} = r(G_3 + z^2G_4), \quad \frac{\mathrm{d}z}{\mathrm{d}t} = -z(2G_3 + r^2G_4), \tag{3.2}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = -zG_3(R). \tag{3.3}$$

Dynamical system (3.2) is smooth everywhere and has invariant line $r = 0$ with equilibrium points $(0,0)$, $(0, \sigma R_n)$, $\sigma = \pm 1$ and invariant line $z = 0$ with equilibrium points $(R_n, 0)$.

Lemma 1. All equilibrium points $(0,0)$, $(0, \sigma R_n)$, $(R_n, 0)$ *are saddles. The quarter-circles* $R = R_n$, $z > 0$, $r > 0$ *and* $R = R_n$, $z < 0$, $r > 0$ *are their separatrices which for* $n = 2k$ *go from point* $(0, \sigma R_n)$ *to point* $(R_n, 0)$ *and for* $n = 2k + 1$ *go from point* $(R_n, 0)$ *to point* $(0, \sigma R_n)$.

Proof. At these equilibrium points system (3.2) has the following eigenvalues:

$$(0,0): \quad \lambda_r = 1/15, \quad \lambda_z = -2/15, \tag{3.4}$$

$$(0, \sigma R_n): \quad \lambda_r = R_n^2 G_4(R_n), \quad \lambda_z = -2R_n^2 G_4(R_n),$$

$$(R_n, 0): \quad \lambda_r = R_n^2 G_4(R_n), \quad \lambda_z = -2R_n^2 G_4(R_n),$$

where we used $G_3(0) = 1/15$ for the point $(0,0)$, see (2.17). Let us find the values $G_4(R_n)$ at the points R_n satisfying equation $G_3(R) = 0$ that is equivalent to Equation (2.25). Substituting Equation (2.25) into formula (2.16) we find

$$G_4(R_n) = \frac{\cos R_n}{R_n^6}\left((6R_n^2 - 15)\frac{\tan R_n}{R_n} - R_n^2 + 15\right) = \frac{\cos R_n}{R_n^2(3 - R_n^2)}.$$

Inserting here the identity $\cos R = \mathrm{sign}(\cos R)(1 + \tan^2(R))^{-1/2}$ and Equation (2.25) we get

$$G_4(R_n) = \frac{-\mathrm{sign}(\cos R_n)}{R_n^2\sqrt{R_n^4 + 3R_n^2 + 9}} \neq 0. \tag{3.5}$$

Since $G_4(R_n) \neq 0$ we get from (3.4) that all equilibrium points (0,0), $(0, \sigma R_n)$, $(R_n, 0)$ are saddles and the quarter-circles $R = R_n$, $z > 0$, $r > 0$ and $R = R_n$, $z < 0$, $r > 0$ are their separatrices. Since $G_3(0) = 1/15$, we find from formulae (3.5) and (2.26) that $G_4(R_1) < 0$, $G_4(R_{2k}) > 0$, $G_4(R_{2k+1}) < 0$. Hence formulae (3.4) yield

$$(0, \sigma R_{2k+1}): \quad \lambda_r < 0,\ \lambda_z > 0, \qquad (0, \sigma R_{2k}): \quad \lambda_r > 0,\ \lambda_z < 0.$$

These formulae imply that dynamics on the separatrices is exactly as shown in Figure 2. □

II. For the spherical radius $R = \sqrt{r^2 + z^2}$ we find from (3.2)

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{\partial R}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{\partial R}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{1}{R}\left(r\frac{\mathrm{d}r}{\mathrm{d}t} + z\frac{\mathrm{d}z}{\mathrm{d}t}\right) = \frac{(r^2 - 2z^2)}{R}G_3(R). \tag{3.6}$$

Equations (2.16) and (3.6) imply

$$\frac{\mathrm{d}}{\mathrm{d}t}G_3(R) = \frac{\mathrm{d}G_3(R)}{\mathrm{d}R}\frac{\mathrm{d}R}{\mathrm{d}t} = (r^2 - 2z^2)G_3(R)G_4(R). \tag{3.7}$$

Equation (3.6) gives another proof of invariance of all spheres $\mathbb{S}_k^2$ $(R = R_k)$ where $G_3(R_k) = 0$.

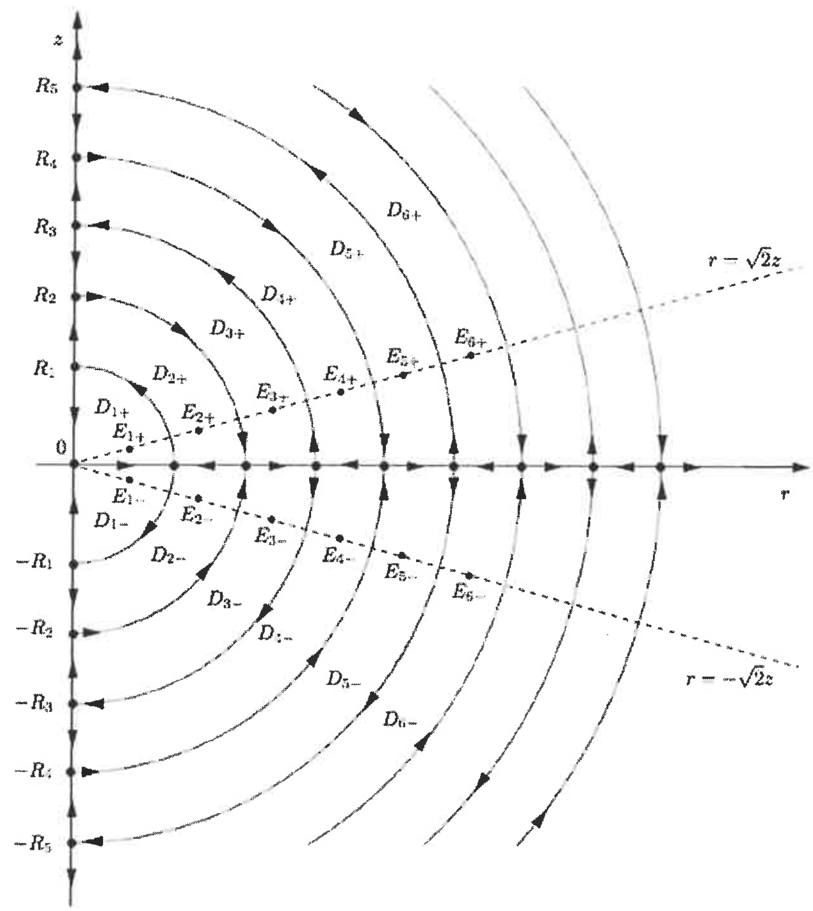

FIG. 2. Invariant domains $\mathcal{D}_{k,\pm}$ of dynamical system (3.2), its equilibrium points and separatrices.

III. Dynamical system (3.2) in view of Equation (3.7) has invariant open domains $\mathcal{D}_{k.\pm}$ defined by equations

$$\mathcal{D}_{k.+}: \quad R_{k-1}^2 < r^2 + z^2 < R_k^2, \quad k \geq 1, \quad r > 0, \quad z > 0, \tag{3.8}$$

where $G_3(R_k) = 0$ and $R_0 = 0$. Invariant domains $\mathcal{D}_{k.-}$ are obtained from (3.8) by the reflection $z \longrightarrow -z$. Boundary $\partial\mathcal{D}_{k.+}$ of domain $\mathcal{D}_{k.+}$ is

$$\partial\mathcal{D}_{k+} = S_k \cup J_k \cup S_{k-1} \cup I_k, \quad k \geq 1, \tag{3.9}$$

$$S_k: \ r^2 + z^2 = R_k^2, \ z > 0; \quad J_k: \ R_{k-1} \leq r \leq R_k, \ z = 0, \tag{3.10}$$

$$I_k: \ R_{k-1} \leq z \leq R_k, \ r = 0.$$

The invariant domains $\mathcal{D}_{k.+}$ and $\mathcal{D}_{k.-}$ of system (3.2) are shown in Figure 2.

Proposition 1. All trajectories of dynamical system (3.2) in the invariant domains $\mathcal{D}_{k.\pm}$ are closed curves C_H: $H_2(r,z) = H \neq 0$. The corresponding trajectories of dynamical system (3.2), (3.3) move on invariant tori $\mathbb{T}_H^2 = C_H \times S^1$ and are either periodic or quasi-periodic.

Proof. Let us show first that inside each invariant domain $\mathcal{D}_{k.\pm}$ system (3.2) has one and only one equilibrium point $E_{k\pm}$. Indeed, the equilibrium points $(r_j \neq 0, z_j \neq 0)$ of system (3.2) are defined by two equations,

$$G_3(R) + z^2G_4(R) = 0, \quad 2G_3(R) + r^2G_4(R) = 0. \tag{3.11}$$

Equations (3.11) imply $r_j = \pm\sqrt{2}z_j$, $\tilde{R}_j^2 = r_j^2 + z_j^2 = 3z_j^2$ and equation

$$3G_3(\tilde{R}_j) + (r_j^2 + z_j^2)G_4(\tilde{R}_j) = 3G_3(\tilde{R}_j) + \tilde{R}_j^2G_4(\tilde{R}_j) = 0. \tag{3.12}$$

Equation (3.12) yields the equation

$$3G_3(R) + R^2G_4(R) = \frac{1}{R^4}\left((3R^2 - 6)\frac{\sin R}{R} - (R^2 - 6)\cos R\right) = 0,$$

which has the equivalent form

$$\tan R = R\frac{R^2 - 6}{3R^2 - 6}. \tag{3.13}$$

Hence the equilibrium points of system (3.2) are defined by the relations

$$r_j = \sqrt{2/3}\tilde{R}_j, \quad z_j = \pm\sqrt{1/3}\tilde{R}_j, \tag{3.14}$$

where $\tilde{R}_j$ are all positive roots of Equation (3.13) that has infinitely many solutions $\tilde{R}_j$ with the asymptotics

$$\tilde{R}_j \approx (j + 1/2)\pi, \quad j \longrightarrow \infty. \tag{3.15}$$

The first four solutions to Equation (3.13) have the form

$$\tilde{R}_1 \approx 3.8702, \quad \tilde{R}_2 \approx 7.4431, \quad \tilde{R}_3 \approx 10.7130, \quad \tilde{R}_4 \approx 13.9205. \tag{3.16}$$

The asymptotics (2.26) and (3.15) together with the formulae (2.27) and (3.16) prove that between any two points R_{k-1} and R_k there is one and only one point $\tilde{R}_k$. Hence, in each invariant domain $\mathcal{D}_{k.+}$, there is only one equilibrium point E_{k+} with positive coordinates (r_k, z_k) (3.14) and in each invariant domain $\mathcal{D}_{k.-}$ there is only one equilibrium point E_{k-} with coordinates $(r_k, -z_k)$.

Equations (2.10) for $H = H_2$ imply that at each equilibrium point $E_{k\pm}$ we have $\partial H_2/\partial z(r_k, \pm z_k) = 0$, $\partial H_2/\partial r(r_k, \pm z_k) = 0$. Hence each equilibrium point $E_{k\pm}$ is a point of extremum of function $H_2(r,z) = -zr^2G_3(R)$. Since at the boundary of each invariant domain $\mathcal{D}_{k.\pm}$ function $H_2(r,z)$ is zero and inside each domain $\mathcal{D}_{k.\pm}$ it has only one point of extremum $E_{k\pm}$; we get that this point is either global maximum of function $H_2(r,z)$ in the domain $\mathcal{D}_{k.\pm}$ or its global minimum. In both cases the curves C_H of constant levels of function $H_2(r,z) = H \neq 0$ are closed curves. Since function

$H_2(r,z) = -zr^2G_3(R)$ is first integral of system (3.2), we get that all trajectories of system (3.2) in the invariant domains $\mathcal{D}_{k.\pm}$ are closed curves C_H.

