Enrichment Mathematics for Grades Seven and Eight
(Part II - Geometry)

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Preface to the First Edition

This is the second in a two volume manual for enrichment mathematics classes at the senior elementary level. The enrichment program is built around a two year cycle, so that students can participate in it through grades 7 and 8 without repeating material. Each of the two volumes corresponds to one year in the program. The order in which they are taken up is optional. However, within each volume the material should generally be covered in the order in which it appears. There should be enough material in each volume to fill a program that runs from the beginning of October to the end of May, if the enrichment class meets once a week.

I think of the subject of mathematics as roughly divided into two domains: discrete mathematics and continuous mathematics. Of course, as a description of the field as it is explored by professional mathematicians, this is a very inadequate division, and there are many other criteria by which to classify the different parts of the discipline. Nevertheless, I think it is fair to say that the kind of intuition involved in the understanding of mathematics is of two sorts. On one side we have the intuition involved in the understanding of discrete objects such as whole numbers and finite or even countable collections. On the other we have the intuition involved in the understanding of continuous objects such as lines, planes, and surfaces, and infinite (especially non-countable) collections such as the collection of all real numbers. The first of these is associated with arithmetic, while the second is associated with geometry.

In the first volume I have collected topics that fall under the heading of arithmetic. There you find units on patterned sequences, prime numbers and prime factors, rational and irrational numbers, modular arithmetic, counting large sets, and probability. This second volume focuses on geometry. It opens with a chapter on the Theorem of Pythagoras. From there is moves to a study of angles in various geometric objects, area and volume, geometric patterns, and topology. As in the first volume, there are lessons that include elementary discussions of conjectures and proofs.

One of the current realities in elementary and secondary mathematical education of students is that geometry receives short shrift. For one reason or another, teachers tend to leave units on geometry to the end, partly because in textbooks and in education department syllabi, geometry is included at or near the end. By the time they get to it, both the students and the teachers are tired. Furthermore, too much of it consists of “naming of parts”, and not enough centers on genuine problems. I do not know all the reasons why geometry has suffered this neglect. I do know that a lot of it has to do with the ascendancy of the ”new mathematics” in the late sixties, during which both geometric intuition and logical reasoning were wiped from the high
school curriculum. To this day, the curriculum has not recovered from the loss. Clearly a one year long focus on geometry for one hour a week is not going to correct this. More needs to be done, especially at the high school level. However, senior elementary students are at an age when geometric intuition should begin to be challenged and disciplined. I hope this course will be a helpful step in that direction.

I have tried to write the material so that it can be used by a teacher or by a parent. All that is required of the person directing the class is a curiosity about mathematics, combined with some confidence. The instructor should not feel that she has to be an expert in mathematics before attempting to direct a version of this enrichment program. The material in the course can be (re-) learned as it is presented. Nothing is more infectious than the enthusiasm of an instructor who communicates material that she recently learned herself. None of the material is difficult, and in every case I have tried to explain it as simply as possible. The chapters are meant to be used by the instructor rather than by the student. The instructor should think of the books as sources of suggestions on what material to cover each time, what to aim for, and how to construct the lesson.

In my own implementation of the program I try to create an atmosphere in which the students will feel that the activity is a break from their regular classes. I want them to think of the enrichment classes as “fun”, at least relatively speaking. For these reasons I give them no homework, no tests, and no marks. I do arrange for the students to participate in (at least) one mathematics contest, and usually have them practice on one or two old tests by way of preparation.


In this 2007 edition, the format has been changed considerably to make it easier to use by university students taking Math 010. The authors acknowledge the support of the PromoScience program of the National Science and Engineering Research Council of Canada, as well as Queen’s University’s Centre for Teaching and Learning, in funding the creation of this new edition.