For each closed trajectory C_H, the angular variable φ due to Equation (3.3) changes monotonously along the circle S^1. Hence we get that dynamics of the 3-dimensional system (3.2) and (3.3) at $H_2(r,z) = H \neq 0$ occurs on the invariant tori $\mathbb{T}^2 = C_H \times S^1$ and is either periodic or quasi-periodic. □

Remark 5. The first three equilibrium points E_{k+} have the following approximate coordinates (r_k, z_k) (3.14):

$$E_{1+} : (3.1600,\ 2.2345), \quad E_{2+} : (6.0773,\ 4.2973), \quad E_{3+} : (8.7471,\ 6.1852).$$

The equilibrium points E_{k-} have coordinates $(r_k, -z_k)$ and are reflections of the points E_{k+}. The equilibrium points are shown in Figure 2.

Corollary 1. Rotation of the closed trajectories of dynamical system (3.2) in invariant domains $\mathcal{D}_{2n.+}$, $\mathcal{D}_{2n+1.-}$ is clockwise and in invariant domains $\mathcal{D}_{2n+1.+}$, $\mathcal{D}_{2n.-}$ is counter-clockwise.

Proof. Dynamics of separatrices is shown in Figure 2 based on Lemma 2. Hence by the continuity we get the described directions of rotation of the closed trajectories. □

IV. METHOD FOR CONSTRUCTION OF THE MODULI SPACE $\mathcal{S}(\mathcal{D})$

I. It is evident that topology of trajectories does not depend on their parametrization. Therefore we choose a new time variable τ that makes the analysis simpler

$$\frac{d\tau}{dt} = \frac{H_2}{2\pi r^2}. \tag{4.1}$$

In the new time τ, the dynamical system (2.10) for $H = H_2$ takes the form

$$\frac{dr}{d\tau} = -\frac{2\pi r}{H_2}\frac{\partial H_2}{\partial z}, \qquad \frac{dz}{d\tau} = \frac{2\pi r}{H_2}\frac{\partial H_2}{\partial r}, \tag{4.2}$$

$$\frac{d\varphi}{d\tau} = 2\pi. \tag{4.3}$$

The main advantage of the time change (4.1) is that the τ-derivative (4.3) of the angular variable φ is constant (and equals to 2π).

Remark 6. The time change (4.1) preserves direction of time in invariant domains $\mathcal{D}_{2n.+}$ and $\mathcal{D}_{2n+1.-}$ where $H_2(r,z) = -zr^2G_3(R) > 0$ and reverses it in the domains $\mathcal{D}_{2n+1.+}$ and $\mathcal{D}_{2n.-}$ where $H_2(r,z) = -zr^2G_3(R) < 0$. Hence using Corollary 3 we get that after the time change (4.1) all trajectories of system (4.2) rotate clockwise, as shown in Figure 3.

System (4.2) evidently is a reparametrized Hamiltonian system with the Hamiltonian function $H_2(r,z)$. Equation (4.3) defines rotation of the angular variable φ with constant speed 2π.

Substituting formulae (3.1) and $H_2(r,z) = -zr^2G_3(R)$, we get the explicit form of the dynamical system (4.2),

$$\frac{dr}{d\tau} = -\frac{2\pi r}{z}\left(1 + z^2\frac{G_4}{G_3}\right), \quad \frac{dz}{d\tau} = 4\pi + 2\pi r^2\frac{G_4}{G_3}. \tag{4.4}$$

System (4.4) evidently has an invariant line $r = 0$. System (4.4) as well as system (4.2) have first integral $H_2(r,z)$. The invariant with respect to the system (3.2) spheres $\mathbb{S}^2_k : G_3(R_k) = 0$ and plane $\mathbb{R}^2 : z = 0$ defined by equation $H_2(r,z) = 0$ are the singular subsets $r^2 + z^2 = R_k^2$ and $z = 0$ of system (4.2) - (4.4).

II. Applying Proposition 1 we get that all trajectories of system (4.4) in invariant domains $\mathcal{D}_{k.\pm}$ are closed curves C_H satisfying equation $H_2(r,z) = H \neq 0$. Let us define for each invariant domain $\mathcal{D}_{k.\pm}$ a continuous function $\tau_k(H)$ of one variable H.

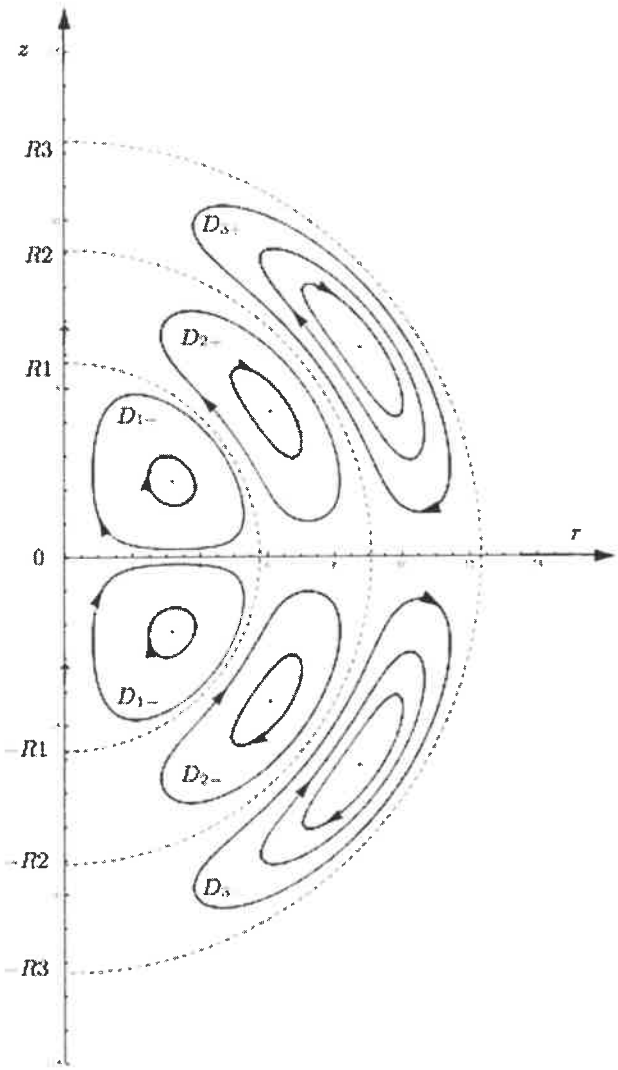

FIG. 3. Phase portrait of dynamical system (4.2) in invariant domains $\mathcal{D}_{1.\pm}$, $\mathcal{D}_{2.\pm}$, and $\mathcal{D}_{3.\pm}$. All rotations are clockwise.

For any constant $H \neq 0$ we consider the closed trajectory $C_H \subset \mathcal{D}_{k.\pm} \subset \mathbb{R}^2$: $H_2(r,z) = H$ and define $\tau_k(H)$ as the period of the trajectory C_H of dynamical system (4.4) in $\mathcal{D}_{k.\pm}$. Thus we get a function $\tau_k(H)$ that is continuous in its domain due to the general theory of dynamical systems.[38] Since dynamical systems (4.4) in two domains $\mathcal{D}_{k.+}$ and $\mathcal{D}_{k.-}$ are equivalent by the automorphism $z \longrightarrow -z$, $\tau \longrightarrow -\tau$ induced by the automorphism T (2.28), we get that the functions $\tau_k(|H|)$ for them coincide.

Trajectories of system (4.2) and (4.3) move on invariant tori $\mathbb{T}^2_H = C_H \times S^1$ in the 3-dimensional space r, z, φ where the circle S^1 corresponds to the angular variable $\varphi \text{mod}(2\pi)$.

Proposition 2. Topology of trajectories in an invariant domain $\mathcal{D}_{k.\pm} \times S^1$ is changing from one torus $\mathbb{T}^2_{H_1}$ to another $\mathbb{T}^2_{H_2}$ if and only if the function of periods $\tau_k(H)$ is not constant.

Proof. If the continuous function $\tau_k(H) \neq \text{const}$ then it takes all rational and all irrational values in some interval (a,b).

Let a closed trajectory C_{H_1} have a rational period $\tau_k(H_1) = p/q$. During the time $\tau_k(H_1)$ the angular variable φ is changed for $2\pi\tau_k(H_1)$ because $d\varphi/d\tau = 2\pi$. After q complete turns of trajectory around the closed curve C_{H_1} the angular variable φ is changed for $q(2\pi\tau_k(H_1)) = 2\pi p$, because $\tau_k(H_1) = p/q$. Hence all trajectories on the torus $T^2_{H_1}$ are closed curves.

Now let a closed trajectory C_{H_2} in $\mathcal{D}_{k.\pm}$ have an irrational period $\tau_k(H_2)$. Then after any N complete turns of trajectory around the closed curve C_{H_2} the angular variable φ is changed for $2\pi N\tau_k(H_2)$. For any integers N and M we have $2\pi N\tau_k(H_2) \neq 2\pi M$ because $\tau_k(H_2)$ is irrational, $\tau_k(H_2) \neq M/N$. Hence all trajectories on the torus $T^2_{H_2}$ are non-closed infinite quasiperiodic curves. As known any quasiperiodic trajectory is dense on $T^2_{H_2}$.

Hence, the topology of trajectories is changing from one torus to another if the function of periods $\tau_k(H)$ is not constant.

If $\tau_k(H) \equiv \tau_1 = \text{const}$ for all H then trajectories on all tori either are all closed (if τ_1 is rational) or are all dense (if τ_1 is irrational). This means that if function $\tau(H)$ is constant then all trajectories have the same topology. □

A. Structure of knots

All trajectories on the torus T^2_H with the period $\tau_k(H) = p/q$ (p and q are coprime) are closed curves which make q complete turns around the meridians and p complete turns around the longitudes of the torus T^2_H. These closed curves form knots in $\mathbb{R}^3$ which are called torus knots $K_{p,q}$.[5]

All trajectories with period $\tau_k(H) = 2/5$ make 5 turns around the meridians and 2 turns around the longitudes. They form a non-trivial knot $K_{2,5}$ shown in Figure 7 of Section VIII.

Corollary 2. If for some integers p and q a torus knot $K_{p,q}$ is realized by vortex lines for the fluid flow (2.21) then its mirror image $\tilde{K}_{p,q}$ is not realized.

Proof. Indeed, any torus knot $K_{p,q}$ and its mirror image $\tilde{K}_{p,q}$ have opposite directions of rotation around the meridians. As shown in Remark 6 all closed trajectories C_H of dynamical system (4.2) rotate in the clockwise direction, see Figure 3. So the opposite (counter-clockwise) rotation is not realized and hence the mirror image $\tilde{K}_{p,q}$ does not appear. □

III. **The main method.** To construct the moduli space $\mathcal{S}(\mathcal{D})$ of vortex knots for the fluid flow (2.21) it is necessary to find the ranges of all functions of periods $\tau_k(H)$ for the invariant domains $\mathcal{D}_{k.\pm}$ for $k = 1, 2, 3, \ldots$. Indeed, using the proof of Proposition 2 and Corollary 4 we get that all rational numbers p/q from those ranges define all torus knots $K_{p,q}$ realized by the closed vortex lines for the flow (2.21). To find the ranges of the continuous functions $\tau_k(H)$ we calculate in Sections V and VI their limits

$$\lim_{H\longrightarrow H_{k\pm}} \tau_k(H), \quad \lim_{H\longrightarrow 0} \tau_k(H), \tag{4.5}$$

where $H_{k\pm}$ are the values of function $H_2(r,z)$ at the equilibrium points $E_{k\pm} : (r_k, \pm z_k)$. Then since the limits occur to be different and the functions $\tau_k(H)$ occur to be monotonous in their domains we get their ranges between the above limits. This leads to the construction of the moduli spaces $\mathcal{S}(\mathbb{R}^3)$ and $\mathcal{S}_m(\mathbb{B}^3_a)$ in Sections VIII and IX.

V. LIMITS OF FUNCTIONS $\tau_k(H)$ AT $H \longrightarrow H_k$

I. For the rest of the paper, the function $H(r,z)$ means $H_2(r,z) = -zr^2G_3(R)$.

Lemma 2. All maxima and minima of function $H_2(r,z)$ at the points $E_{k\pm}$ (r_k, z_k) (3.14) are non-degenerate.

Proof. Let $H_k = H_2(r_k, z_k)$ and $\tilde{R}_k = \sqrt{r_k^2 + z_k^2}$. At the two points E_{k+} and E_{k-} the values of H_k differ only by the sign. For the function $H_2(r,z) = -zr^2G_3(R)$ we have

$$\frac{\partial H}{\partial r} = \frac{2H}{r} - \frac{zr^3}{R}G_3', \qquad \frac{\partial H}{\partial z} = \frac{H}{z} - \frac{z^2r^2}{R}G_3', \qquad G_3' = \frac{\mathrm{d}G_3}{\mathrm{d}R}. \tag{5.1}$$

Hence at the equilibrium points (3.14) the relation holds,

$$z_k r_k^4 \tilde{R}_k^{-1} G_3'(\tilde{R}_k) = 2H_k. \tag{5.2}$$

Differentiating Equations (5.1), we find

$$\frac{\partial^2 H}{\partial r^2} = -\frac{2H}{r^2} + \frac{2}{r}\frac{\partial H}{\partial r} - \frac{3zr^2}{R}G_3' + \frac{zr^4}{R^3}G_3' - \frac{zr^4}{R^2}G_3'',$$

$$\frac{\partial H}{\partial z} = -\frac{H}{z^2} + \frac{1}{z}\frac{\partial H}{\partial z} - \frac{2zr^2}{R}G_3' + \frac{z^3r^2}{R^3}G_3' - \frac{z^3r^2}{R^2}G_3'',$$

$$\frac{\partial^2 H}{\partial r \partial z} = \frac{1}{z}\frac{\partial H}{\partial r} - \frac{2z^2r}{R}G_3' + \frac{z^2r^3}{R^3}G_3' - \frac{z^2r^3}{R}G_3''.$$

Substituting $\partial H/\partial r(r_k, z_k) = 0$, $\partial H/\partial z(r_k, z_k) = 0$ and formula $G_3'' = -G_3 - 6G_3'/R$ (2.26), we get

$$\frac{\partial^2 H}{\partial r^2}(r_k, z_k) = -\frac{2H_k}{r_k^2} + \frac{1}{\tilde{R}_k}G_3'\left(-3z_k r_k^2 + \frac{7z_k r_k^4}{\tilde{R}_k^2}\right) + \frac{z_k r_k^4}{\tilde{R}_k^2}G_3,$$

$$\frac{\partial H}{\partial z}(r_k, z_k) = -\frac{H_k}{z_k^2} + \frac{1}{\tilde{R}_k}G_3'\left(-2z_k r_k^2 + \frac{7z_k^3 r_k^2}{\tilde{R}_k^2}\right) + \frac{z_k^3 r_k^2}{\tilde{R}_k^2}G_3,$$

$$\frac{\partial^2 H}{\partial r \partial z}(r_k, z_k) = \frac{1}{\tilde{R}_k}G_3'\left(-2z_k^2 r_k + \frac{7z_k^2 r_k^3}{\tilde{R}_k^2}\right) + \frac{z_k^2 r_k^3}{\tilde{R}_k^2}G_3.$$

Substituting here formula (5.2) and $G_3 = -H/(zr^2)$ we derive

$$\frac{\partial^2 H}{\partial r^2}(r_k, z_k) = H_k\left(-\frac{8}{r_k^2} + \frac{14}{\tilde{R}_k^2} - \frac{r_k^2}{\tilde{R}_k^2}\right),$$

$$\frac{\partial H}{\partial z}(r_k, z_k) = H_k\left(-\frac{1}{z_k^2} - \frac{4}{r_k^2} + \frac{14z_k^2}{r_k^2\tilde{R}_k^2} - \frac{z_k^2}{\tilde{R}_k^2}\right),$$

$$\frac{\partial^2 H}{\partial r \partial z}(r_k, z_k) = z_k r_k H_k\left(-\frac{4}{r_k^4} + \frac{14}{r_k^2\tilde{R}_k^2} - \frac{1}{\tilde{R}_k^2}\right).$$

Inserting here formulae (3.14) $r_k^2 = \frac{2}{3}\tilde{R}_k^2$, $z_k^2 = \frac{1}{3}\tilde{R}_k^2$ we find at the equilibrium points (r_k, z_k),

$$\frac{\partial^2 H}{\partial r^2} = \frac{2H_k}{3\tilde{R}_k^2}(3 - \tilde{R}_k^2), \quad \frac{\partial^2 H}{\partial z^2} = -\frac{H_k}{3\tilde{R}_k^2}(6 + \tilde{R}_k^2), \quad \frac{\partial^2 H}{\partial r \partial z} = \frac{z_k r_k H_k}{\tilde{R}_k^4}(12 - \tilde{R}_k^2).$$

Hence we find using (3.14) for the Hessian

$$\mathcal{H}(r_k, z_k) = \frac{\partial^2 H}{\partial r^2}(r_k, z_k)\frac{\partial^2 H}{\partial z^2}(r_k, z_k) - \left(\frac{\partial^2 H}{\partial r \partial z}(r_k, z_k)\right)^2 \tag{5.3}$$

$$\frac{2H_k^2}{9\tilde{R}_k^2}\left((\tilde{R}_k^2 - 3)(\tilde{R}_k^2 + 6) - (\tilde{R}_k^2 - 12)^2\right) = \frac{6H_k^2}{\tilde{R}_k^4}(\tilde{R}_k^2 - 6).$$

The numbers $\tilde{R}_k$ (roots of Equation (3.13)) are given by formulae (3.15) and (3.16). Hence $\tilde{R}_k^2 > 6$. Therefore the Hessian $\mathcal{H}(r_k, z_k) > 0$ which means that the extremum of the Hamiltonian function $H_2(r, z)$ at each equilibrium point (r_k, z_k) is non-degenerate and is either maximum or minimum. □

II. Formula (5.3) leads to a formula for the limit of the function of periods $\tau_k(H)$.

Lemma 3. The limit value of function of periods $\tau_k(H)$ at $H \longrightarrow H_k$ is

$$\lim_{H \longrightarrow H_*} \tau_k(H) = \tau_k(H_k) = \frac{1}{2\sqrt{1 - 6\tilde{R}_k^{-2}}}. \tag{5.4}$$

Proof. At the equilibrium point (r_k, z_k) we find from system (4.2) $\partial H/\partial r = 0$, $\partial H/\partial z = 0$. Since each equilibrium point (r_k, z_k) is a non-degenerate extremum of the Hamiltonian function

$H_2(r,z)$, dynamical system (4.2) in a small neighborhood of the point (r_k, z_k) is approximated by its linear part

$$\frac{dr}{d\tau} = -a_{11}(z - z_k) - a_{12}(r - r_k), \quad \frac{dz}{d\tau} = a_{12}(z - z_k) + a_{22}(r - r_k), \tag{5.5}$$

where constant coefficients a_{ij} have the form

$$a_{11} = \frac{2\pi r_k}{H_k}\frac{\partial^2 H}{\partial z^2}(r_k, z_k), \quad a_{12} = \frac{2\pi r_k}{H_k}\frac{\partial^2 H}{\partial r \partial z}(r_k, z_k), \quad a_{22} = \frac{2\pi r_k}{H_k}\frac{\partial^2 H}{\partial r^2}(r_k, z_k). \tag{5.6}$$

Using formulae (5.6) and (5.3), we get

$$D_k = a_{11}a_{22} - a_{12}^2 = (2\pi r_k/H_k)^2 \mathcal{H}(r_k, z_k) > 0. \tag{5.7}$$

Linear system (5.5) has first integral $F(r,z) = a_{22}r^2 + 2a_{12}rz + a_{11}z^2$. Therefore all its trajectories for $D_k > 0$ are ellipses $F(r,z) = \text{const}$. As known,[38] all solutions to the linear system (5.5) with $D_k > 0$ are periodic with the same period $\tau_k = 2\pi/\sqrt{D_k}$.

The limit of the period $\tau_k(H)$ when $H \longrightarrow H_k$ evidently is τ_k because system (4.2) is approximated by the linear system (5.5) when $(r,z) \longrightarrow (r_k, z_k)$.[38] Using formula (5.7) and expression (5.3) for $\mathcal{H}(r_k, z_k)$, we find for the limit value of function $\tau_k(H)$ at the point $H_k = H_2(r_k, z_k)$,

$$\tau_k(H_k) = \lim_{H \longrightarrow H_k} \tau_k(H) = \tau_k = \frac{2\pi}{\sqrt{D_k}} = \frac{|H_k|}{r_k\sqrt{\mathcal{H}(r_k, z_k)}} = \frac{\tilde{R}_k^2}{r_k\sqrt{6(\tilde{R}_k^2 - 6)}}.$$

Substituting here the expression $r_k = \sqrt{2/3}\tilde{R}_k$ (3.14) we get formula (5.4). □

Remark 7. Using numerical values of $\tilde{R}_k$ (3.16) we find from (5.4) the first four values of limit periods $\tau_k(H_k)$,

$$\tau_1(H_1) \approx 0.5847, \quad \tau_2(H_2) \approx 0.5295, \quad \tau_3(H_3) \approx 0.5136, \quad \tau_4(H_4) \approx 0.5079. \tag{5.8}$$

It is evident that the function $\tau_k(H_k)$ (5.4) is monotonously decreasing when $k \longrightarrow \infty$ and has the limit

$$\lim_{k \longrightarrow \infty} \tau_k(H_k) = 0.5. \tag{5.9}$$

III. Let us derive a formula for the $H_k = H_2(r_k, z_k)$.

Lemma 4. Function $|H_2(r,z)|$ in each invariant domain $\mathcal{D}_{k\pm}$ takes the maximal value

$$|H_k| = \frac{2}{3\sqrt{3}}\frac{1}{\sqrt{1 - 3/\tilde{R}_k^2 + 36/\tilde{R}_k^6}}. \tag{5.10}$$

Proof. Substituting formulae (3.14) into $H_2(r_k, z_k) = -z_k r_k^2 G_3(\tilde{R}_k)$ (2.25) we find $H_k = \sigma\frac{2}{3\sqrt{3}}\tilde{R}_k^3$ $G_3(\tilde{R}_k)$, $\sigma = -\text{sign}(z_k)$. Substituting here formula (2.15) we get

$$H_k = \sigma\frac{2}{3\sqrt{3}}\frac{\cos\tilde{R}_k}{\tilde{R}_k}\left((3 - \tilde{R}_k^2)\frac{\tan\tilde{R}_k}{\tilde{R}_k} - 3\right). \tag{5.11}$$

At the equilibrium points $E_{k\pm}$, Equation (3.13) yields

$$\frac{\tan\tilde{R}_k}{\tilde{R}_k} = \frac{\tilde{R}_k^2 - 6}{3\tilde{R}_k^2 - 6}. \tag{5.12}$$

Inserting this into (5.11) we find

$$H_k = \sigma\frac{2}{3\sqrt{3}}\frac{\cos\tilde{R}_k}{\tilde{R}_k}\left(\frac{(3 - \tilde{R}_k^2)(\tilde{R}_k^2 - 6)}{3\tilde{R}_k^2 - 6} - 3\right) = -\sigma\frac{2}{3\sqrt{3}}\frac{(\cos\tilde{R}_k)\tilde{R}_k^3}{3\tilde{R}_k^2 - 6}.$$

Substituting here formula $\cos\tilde{R}_k = \mathrm{sign}(\cos\tilde{R}_k)(1+\tan^2\tilde{R}_k)^{-1/2}$, formula (5.12), and $\sigma = -\mathrm{sign}(z_k)$ we get

$$H_k = \frac{2}{3\sqrt{3}}\frac{\mathrm{sign}(z_k\cos\tilde{R}_k)\tilde{R}_k^3}{\sqrt{(3-\tilde{R}_k^2)^2+\tilde{R}_k^2(\tilde{R}_k^2-6)^2}} = \frac{2}{3\sqrt{3}}\frac{\mathrm{sign}(z_k\cos\tilde{R}_k)}{\sqrt{1-3/\tilde{R}_k^2+36/\tilde{R}_k^6}}. \tag{5.13}$$

Since $H_k = H_2(r_k, z_k)$ is either a maximum or minimum of function $H_2(r,z)$ in the domain $\mathcal{D}_{k\pm}$ we get from (5.13) formula (5.10). □

Remark 8. Function $H_2(r,z) = -zr^2G_3(R)$ is negative in the invariant domain $\mathcal{D}_{1+}$. Indeed, this follows from the formula $G_3(0) = 1/15$, see (2.17). Function $H_2(r,z) = -zr^2G_3(R)$ has opposite signs in any two neighbouring domains $\mathcal{D}_{k\pm}$, because it is zero at their common border. Hence using numerical values of $\tilde{R}_k$ (3.16) we get from (5.13) the numerical values of H_k at the equilibrium points E_{k+},

$$H_1 \approx -0.4276, \quad H_2 \approx 0.3957, \quad H_3 \approx -0.3900, \quad H_4 \approx 0.3879. \tag{5.14}$$

At the equilibrium points E_{k-}, the values of H_k are opposite to (5.14).