About This Manual

The new format of this manual was designed to better structure the content of the enrichment program, which will be the focus of discussion in the university lectures and the basis of the enrichment sessions that university students will then conduct. It is important to note that the fundamental mathematical content and problems of previous editions have not changed. The manual is divided into six chapters:

1. The Theorem of Pythagoras
2. Applications of Pythagoras’ Theorem
3. Similarity
4. Plato and Polyhedra
5. Perimeter, Area, and Volume: Linear Shapes
6. Perimeter and Area: Circles
Within each chapter, the content is organized into a series of three to five **Enrichment Activities**, each of which has its own **Lesson Goals, Materials, Background** (if applicable), **Problem Statement, Lesson Sequence**, and **Extensions/Modifications** (if applicable) clearly stated. The new layout of the Enrichment Activities should help university students (or tutors) (1) locate a given activity quickly and (2) better prepare for their own enrichment sessions.

In addition to the Enrichment Activities, some chapters contain “Problem Sets”. In general, these are meant to supplement the Enrichment Activities by giving students practice doing certain important calculations. The university student (or tutor) has the choice of using a Problem Set to fill an entire enrichment session or to spread the problems over several weeks as “warm-up” exercises.

The Appendix has been revised to reflect changes in the manual. In particular, some overhead transparency and handout templates have been added. Also, Appendix B, which previously contained puzzles, has been removed because such puzzles are now available on the course website, and an Appendix containing **Spot Questions** has been added. Details concerning Spot Questions are provided in the next section.

In terms of preparing for university lectures, it is better not to read ahead in the manual so that you (the university student) can get a sense of the initial responses and challenges that students might have when first exposed to a concept or problem. After the material has been discussed in the lectures, you should read the manual carefully to help fill in any gaps and to prepare for your own presentation of the activity.

**Structuring an Enrichment Session**

In general, a good enrichment session

1. engages students from the beginning

2. consists of challenging and fun mathematical problems

3. ends with something that all students can feel good about.

**Beginning the Class**

A popular way to engage students from the beginning is through the use of **good games and/or puzzles**. Such games and puzzles should not only be fun, but also focus the attention of the students on the rich mathematics that is to follow in the lesson. Much effort has been devoted to finding games and puzzles appropriate for this purpose, which are now available on the StepAhead website. For the most part, the games and puzzles are organized according to the chapters and enrichment activities within this manual, which should help university students (or instructors) find relevant puzzles and/or games quickly.

Another effective way to begin a class is to use a **short problem** that either introduces the topic of the current session or reviews a concept from the previous session. These questions should be done individually by students and should not take more than 3-5 minutes. You will find that having something for students to work on when they enter the classroom naturally focuses their attention. It also helps you as the tutor to assess student understanding in an informal way. In the 2006 version of Math 010, these opening problems were called **Spot Questions**.
These questions are included in Appendix C and are also referenced in the Materials and Lesson Sequence sections of relevant Enrichment Activities.

The Main Lesson
The bulk of the “challenging and fun mathematical problems” are set out in the Enrichment Activities in this manual. These activities will be the focus of your enrichment sessions. The activities are presented in a well-planned sequence and often contain modifications and/or extensions to reflect the differing abilities among classes, or even among students within a class. Some useful lesson materials, such as templates for overhead transparencies and handouts, can be found in the Appendices and are clearly referenced in the Materials section of each activity.

In terms of timing, most of the enrichment activities are designed for a single session. However, depending on the depth of discussion, the number of tangents taken, and the abilities/interest of the class, the activities may take more or less time. For example, if your opening activity takes 15 minutes, then you will most likely not finish the main lesson, or will end up rushing to finish, which is not good. This brings us to the last part of the lesson.

Ending the Class
In terms of ending the session, it is important that you do not end the lesson with something difficult that is likely to frustrate some students. If there is difficult material in a lesson, try to schedule it around the middle of the session. Do not start something that will require much thought with only 5-10 minutes remaining. It is better to continue the activity at the next session and have students summarize one or two things that they have learned thus far in the activity. A short game that reviews something relatively simple can also serve as a nice ending to the session.

Of course, you should also keep in mind that there are so many variables that determine the dynamics of a lesson that not every lesson is going to go well, and certainly not every lesson is going to go exactly as you planned. The important thing is that you learn from each class and make modifications as necessary.
Ideas for the First Enrichment Session

Here is an outline of the types of things you should talk about in the first enrichment session.