Remark 9. Formula (5.10) implies that the values of $|H_k|$ monotonously decrease at $k \longrightarrow \infty$ and have the limit $\lim_{k\longrightarrow\infty}|H_k| = 2/(3\sqrt{3}) \approx 0.3849$. Hence the functions of periods $\tau_k(|H|)$ for all k are defined in domains which contain the segment $[0, 0.3849]$.

VI. LIMITS OF FUNCTIONS $\tau_k(H)$ AT $H \longrightarrow 0$

I. In view of the automorphism T (2.28), the limits of functions of periods $\tau_k(|H|)$ are the same in the domains $\mathcal{D}_{k,+}$ and $\mathcal{D}_{k-}$ (3.8). Function $H_2(r,z) = -zr^2G_3(R)$ (2.25) vanishes at the boundaries $\partial\mathcal{D}_{k,\pm}$(3.9) of these domains. Hence the limit of $\tau_k(H)$ at $H \longrightarrow 0$ is equal to the limit of periods of closed trajectories C_H ($|H_2(r,z)| = \varepsilon << 1$) near the boundaries $\partial\mathcal{D}_{k,\pm}$.

The speed V of dynamics defined by system (4.4) satisfies the equation

$$\frac{1}{4\pi^2}V^2 = \frac{1}{4\pi^2}\left(\left(\frac{dr}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2\right) = 4 + \frac{r^2}{z^2} + \frac{r^2}{G_3^2}(R^2G_4^2 + 6G_3G_4). \tag{6.1}$$

On the circles $R = R_k$ we obtain from (3.5),

$$R^2G_4^2 + 6G_3G_4 = R_k^2G_4^2(R_k) = (R_k^2(R_k^4 + 3R_k^2 + 9))^{-1} > 0. \tag{6.2}$$

Formulae (6.1) and (6.2) imply that: *(a)* In the small neighborhoods of points

$$P_0 : (r = 0, z = 0), \quad P_k : (r = 0, z = R_k), \quad Q_k : (r = R_k, z = 0) \tag{6.3}$$

we have $V > 4\pi$; *(b)* at $(r^2+z^2) \longrightarrow R_k^2$ we have $V \longrightarrow \infty$ in view of Equation (6.2) and $G_3(R_k) = 0$; *(c)* at $z \longrightarrow 0$, the term r^2/z^2 in (6.1) yields $V \longrightarrow \infty$, *(d)* at $r \longrightarrow 0$, formula (6.1) yields $V \longrightarrow 4\pi$.

II. A trajectory C_H in the invariant domain $\mathcal{D}_{1,+}$ at $H \longrightarrow 0$ moves near the boundary $\partial\mathcal{D}_{1,+}$ (3.9) (see Figure 2) and consists of an arc A_1 near the quartercircle S_1 (3.10) having length $\approx \frac{1}{2}\pi R_1$, a small arc of length ε near point Q_1 (6.3), an arc B_1 near the segment J_1 (3.10) of length $\approx R_1$, a small arc of length ε near point P_0, an arc C_1 near the segment I_1 (3.10) of length $\approx R_1$, and a small arc of length ε near point P_1 (6.3). The velocity V of dynamics along these arcs in view of (6.1) and (6.2) has the following limits at $H \longrightarrow 0$:

$$V_{A_1} \longrightarrow \infty, \quad V_{P_0}, V_{P_1}, V_{Q_1} \geq 4\pi, \quad V_{B_1} \longrightarrow \infty, \quad V_{C_1} \longrightarrow 4\pi. \tag{6.4}$$

The total time of trajectory travel along the closed curve C_H at $H \longrightarrow 0$ is

$$\tau_1(H) = \frac{\pi R_1}{2V_{A_1}} + \frac{\varepsilon}{V_{Q_1}} + \frac{R_1}{V_{B_1}} + \frac{\varepsilon}{V_{P_0}} + \frac{R_1}{V_{C_1}} + \frac{\varepsilon}{V_{P_1}}.$$

Using here the limits (6.4) and $\varepsilon \longrightarrow 0$ at $H \longrightarrow 0$ we derive

$$\lim_{H\longrightarrow 0} \tau_1(H) = p_1 = R_1/(4\pi). \tag{6.5}$$

III. A closed trajectory C_H in the invariant domain $\mathcal{D}_{k,+}$ (see Figure 2) at $H \longrightarrow 0$ consists of an arc A_k near the quartercircle S_k (3.9) of length $\approx \frac{1}{2}\pi R_k$, a small arc of length ε near point Q_k (6.3), an arc B_k of length $\approx R_k - R_{k-1}$ near segment J_k (3.9), a small arc of length ε near point Q_{k-1} (6.3), an arc A_{k-1} of length $\approx \frac{1}{2}\pi R_{k-1}$ near the quartercircle S_{k-1}, a small arc of length ε near point P_{k-1} (6.3), an arc C_k of length $\approx R_k - R_{k-1}$ near segment I_k (3.9), and a small arc of length ε near point P_k (6.3). The velocity V of dynamics along these arcs in view of (6.1) and (6.2) has the following limits at $H \longrightarrow 0$:

$$V_{A_k}, V_{A_{k-1}}, V_{B_k} \longrightarrow \infty, \quad V_{Q_k}, V_{Q_{k-1}}, V_{P_{k-1}}, V_{P_k} \geq 4\pi, \quad V_{C_k} \longrightarrow 4\pi. \tag{6.6}$$

The total time of trajectory travel along the closed curve C_H at $H \longrightarrow 0$ is

$$\tau_1(H) = \frac{\pi R_k}{2V_{A_k}} + \frac{\varepsilon}{V_{Q_k}} + \frac{R_k - R_{k-1}}{V_{B_k}} + \frac{\varepsilon}{V_{Q_{k-1}}} + \frac{\pi R_k}{2V_{A_{k-1}}} + \frac{\varepsilon}{V_{P_{k-1}}} + \frac{R_k - R_{k-1}}{V_{C_k}} + \frac{\varepsilon}{V_{P_k}}.$$

Using here the limits (6.6) and $\varepsilon \longrightarrow 0$ at $H \longrightarrow 0$ we find

$$\lim_{H\longrightarrow 0} \tau_k(H) = p_k = (R_k - R_{k-1})/(4\pi). \tag{6.7}$$

IV. Substituting into (6.5) and (6.7) the numerical values of R_k (2.26), we get

$$p_1 \approx 0.4586, \quad p_2 \approx 0.2651, \quad p_3 \approx 0.2569, \quad p_4 \approx 0.2540, \quad p_5 \approx 0.2526. \tag{6.8}$$

From the asymptotics (2.26) and formula (6.7) we find

$$\lim_{k\longrightarrow\infty} p_k = \lim_{k\longrightarrow\infty} (R_k - R_{k-1})/(4\pi) = 0.25. \tag{6.9}$$

Formulae (6.8) demonstrate that convergence to the limit (6.9) is rather fast.

VII. PITCH FUNCTION FOR GENERAL AXISYMMETRIC FLOWS

I. Axisymmetric steady fluid flows have velocity vector fields

$$\mathbf{V}(r,z) = -\frac{1}{r}\frac{\partial\psi}{\partial z}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\psi}{\partial r}\hat{\mathbf{e}}_z + \frac{w(r,z)}{r}\hat{\mathbf{e}}_\varphi, \tag{7.1}$$

where $\psi(r,z)$ is a stream function and $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_z$, $\hat{\mathbf{e}}_\varphi$ are unit orts in the cylindrical coordinates r, z, φ. The steady axisymmetric Euler equations (1.6) are reduced to the nonlinear Grad - Shafranov equation (1.12) for the stream function $\psi(r,z)$ where arbitrary smooth functions $F(\psi)$ and $G(\psi)$ are connected with vector field $\mathbf{V}$ (7.1) and pressure p by the relations[4]

$$\frac{p}{\rho} + \frac{1}{2}|\mathbf{V}|^2 = F(\psi), \quad w(r,z) = G(\psi).$$

The vortex field of the fluid flow $\mathbf{V}(r,z)$ (7.1) has the form

$$\operatorname{curl}\mathbf{V}(r,z) = -\frac{G(\psi)_z}{r}\hat{\mathbf{e}}_r + \frac{G(\psi)_r}{r}\hat{\mathbf{e}}_z - \frac{1}{r}(\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz})\hat{\mathbf{e}}_\varphi. \tag{7.2}$$

Therefore, the dynamical system of vortex lines

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z) = \dot{r}\hat{\mathbf{e}}_r + r\dot{\varphi}\hat{\mathbf{e}}_\varphi + \dot{z}\hat{\mathbf{e}}_z = \operatorname{curl}\mathbf{V} \tag{7.3}$$

by virtue of Equations (1.12) and (7.2) takes the form

$$\dot{r} = -\frac{1}{r}G'(\psi)\psi_z, \quad \dot{z} = \frac{1}{r}G'(\psi)\psi_r, \tag{7.4}$$

$$\dot{\varphi} = -F'(\psi) + \frac{1}{r^2}G(\psi)G'(\psi). \tag{7.5}$$

II. Suppose that the stream function $\psi(r,z)$ has a local non-degenerate maximum or minimum $\psi_m = \psi(a_m)$ at a point a_m with coordinates (r_m, z_m) and $G'(\psi_m) \neq 0$,

$$\psi_r(a_m) = 0, \quad \psi_z(a_m) = 0, \quad \mathcal{H}_m = \psi_{rr}(a_m)\psi_{zz}(a_m) - \psi_{rz}^2(a_m) > 0. \tag{7.6}$$

The function $G(\psi)$ also has a non-degenerate maximum or minimum at the point a_m: the corresponding Hessian is $\mathcal{H}_{G(\psi)} = (G'(\psi_m))^2\mathcal{H}_m > 0$. Therefore system (7.4) has the center equilibrium point a_m and system (7.4) and (7.5) has a stable trajectory –vortex axis $\mathbb{S}_m$: $r = r_m$, $z = z_m$, $0 \le \varphi < 2\pi$. All trajectories of system (7.4) near the center are closed curves $C_\psi : \psi(r,z) = \text{const}$ encircling the point a_m. The corresponding trajectories of system (7.4) and (7.5) are either infinite helices or closed curves—knots—lying on the invariant tori $\mathbb{T}^2_\psi = C_\psi \times \mathbb{S}^1$ where circle $\mathbb{S}^1$ corresponds to the angle φ. Let $t(\psi)$ be the period of the closed trajectory C_ψ. The pitch function $p(\psi)$ of the helices on the torus $\mathbb{T}^2_\psi$ is defined by the formula

$$p(\psi) = \int_0^{t(\psi)} \frac{d\varphi}{dt} dt. \tag{7.7}$$

Formula (7.7) agrees with Moffatt's definition[33] of function $p(\psi)$. If $p(\psi)/(2\pi) = p/q$ where p and q are coprime integers then all helices on the torus $\mathbb{T}^2_\psi$ after q turns around meridians make p turns around the latitudes and hence are closed curves that are called the torus knots $K_{p,q}$.

Substituting Equation (7.5) into formula (7.7) we get

$$p(\psi) = -F'(\psi)t(\psi) + G(\psi)G'(\psi) \int_0^{t(\psi)} \frac{dt}{r^2(t)}.$$

In the limit $\psi \longrightarrow \psi_m$ we have $r(t) \longrightarrow r_m$ for all t, hence

$$p(\psi_m) = \lim_{\psi \longrightarrow \psi_m} p(\psi) = t(\psi_m)\left[-F'(\psi_m) + G(\psi_m)G'(\psi_m)/r_m^2\right], \tag{7.8}$$

where $t(\psi_m) = \lim_{\psi \longrightarrow \psi_m} t(\psi)$.

III. Dynamical system (7.4) near the equilibrium point (r_m, z_m) is approximated by the system in variations[38]

$$\frac{d\delta r}{dt} = -a_{11}\delta z - a_{12}\delta r, \quad \frac{d\delta z}{dt} = a_{12}\delta z + a_{22}\delta r, \tag{7.9}$$

$$a_{11} = c_m\psi_{zz}(a_m),\ a_{12} = c_m\psi_{rz}(a_m),\ a_{22} = c_m\psi_{rr}(a_m),\ c_m = \frac{G'(\psi_m)}{r_m}, \tag{7.10}$$

where $\delta r(t) = r(t) - r_m$, $\delta z(t) = z(t) - z_m$. From Equations (7.6), (7.10) we get

$$D_m = a_{11}a_{22} - a_{12}^2 = c_m^2\mathcal{H}_m > 0. \tag{7.11}$$

Linear system (7.9) has a quadratic first integral $Q(\delta r, \delta z) = a_{22}(\delta r)^2 + 2a_{12}(\delta r)(\delta z) + a_{11}(\delta z)^2$ that in view of (7.11) is either positive or negative definite. Hence its level curves $Q(\delta r, \delta z) = \text{const}$ are nested ellipses and therefore all solutions to (7.9) are periodic. Due to the scaling invariance of system (7.9) all its solutions have the same period $t_m = 2\pi/\sqrt{D_m}$.

From the general theory of dynamical systems,[38] it follows that the limit at $\psi \longrightarrow \psi_m$ of the function of periods $t(\psi)$ is the period t_m of the system in variations (7.9). Using formula (7.11) we find

$$t(\psi_m) = \lim_{\psi \longrightarrow \psi_m} t(\psi) = \frac{2\pi}{\sqrt{D_m}} = \frac{2\pi}{|c_m|\sqrt{\mathcal{H}_m}}. \tag{7.12}$$

Substituting (7.12) and (7.10) into (7.8) we get

$$p(\psi_m) = \lim_{\psi \longrightarrow \psi_m} p(\psi) = \frac{2\pi r_m}{|G'(\psi_m)|\sqrt{\mathcal{H}_m}} \left[-F'(\psi_m) + \frac{1}{r_m^2}G(\psi_m)G'(\psi_m)\right]. \tag{7.13}$$

This formula proves that for the case of arbitrary functions $F(\psi)$, $G(\psi)$ in Equation (1.12) the pitch function $p(\psi)$ has a finite and non-zero limit at $\psi \longrightarrow \psi_m$. The limit (7.13) is one of the two exact bounds for the range of the pitch function $p(\psi)$. Hence we get one of the two exact bounds $p(\psi_m)/(2\pi)$ for the range of the rationals p/q for the torus knots $K_{p,q}$ which can be realized as vortex knots for the considered fluid flow $\mathbf{V}(r,z)$ (7.1). Since the exact bound $p(\psi_m) \neq 0$, the pitch function $p(\psi)$ cannot change continuously from zero to infinity.

VIII. MODULI SPACE $\mathcal{S}(\mathbb{R}^3)$ OF VORTEX KNOTS

I.

Proposition 3. For the exact solution to the steady Euler equations

$$\mathbf{V}_2(\mathbf{x}) = ((x + yz)G_3 + xz^2G_4)\hat{\mathbf{e}}_x + ((y - xz)G_3 + yz^2G_4)\hat{\mathbf{e}}_y + \tag{8.1}$$

$$z(G_2 + 3G_3 + z^2G_4)\hat{\mathbf{e}}_z, \qquad p(\mathbf{x}) = C - \rho|\mathbf{V}_2(\mathbf{x})|^2/2,$$

the dynamics of its vortex lines is non-degenerately integrable in each invariant domain $\mathcal{D}_{k.\pm} \times S^1$, $k = 1,2,\ldots$. Here G_2, G_3, G_4 are analytic functions (2.14), (2.15), and (2.16) of the spherical radius $R = \sqrt{x^2 + y^2 + z^2}$.

Proof. Vector field $\mathbf{V}_2(\mathbf{x})$ (8.1) has form (2.9) in the cylindrical coordinates r, z, φ with function $f(r,z) = zG_2(R)$. Hence vector field (8.1) satisfies Beltrami equation (2.1) and therefore together with the pressure $p(\mathbf{x}) = C - \rho|\mathbf{V}_2(\mathbf{x})|^2/2$ is an exact solution to the steady Euler Equations (1.6). The dynamical system $d\mathbf{x}/dt = \operatorname{curl}\mathbf{V}_2(\mathbf{x})$ was transformed into system (4.2) and (4.3) in Section IV where we proved in Proposition 2 that the system is non-degenerately integrable if and only if the functions of periods $\tau_k(H)$ are not constant. In Sections V and VI we derived the limits of functions $\tau_k(H)$ when the Hamiltonian function $H_2(r,z)$ is changing between its maximum or minimum H_k in the invariant domains $\mathcal{D}_{k.\pm}$ and the zero value at their boundaries $\partial\mathcal{D}_{k.\pm}$. Since functions $\tau_k(H)$ are continuous, we conclude that each function $\tau_k(H)$ takes all values between its limits (6.5), (6.7), and (5.4),

$$p_k = \frac{1}{4\pi}(R_k - R_{k-1}) < \tau_k(H) < \frac{1}{2\sqrt{1 - 6\tilde{R}_k^{-2}}} = \tau_k(H_k), \tag{8.2}$$

where $p_1 = R_1/(4\pi)$. Numerical calculations show that functions $\tau_k(|H|)$ are changing monotonously between their limits p_k (6.7) (or p_1 (6.5)) and $\tau_k(H_k)$ (5.4). Results of the numerical calculations are presented in Figure 4 for $k = 1,2,3,4$.

Using the numerical values (6.8) and (5.8) we find for the invariant domains $\mathcal{D}_{k.\pm}$,

$$\mathcal{D}_{1.\pm}: \quad 0.4586 < \tau_1(H) < 0.5847, \quad \mathcal{D}_{2.\pm}: \quad 0.2681 < \tau_2(H) < 0.5295, \tag{8.3}$$

$$\mathcal{D}_{3.\pm}: \quad 0.2569 < \tau_3(H) < 0.5136, \quad \mathcal{D}_{4.\pm}: \quad 0.2540 < \tau_4(H) < 0.5079.$$

Remark 10. The first inequalities (8.3) prove that all values of the safety factor $\mathbf{q}(H) = \tau_1(H)$ for the flow $\mathbf{V}_2(\mathbf{x})$ (8.1) in the first invariant ball $\mathbb{B}_1^3$ belong to those presented in (8.3) interval of small length $\ell \approx 0.1261$.

The plots of functions $\tau_k(|H|)$ are shown in Figure 5 for $k = 1$ and in Figure 6 for $k \geq 2$. The sequences of numbers p_k and $\tau_k(H_k)$ monotonously decreases as $k \longrightarrow \infty$ to the limits (6.9) and (5.9),

$$\lim_{k\longrightarrow\infty} p_k = 0.25, \qquad \lim_{k\longrightarrow\infty} \tau_k(H_k) = 0.5. \tag{8.4}$$

The limits (8.2)–(8.4) prove that the functions $\tau_k(H)$ are not constant for all $k \geq 1$. Hence applying Proposition 2 we get that dynamical system $d\mathbf{x}/dt = \operatorname{curl}\mathbf{V}_2(\mathbf{x})$ (4.2) and (4.3) is *non-degenerately* integrable in all invariant domains $\mathcal{D}_{k.\pm} \times S^1$. □

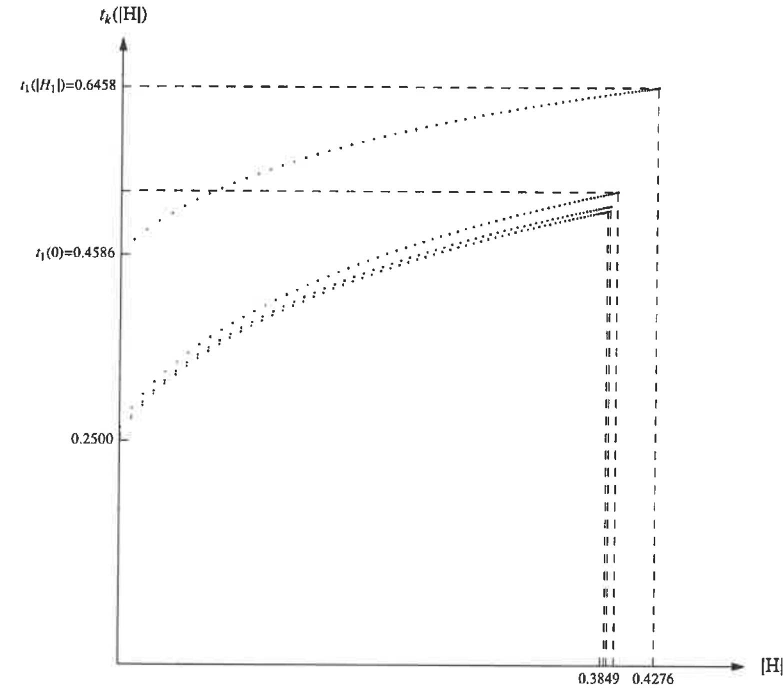

FIG. 4. Numerical calculations of functions of periods $\tau_1(|H|)$, $\tau_2(|H|)$, $\tau_3(|H|)$, and $\tau_4(|H|)$.

Proposition 4. The functions $\hat{f}(r,z)$ (2.29) define vector fields $\mathbf{V}_{\hat{f}}(\mathbf{x})$ (2.9) which are exact solutions to the Euler equations (1.6) and provide the non-degenerate integrability of system $\mathrm{d}\mathbf{x}/\mathrm{d}t = \mathrm{curl}\,\mathbf{V}_{\hat{f}}(\mathbf{x})$ *if*

$$a_1 + \cdots + a_n = 1, \quad a_k \geq 0, \quad |z_k| < \varepsilon << 1. \tag{8.5}$$

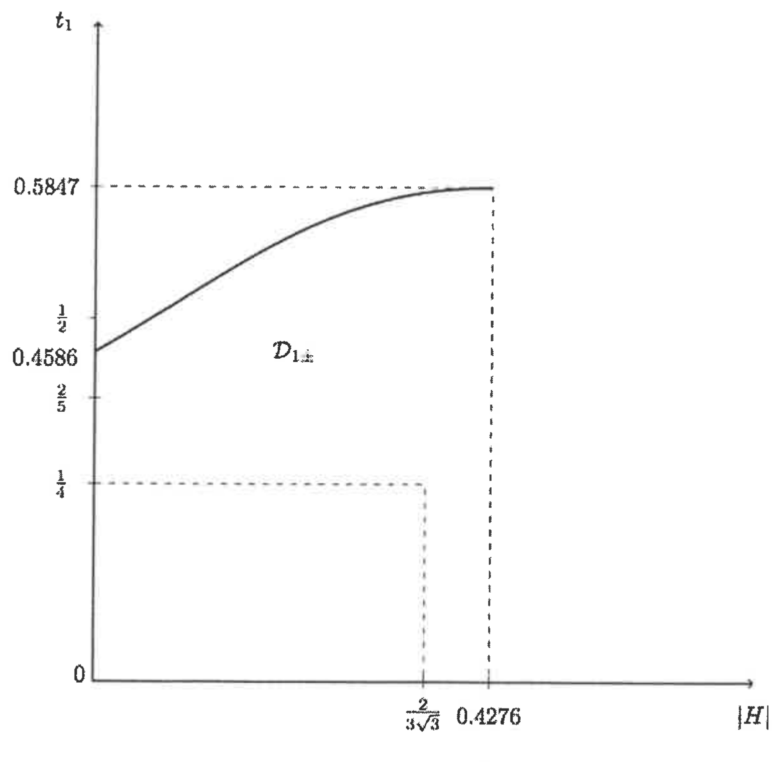

FIG. 5. Function of periods $\tau_1(|H|)$ for invariant domains $\mathcal{D}_{1,\pm}$.

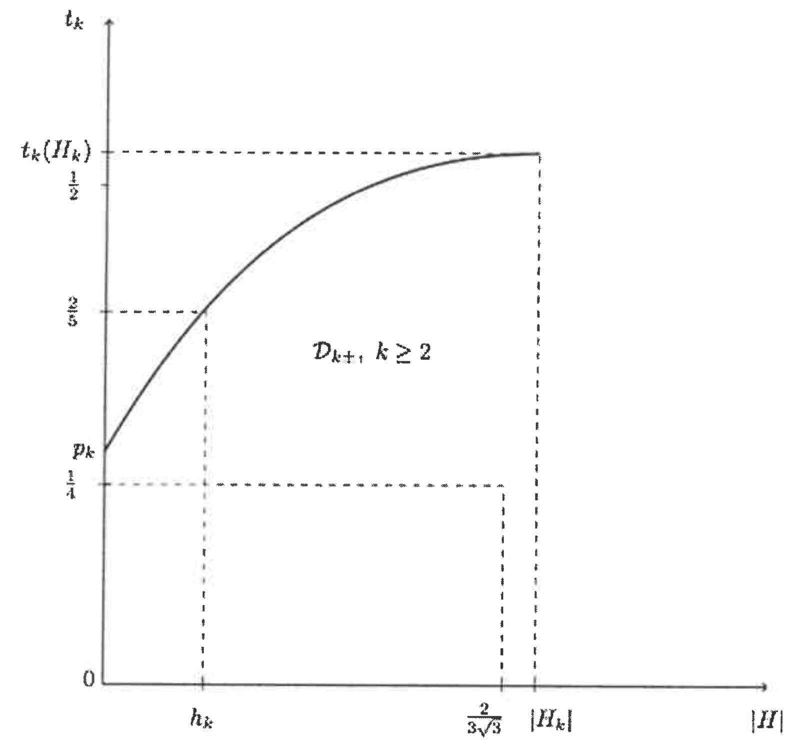

FIG. 6. Function of periods $\tau_k(|H|)$ for invariant domains $\mathcal{D}_{k,\pm}$, $k \geq 2$.

Proof. Vector fields $\mathbf{V}_{\hat{f}}(\mathbf{x})$ (2.9) satisfy Beltrami equation (2.1) and hence define solutions to Euler equations (1.6) with $p(\mathbf{x}) = C - \rho|\mathbf{V}_{\hat{f}}(\mathbf{x})|^2/2$. Function $\hat{f}(r,z)$ (2.29) at the conditions (8.5) is a small ε-perturbation of function $\tilde{f}(r,z) = zG_2(R)$ which corresponds to the exact solution (8.1). Hence the function $H_{\hat{f}}(r,z) = -r\hat{f}_r(r,z)$ also is a small ε-perturbation of function $H_2(r,z) = -r\tilde{f}_r(r,z)$ (2.25). Hence trajectories $C_{H_{\hat{f}}}$ of the perturbed dynamical system (4.2) in the plane (r,z) also are closed curves. Since function of periods $\tau(H_{\hat{f}})$ of closed trajectories $C_{H_{\hat{f}}}$ continuously depends on perturbations we get that $\tau(H_{\hat{f}})$ is not constant since all functions $\tau_k(H)$ (8.2) are not constant. Therefore applying Proposition 2 we get that the corresponding dynamical system (4.2) and (4.3) is *non-degenerately* integrable. □

II. Let us consider trajectories of dynamical system (4.2) and (4.3) on the invariant tori $T^2_H = C_H \times S^1$ where closed curves $C_H \subset R^2$ are defined by equations $H_2(r,z) = H = \text{const}$ and circle S^1 corresponds to the angular variable φ. Suppose that for $H = H_0$ the function of periods $\tau(H)$ has a rational value $\tau(H_0) = p/q$ with relatively prime p and q. Then all trajectories of system (4.2) and (4.3) on torus T_{H_0} make q complete turns over meridians and p complete turns over the longitudes. Hence all these trajectories form a torus knot $K_{p,q}$.

One of the topological invariants of any knot is its Alexander polynomial that is defined up to an arbitrary factor $\pm t^{n_1}$. The equality of the Alexander polynomials $\Delta(\mathcal{K})$ and $\Delta(\mathcal{L})$ of two knots $\mathcal{K}$ and $\mathcal{L}$ is the necessary condition for their isotopy equivalence but not the sufficient one.

The Alexander polynomial for a torus knot $K_{p,q}$ has the form[5]

$$\Delta_{p,q}(t) = \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)}. \tag{8.6}$$

The polynomial $\Delta_{p,q}(t)$ has degree $n = pq + 1 - p - q = (p-1)(q-1)$. Since p and q are relatively prime, the degree n is always even. For $p/q = 1/q$ and for $p/q = p/1$ the polynomial $\Delta_{p,q}(t) \equiv 1$ and the corresponding closed curves are unknots.

Lemma 5. If two torus knots $K_{p,q}$ and $K_{\tilde{p},\tilde{q}}$ are equivalent then either $\tilde{p}/\tilde{q} = p/q$ or $\tilde{q}/\tilde{p} = p/q$.

Proof. If the two knots are equivalent then their Alexander polynomials (8.6) after multiplication by factors $\pm t^{n_i}$ coincide

$$\frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)} = \frac{(t^{\tilde{p}\tilde{q}}-1)(t-1)}{(t^{\tilde{p}}-1)(t^{\tilde{q}}-1)}. \tag{8.7}$$

Polynomial $\Delta_{p,q}(t)$ (8.6) does not have any real roots and all its complex roots lie on the unit circle $|t| = 1$. The root with minimal argument is $\tau_1 = \exp(2\pi i/(pq))$. The equality of polynomials (8.7) yields $\exp(2\pi i/(pq)) = \exp(2\pi i/(\tilde{p}\tilde{q}))$. Hence we get $pq = \tilde{p}\tilde{q}$. Since the degrees of polynomials (8.7) coincide, we have $pq + 1 - p - q = \tilde{p}\tilde{q} + 1 - \tilde{p} - \tilde{q}$. Therefore $p + q = \tilde{p} + \tilde{q}$. The two equalities $pq = \tilde{p}\tilde{q}$ and $p + q = \tilde{p} + \tilde{q}$ imply that either $\tilde{p} = p, \tilde{q} = q$ or $\tilde{q} = p, \tilde{p} = q$. □

Lemma 6 evidently implies the following

Corollary 3. The torus knots $K_{p,q}$ and $K_{\tilde{p},\tilde{q}}$ with $p/q < 1$ and $\tilde{p}/\tilde{q} < 1$ are not isotopic if $p/q \neq \tilde{p}/\tilde{q}$.

Remark 11. Lemma 6 was first published in Ref. 39 as Theorem 2.2.2. Its proof is given in sections 6.1.17 and 12.2.15 of Ref. 39 and is based on "Kurosh subgroup theorem" and uses the notions of "non-slice and non-amphicheiral" knots. Neither of those are necessary to prove Lemma 6, as follows from the above presented straightforward proof.

Lemma 6. For the exact fluid flow (2.21), the vortex knots in the invariant upper half-space $\mathbb{H}^3_+$, $z > 0$, are isotopic to the vortex knots in the invariant lower half-space $\mathbb{H}^3_-$, $z < 0$.

Proof. Evidently, the plane $z = 0$ is an invariant submanifold of the dynamical system (2.22) corresponding to the flow (8.1) for which $\operatorname{curl}\mathbf{V}_2(\mathbf{x}) = \mathbf{V}_2(\mathbf{x})$. Hence all closed trajectories of this system lie either in the upper half-space $\mathbb{H}^3_+$ or in the lower half-space $\mathbb{H}^3_-$. Dynamical system (2.22) is invariant under the automorphism T (2.28) which transforms trajectories in domain $z > 0$ into trajectories in domain $z < 0$. The automorphism T can be included into the isotopy $\mathcal{T}_\theta(x, y, z) = (x, \cos\theta y - \sin\theta z, \sin\theta y + \cos\theta z)$, $0 \le \theta \le \pi$. Indeed, for $\theta = 0$ we get $\mathcal{T}_0 = \mathrm{id}$ and for $\theta = \pi$ we find $\mathcal{T}_\pi = T$. Therefore any vortex knot K in the upper half-space $\mathbb{H}^3_+$ is smoothly isotopic by the transformations $\mathcal{T}_\theta(K)$ to the vortex knot $T(K)$ in the lower half-space $\mathbb{H}^3_-$ and *vice versa*. □

Theorem 1. *For the exact fluid flow (8.1) in the Euclidean space $\mathbb{R}^3$ and in the half-spaces $\mathbb{H}^3_+$, $z > 0$ and $\mathbb{H}^3_-$, $z < 0$, the moduli spaces $\mathcal{S}(\mathbb{R}^3)$, $\mathcal{S}(\mathbb{H}^3_+)$, and $\mathcal{S}(\mathbb{H}^3_-)$ of vortex knots coincide and are naturally isomorphic to the set of all rational numbers p/q in the interval*

$$J_1: \quad \frac{1}{4} < \tau < \tilde{M}_1 = \frac{1}{2\sqrt{1 - 6\tilde{R}_1^{-2})}} = \tau_1(H_1) \approx 0.5847, \tag{8.8}$$

where $\tilde{R}_1 \approx 3.8702$ is the first positive solution (3.16) to the equation $\tan R = R(R^2 - 6)/(3R^2 - 6)$. *The vortex torus knots $K_{p,q}$ with $p/q \in J_1$ are mutually non-equivalent. All vortex knots $K_{p,q}$ have a clockwise rotation around the meridians; the mirror images of the knots $K_{p,q}$ are not realized by the vortex lines.*

Proof. Applying Lemma 7 we get that the vortex knots in the upper half-space $\mathbb{H}^3_+$, $z > 0$ are isotopic to the vortex knots in the lower half-space $\mathbb{H}^3_-$, $z < 0$. Hence the corresponding moduli spaces for $\mathbb{R}^3$, $\mathbb{H}^3_+$, and $\mathbb{H}^3_-$ coincide.

Proposition 3 and Equation (8.2) imply that for all fractions p/q from the range of a function of periods $\tau_k(H)$ the corresponding torus knots $K_{p,q}$ are realized by the vortex lines in the invariant domain $\mathcal{D}_{k,\pm} \times S^1$, $k \ge 1$. We have shown in Sections V and VI that the lower and upper limits of ranges (8.2) of functions $\tau_k(H)$ monotonously decrease when $k \longrightarrow \infty$,

$$p_k \downarrow \frac{1}{4}, \quad \tau_k(H_k) = \frac{1}{2\sqrt{1 - 6\tilde{R}_k^{-2}}} \downarrow \frac{1}{2}. \tag{8.9}$$

Hence the union of the ranges (8.2) of all functions $\tau_k(H)$ is the interval J_1 (8.8). Hence for any rational number $p/q \in J_1$ the corresponding torus knot $K_{p,q}$ is realized by the closed vortex lines.

The mutual non-equivalence of the torus knots $K_{p,q}$ for $p/q \in J_1$ follows from Corollary 5 since $p/q < \tau_1(H_1) < 1$. Their clockwise rotation around the meridians follows from Remark 6 of Section IV and the non-realisation of the mirror images of the vortex knots $K_{p,q}$ follows from Corollary 4.

Hence we have a one-to-one correspondence between all non-isotopic vortex knots for the fluid flow (8.1) in $\mathbb{R}^3$, or $\mathbb{H}^3_+$, or $\mathbb{H}^3_-$ and the set of all rational numbers from the interval J_1. This proves the above description of the moduli space $\mathcal{S}$. □

III. Let us define two intervals J_2 and J_3,

$$J_2: \quad (0.25,\ 0.5), \qquad J_3: \quad (0.5,\ \tilde{M}_1). \tag{8.10}$$

Let $\mathcal{F} \subset R^1$ be the union of three intervals $\mathcal{F} = (0,\ 0.25) \bigcup (\tilde{M}_1,\ \tilde{M}_1^{-1}) \bigcup (4,\ \infty)$. For any $x \in \mathcal{F}$, we have $x \notin J_1$ and $x^{-1} \notin J_1$.

Proposition 5. (a) Any torus knot $K_{p,q}$ with p/q from the interval J_2 (8.10) is realized by the closed vortex lines in infinitely many invariant domains $\mathcal{D}_{k.\pm} \times S^1$.

(b) Any torus knot $K_{\tilde{p},\tilde{q}}$ with $\tilde{p}/\tilde{q}$ from the interval J_3 (8.10) is realized only in finitely many invariant domains $\mathcal{D}_{k.\pm} \times S^1$.

(c) All torus knots $K_{\tilde{p},\tilde{q}}$ with rational numbers $\tilde{p}/\tilde{q}$ belonging to the set $\mathcal{F}$ (8.10) are not isotopy equivalent to any torus knot realized by closed vortex lines for the exact solution (8.1).

Proof. (a) Since $\lim_{k\to\infty} p_k = 1/4$ (8.9) and $\tau_k(H_k) > 1/2$, we get that any fraction $p/q \in J_2$ belongs to the ranges of all functions $\tau_k(H)$ starting from some k_1. Therefore the corresponding knot $K_{p,q}$ is realized in infinitely many invariant domains $\mathcal{D}_k \times S^1$ for $k \geq k_1$.

(b) Since $\lim_{k\to\infty} \tau_k(H_k) = 1/2$ (8.9) we find from (8.2) that any $\tilde{p}/\tilde{q} > 1/2$ does not belong to the ranges of all functions $\tau_k(H)$ starting from some integer k_2. Hence applying Corollary 5 we find that the torus knot $K_{\tilde{p},\tilde{q}}$ for $\tilde{p}/\tilde{q} \in J_3$ is realized by the closed vortex lines only in finitely many invariant domains $\mathcal{D}_{k.\pm} \times S^1$ for $1 \leq k < k_2$.

(c) For $\tilde{p}/\tilde{q} \in \mathcal{F}$ we have that both $\tilde{p}/\tilde{q} \notin J_1$ and $\tilde{q}/\tilde{p} \notin J_1$. Hence by Lemma 6 the Alexander polynomial $\Delta_{\tilde{p},\tilde{q}}(t)$ (8.6) is different from the Alexander polynomials $\Delta_{p,q}(t)$ for all torus knots $K_{p,q}$ with $p/q \in J_1$. Therefore the torus knot $K_{\tilde{p},\tilde{q}}$ is not isotopic to any torus knot $K_{p,q}$ with $p/q \in J_1$. □

Example 1. The torus knot $K_{2,5}$ (see Figure 7) is realized by the closed vortex lines in all domains $\mathcal{D}_{k.\pm} \times S^1$ for $k \geq 2$ because the fraction $2/5 \in J_2$ is in the ranges of all functions $\tau_k(H)$, $k \geq 2$, see formulae (8.3) and Figure 6. Applying Corollary 5 we find that the knot $K_{2,5}$ is not realized in the two domains $\mathcal{D}_{1.\pm} \times S^1$ because $2/5$ does not belong to the range of function $\tau_1(|H|)$, see formulae (8.3) and Figure 5. The Alexander polynomial of the torus knot $K_{2,5}$ has the form $\Delta_{2,5}(t) = t^4 - t^3 + t^2 - t + 1$. This is the only quartic Alexander polynomial possible for the torus knots.

Example 2. Another simple torus knot that is realized in all domains $\mathcal{D}_{k.\pm} \times S^1$ for $k \geq 2$ and is not realized in the two domains $\mathcal{D}_{1.\pm} \times S^1$ is the knot $K_{2,7}$. Indeed, the fraction $2/7 \approx 0.2857 \in J_2$ is in the ranges of all functions $\tau_k(H)$ (8.3) for $k \geq 2$ and is not in the range of function $\tau_1(|H|)$. The corresponding Alexander polynomial (8.6) has degree 6: $\Delta_{2,7}(t) = t^6 - t^5 + t^4 - t^3 + t^2 - t + 1$.

The simplest torus knots $K_{2,3}$ (the trefoil knot) and $K_{3,4}$ are not realized by the closed vortex lines for the exact solution (8.1) because the fractions $2/3$ and $3/4$ do not belong to the interval (8.8).

IX. MODULI SPACES $\mathcal{S}_m(\mathbb{B}^3_a)$ OF VORTEX KNOTS

I. In this section we present the moduli spaces of vortex knots for the fluid flows $\mathbf{V}(\mathbf{x})$ inside a ball $\mathbb{B}^3_a$ $(x^2 + y^2 + z^2 \leq a^2)$ which are solutions to the boundary eigenvalue problem[26–30] for the curl

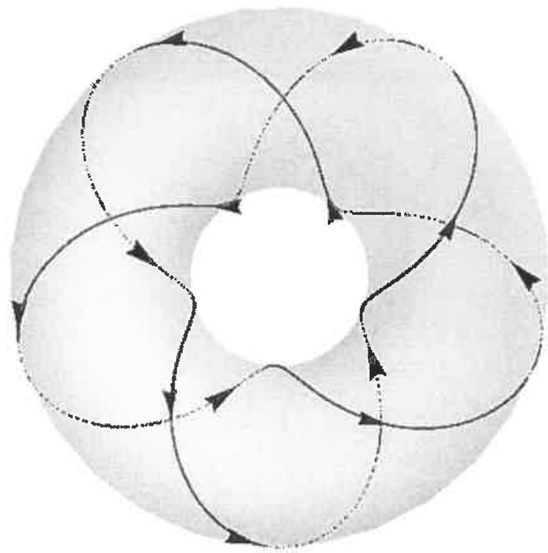

FIG. 7. The torus knot $K_{2,5}$ with period $\tau_k(h_k) = 2/5$ is realized by the closed vortex lines in each invariant domain $\mathcal{D}_{k,\pm}$ for $k \geq 2$.

operator

$$\operatorname{curl} \mathbf{V}(\mathbf{x}) = \lambda \mathbf{V}(\mathbf{x}), \quad (\mathbf{V}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})|_{\partial \mathbb{B}_a^3} = 0. \tag{9.1}$$

Here $\mathbf{n}(\mathbf{x}) = \mathbf{x}/a$ is the unit normal vector field on the boundary sphere $\mathbb{S}_a^2 = \partial \mathbb{B}_a^3$. The second equation in (9.1) is called the non-penetration condition and means that the boundary sphere $\mathbb{S}_a^2$ is an invariant submanifold for the fluid flow $\mathbf{V}(\mathbf{x})$.

Let us construct for the vector field $\mathbf{V}_2(x, y, z)$ (8.1) an infinite series of axisymmetric solutions to the boundary eigenvalue problem (9.1),

$$\mathbf{V}_{2m}(x, y, z) = \mathbf{V}_2(\lambda_m x, \lambda_m y, \lambda_m z), \quad \lambda_m = a^{-1} R_m, \tag{9.2}$$

where R_m is the mth positive solution to the equation $\tan R = 3R/(3 - R^2)$ (2.25). Since the vector field $\mathbf{V}_2(x, y, z)$ (2.21), (8.1) satisfies Equation (2.1), we find that for any λ vector field $\mathbf{V}_2(\lambda x, \lambda y, \lambda z)$ satisfies Beltrami equation $\operatorname{curl} \mathbf{V}(\mathbf{x}) = \lambda \mathbf{V}(\mathbf{x})$. The first integral $H_2(r, z) = -zr^2 G_3(R)$ (2.25) of the vector field $\mathbf{V}_2(x, y, z)$ (2.21) yields the first integral $H_{2\lambda}(r, z) = -\lambda^3 zr^2 G_3(\lambda R)$ of the vector field $\mathbf{V}_2(\lambda x, \lambda y, \lambda z)$. Since all invariant submanifolds $H_2(r, z) = C$ with a non-zero constant $C \neq 0$ are tori $\mathbb{T}_C^2$, we get that the sphere $\mathbb{S}_a^2$ of radius a is an invariant submanifold for the vector field $\mathbf{V}_2(\lambda x, \lambda y, \lambda z)$ if and only if the equation $H_{2\lambda}(r, z) = -\lambda^3 zr^2 G_3(\lambda a) = 0$ holds on $\mathbb{S}_a^2$. Therefore, λa must satisfy equation $G_3(R) = 0$ (2.15), which means that λa must be equal to one of the roots R_m of equation $\tan R = 3R/(3 - R^2)$ (2.25). Hence we get an infinite series of eigenvalues $\lambda_m = a^{-1} R_m$ and eigenvector fields $\mathbf{V}_{2m}(x, y, z)$ (9.2) for the boundary eigenvalue problem (9.1).

Theorem 2. *The moduli space $\mathcal{S}_m(\mathbb{B}_a^3)$ of vortex knots for the mth fluid flow (9.2) inside the ball $\mathbb{B}_a^3$ is naturally isomorphic to the set of all rational numbers p/q in the interval*

$$I_m : \quad \frac{1}{4\pi}(R_m - R_{m-1}) < \tau < \tilde{M}_1 \approx 0.5847, \tag{9.3}$$

where number $\tilde{M}_1$ is defined by Equation (8.8). The space $\mathcal{S}_m(\mathbb{B}_a^3)$ does not depend on the radius a. The vortex torus knots $K_{p,q}$ with $p/q \in J_m$ are mutually non-equivalent. All vortex knots $K_{p,q}$ have a clockwise rotation around the meridians.

Proof. The equation for the vortex lines for the mth flow (9.2) has the form

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \operatorname{curl} \mathbf{V}_2(\lambda_m \mathbf{x}). \tag{9.4}$$

Substituting here Beltrami equation (9.1) and multiplying with λ_m, we get $d(\lambda_m \mathbf{x})/dt = \lambda_m^2 \mathbf{V}_2(\lambda_m \mathbf{x})$. Hence the vortex lines for the vector field (9.2) inside the ball $\mathbb{B}_a^3$ ($|\mathbf{x}| \leq a$) after change of time $d\tau/dt = \lambda_m^2$ and substitution $\lambda_m \mathbf{x} = \mathbf{y}$ satisfy the equation

$$\frac{d\mathbf{y}}{d\tau} = \operatorname{curl} \mathbf{V}_2(\mathbf{y}) = \mathbf{V}_2(\mathbf{y}). \tag{9.5}$$

Since $|\mathbf{y}| = |\lambda_m||\mathbf{x}| \leq R_m$, the vortex lines (9.4) for the flow (9.2) inside the ball $\mathbb{B}_a^3$ are mapped by the diffeomorphism $\mathbf{y} = \lambda_k \mathbf{x}$ into the vortex lines (1.9) and (9.5) for the axisymmetric fluid flow $\mathbf{V}_2(x, y, z)$ (8.1) inside the invariant sphere $\mathbb{S}_m^2$ of radius R_m.

The interior of the invariant with respect to the flow (9.5) sphere $\mathbb{S}_m^2$ is the union of $2m$ invariant domains $\mathcal{D}_{k.\pm} \times S^1$, $k = 1, 2, \ldots, m$, the disk $x^2 + y^2 < R_m^2$ in the plane $z = 0$, the interval $-R_m < z < R_m$, $r = 0$, and $m - 1$ intermediate invariant spheres $\mathbb{S}_k^2$, $k = 1, 2, \ldots, m - 1$. In each domain $\mathcal{D}_{k.\pm} \times S^1$, the function of periods $\tau_k(H)$ (see Section IV) is changing in the interval (8.2): $p_k = (R_k - R_{k-1})/(4\pi) < \tau_k(H) < \tau_k(H_k)$. Since any two subsequent intervals $(p_k, \tau_k(H_k))$ and $(p_{k+1}, \tau_{k+1}(H_{k+1}))$ have non-zero intersection and both bounds p_k and $\tau_k(H_k)$ monotonously decrease when k grows, we find that the union of intervals $(p_k, \tau_k(H_k))$ for $k = 1, 2, \ldots, m$ is the interval I_m (9.3).

As is shown in Section IV, any vortex knot $K_{p,q}$ of the considered axisymmetric flow in $\mathbb{R}^3$ corresponds to the rational value p/q of some function of periods $\tau_k(H)$ and vice versa. All vortex knots $K_{p,q}$ realized by the system (9.4) and (9.5) have a clockwise rotation around the meridians because by virtue of Theorem 1 this is true for all vortex knots for the flow $\mathbf{V}_2(\mathbf{x})$ (8.1). Hence using Theorem 1 we obtain that the moduli space $\mathcal{S}_m(\mathbb{B}_a^3)$ of vortex knots for the mth flow (9.2) is isomorphic the set of all rational numbers p/q in the interval I_m (9.3) and does not depend on the radius a of the ball $\mathbb{B}_a^3$. □

Using formulae (6.8) for the numbers $p_k = (R_k - R_{k-1})/(4\pi)$ for $k = 1, 2, 3, 4$ we find the approximate bounds of the first six intervals I_m (9.3),

$$I_1: \quad (0.4586, 0.5847), \quad I_2: \quad (0.2651, 0.5847), \quad I_3: \quad (0.2569, 0.5847),$$

$$I_4: \quad (0.2540, 0.5847), \quad I_5: \quad (0.2526, 0.5847), \quad I_6: \quad (0.2519, 0.5847).$$

In the limit $m \longrightarrow \infty$ we find from Equation (6.9) that $I_m \longrightarrow J_1$ where the interval J_1 : $(1/4, 0.5847)$ is defined by formulae (8.8). Hence we get from Theorems 1 and 2 that each vortex knot of the axisymmetric fluid flow (8.1) in the whole Euclidean space $\mathbb{R}^3$ is realized for a sufficiently large m as a vortex knot for the eigenvector field $\mathbf{V}_{2m}(x, y, z)$ (9.2) inside the ball $\mathbb{B}_a^3$.

X. CONCLUSION

I. We have derived exact axisymmetric solutions to the steady Euler equations (1.6) with velocity vector fields

$$\mathbf{V}_{\hat{f}}(\mathbf{x}) = \frac{1}{r}\frac{\partial}{\partial z}\left(r\frac{\partial \hat{f}}{\partial r}\right)\hat{\mathbf{e}}_r - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \hat{f}}{\partial r}\right)\hat{\mathbf{e}}_z - \frac{\partial \hat{f}}{\partial r}\hat{\mathbf{e}}_\varphi, \tag{10.1}$$

and pressure $p = C - \rho_c|\mathbf{V}_{\hat{f}}(\mathbf{x})|/2$, where functions $\hat{f}(r, z)$ have the form

$$\hat{f}(r, z) = \sum_{k=1}^{n} \frac{a_k(z - z_k)}{r^2 + (z - z_k)^2}\left(\cos\sqrt{r^2 + (z - z_k)^2} - \frac{\sin\sqrt{r^2 + (z - z_k)^2}}{\sqrt{r^2 + (z - z_k)^2}}\right), \tag{10.2}$$

and a_k, z_k are arbitrary constants. Functions (10.2) satisfy the Helmholtz equation $\Delta \hat{f} = -\hat{f}$. The vector fields $\mathbf{V}_{\hat{f}}(\mathbf{x})$ (10.1) - (10.2) obey also the Beltrami equation $\operatorname{curl} \mathbf{V} = \mathbf{V}$.

We have proved in Propositions 3 and 4 that dynamical systems of vortex lines $d\mathbf{x}/dt = \operatorname{curl} \mathbf{V}_{\hat{f}}(\mathbf{x})$ (1.9) for $a_1 + \cdots + a_n = 1$, $a_k \geq 0$, $|z_k| < \varepsilon \ll 1$ are non-degenerately integrable and their dynamics occurs on invariant tori $\mathbb{T}_H^2 = C_H \times S^1$, defined by equation $H(\mathbf{x}) = H = \text{const}$, $H(\mathbf{x}) = -r\partial \hat{f}(r, z)/\partial r$. Here $C_H \subset R^2$ is a closed curve $H(r, z) = H = \text{const}$ and circle S^1 corresponds to the angular variable φ.

The simplest of the exact solutions (10.1) and (10.2) is the flow $\mathbf{V}_{f_2}(\mathbf{x})$ (8.1) which has the form (10.1) with function

$$\hat{f} = f_2(r,z) = zG_2(R) = \frac{z}{r^2+z^2}\left(\cos\sqrt{r^2+z^2} - \frac{\sin\sqrt{r^2+z^2}}{\sqrt{r^2+z^2}}\right). \tag{10.3}$$

For the fluid flow $\mathbf{V}_{f_2}(\mathbf{x})$, system (1.9) has a closed subsystem (3.2) in the (r,z)-plane $\mathbb{R}^2$. Subsystem (3.2) has infinitely many invariant domains $\mathcal{D}_{k.\pm}$ bounded by the semi-circles $R = R_k$ and $R = R_{k-1}$, $r \geq 0$, satisfying equation $H_2(r,z) = -r\partial f_2/\partial r = 0$ and the line $z = 0$. Here R_k are roots of equation $\tan R = 3R/(3-R^2)$. For each invariant domain $\mathcal{D}_{k.\pm}$ the function $\tau_k(H)$ of periods of closed trajectories $C_H \subset \mathbb{R}^2$ is not constant. The rational values of functions of periods $\tau_k(H) = p/q$ define tori $\mathbb{T}^2_H$ on which all trajectories of system (1.9) are closed curves that make q complete turns around the meridians and p complete turns around the longitudes. These trajectories form the torus knots $K_{p,q}$ (p and q are coprime).

II. We have proved in Theorem 1 that all vortex knots for the exact axisymmetric flow $\mathbf{V}_2(\mathbf{x}) = \mathbf{V}_{f_2}(\mathbf{x})$ (2.21), (8.1) are torus knots $K_{p,q}$ with $1/4 < p/q < 0.5847$. This gives a counterexample to Moffatt's statement of Ref. 32 from which it would follow that all torus knots $K_{p,q}$ for any p/q are realized as vortex knots for any steady axisymmetric fluid flow. Therefore the claimed uniformity in Ref. 32 (see the quote between Equations (1.12) and (1.13) of Section I) does not correspond to the facts.

For any axisymmetric steady fluid flows for which the stream function $\psi(r,z)$ has a non-degenerate maximum or minimum ψ_m at a point $a_m(r_m, z_m)$, we have proved in Section VII that the pitch function $p(\psi)$ has a finite and non-zero limit at $\psi \longrightarrow \psi_m$,

$$p(\psi_m) = \lim_{\psi\longrightarrow\psi_m} p(\psi) = \frac{2\pi r_m}{|G'(\psi_m)|\sqrt{\mathcal{H}_m}}\left[-F'(\psi_m) + \frac{1}{r_m^2}G(\psi_m)G'(\psi_m)\right]. \tag{10.4}$$

Formula (10.4) implies that presented in Refs. 33 and 34 with no proof Moffatt's statements that for the studied special flows $\lim_{\psi\longrightarrow\psi_{max}} p(\psi) = \infty$ are incorrect.

Formula (10.4) yields a plethora of counterexamples to the concluding part of Moffatt's statement of Ref. 32, p. 29:

"... the pitch of the helix varying continuously from zero ... to infinity ".

Indeed, for the generic functions $F(\psi)$ and $G(\psi)$ in the Grad–Shafranov Equation (1.12) the limit value $p(\psi_m)$ (10.4) evidently is finite and non-zero and is one of the two exact bounds of the range of the function $p(\psi)$. Since the bound $p(\psi_m) \neq 0$ we get that the pitch function $p(\psi)$ does not change "continuously from zero to infinity."

III. We have demonstrated in Theorem 1 of Section VIII that the moduli space $\mathcal{S}(\mathbb{R}^3)$ of all non-isotopic vortex knots for the fluid flow $\mathbf{V}_{f_2}(\mathbf{x})$ (8.1) is naturally isomorphic to the set of all rational numbers p/q in the interval $J_1:\ 0.25 < \tau < \tilde{M}_1 \approx 0.5847$. In Proposition 5 we proved that torus knots $K_{p,q}$ with $0.25 < p/q < 0.5$ are realized on infinitely many invariant tori $\mathbb{T}^2_H \subset \mathcal{D}_{k.\pm} \times S^1$ for $k \geq 2$, while torus knots with $0.5 < p/q < \tilde{M}_1$ are realized only on finitely many tori.

We have shown in Section IX that the axisymmetric fluid flows $\mathbf{V}_{2m}(x,y,z) = \mathbf{V}_{zG_2}(\lambda_m x, \lambda_m y, \lambda_m z)$ are solutions to the boundary eigenvalue problem for the curl operator on a ball $\mathbb{B}^3_a$ of radius a, provided that $\lambda_m = a^{-1}R_m$ and $\tan R_m = 3R_m/(3-R_m^2)$. We have proved in Theorem 2 that the corresponding moduli space $\mathcal{S}_m(\mathbb{B}^3_a)$ of vortex knots is the set of all rational numbers in the interval $I_m:\ (R_m - R_{m-1})/(4\pi) < \tau < \tilde{M}_1 \approx 0.5847$, where R_k is the kth positive root of equation $\tan R = 3R/(3-R^2)$ (2.25). Therefore the moduli spaces $\mathcal{S}_m(\mathbb{B}^3_a)$ do not depend on the radius a of the ball $\mathbb{B}^3_a$, all spaces $\mathcal{S}_m(\mathbb{B}^3_a)$ are different (for different m's) and tend to the moduli space $\mathcal{S}(\mathbb{R}^3)$ when $m \longrightarrow \infty$ because $(R_m - R_{m-1})/(4\pi) \longrightarrow 0.25$.

In view of the equivalence of the steady Equations (1.5) and (1.6) as well as Equations (1.5) and (1.4) the above results are equally applicable to the moduli spaces of magnetic knots formed by the magnetic field lines for the solutions $\mathbf{B}_{\hat{f}}(\mathbf{x})$ and $\mathbf{B}_{f_2}(\mathbf{x})$ (10.1)–(10.3) to the plasma equilibrium Equations (1.5) with pressure $\tilde{p} = \text{const}$ and for the more general MHD equilibria (1.4) with $\mathbf{V}(\mathbf{x}) = \alpha\mathbf{B}(\mathbf{x})$.

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[40] The safety factor $\mathbf{q}(P)$ is connected to the pitch $p(P)$ of the helical magnetic force lines on the torus P by the relation $p(P) = 2\pi\mathbf{q}(P)$. For vortex helical lines, we use both of these notions.