1. You should begin the first session with a general discussion of mathematics and an outline of the enrichment program. Of course, you should introduce yourself to the students if they do not know you already, and write down their names so that you can get to know them. You may want to bring along an ice-breaking game to help them to be comfortable with you. Ask the students what they do or do not like about mathematics. Depending on the way in which the group has been selected, you should be prepared to hear at least some of the students say that they do not like mathematics at all. This is especially the case when the members of the group have been selected by the teacher for their performance in the subject rather than for their interest in it. Do not take this as a discouraging sign that you are off to a difficult start. In fact it is an excellent opportunity to tell the students that a lot of mathematics is not at all like the questions they are asked to do in mathematics class, and that along with the more familiar, the enrichment program involves types of questions that they would not have thought of as belonging to the subject. Certainly there will be enough new and surprising material in the program to give each student a fresh opportunity to come to like the subject.

2. To give them a flavor of the breadth of the subject, there are a number of non-standard mathematical objects you can point out to them. Since this program is focused on geometry, you should try to think especially of those that have an element of geometry in them. There are many web sites that have beautiful visual illustrations of mathematical objects. You can find many of them by doing a web search under the headings "fractal", "pattern", "tessellation", Escher", "tiling", and "symmetry". If you have pictures of fractals, you can bring these along. You can also bring along a book of Escher drawings if you have one. I have on occasion worn my Escher tie. You want the students to recognize that a lot of mathematics is very geometrical, and that it does not need to be dull.

Here is a list of good web sites available at the time of this writing:
3. Describe for the students the way you plan to run the class. Tell them that the purpose of the program is to have fun and to engage in some challenging mathematics. There will be very little pressure to “learn stuff”. In fact the learning will come more or less automatically. I suggest you tell the students that there will be no tests and no homework. Remind them that they are not there to compete. There is absolutely no virtue in getting something done in a hurry, or to finish ahead of your classmates. In fact, one of the most important aspects of learning mathematics is to learn to discuss it, to explain it, to communicate it. A lot of learning takes place in discussions between students. One of your main tasks is to encourage that type of activity in the classroom, and to keep it focused on the topic at hand. If the material is interesting it should be savoured. There are many different ways to do mathematics well. While undoubtedly the ability to do complex calculations effectively has value, there is at least as much value in the ability to turn a problem on its head and see it from a totally new and revealing point of view. Discussions between students are often the best way to tease out students’ understanding. They learn a lot from each other. Discussions between you and students are also very good of course, but you have to be very careful not to create a dynamic in which students simply look to you as the source of solutions. Your goal, as the teacher, is to develop the students’ minds, their love of problem solving, their geometric intuition, but not their reflexes.

4. One further comment should be made before we begin. This is a geometry class. As much as possible, give your own spatial intuition free rein. The world of geometry is a world of the imagination. Draw lots of diagrams; make use of pictures and physical objects; use gesture; use movement. Encourage students to do the same. Geometry is above all embodied mathematics.

You are now ready to start planning the first enrichment activity. Below is an activity you may wish to use.

**Introductory Activity - Cubes and Spheres**

1. Bring models of a cube and a sphere to the class. You should begin with a discussion of the meanings of the words “ball” and “sphere” in mathematics. A sphere is a perfectly round shape whose interior is empty, while a ball is one whose interior is filled in. One way to put it is “a sphere is the surface of a ball”. This is, of course, a little different from the way we use the word “ball” in every-day English, for many of our balls have nothing (other than air) in them. For a mathematician, however, when she is talking mathematics, a soccer ball is really a sphere, while a baseball is a ball. To make the distinction clear to the students, it is a good idea to introduce the concept of dimension. None of the definitions you give need to be rigorous or precise. In fact, until you are well
into the body of a mathematical topic, precision is impossible. One-dimensional objects are those that have no thickness, but only length. In the real world such objects do not exist, but things like lines, pencils and clotheslines are approximations, and can therefore be thought of as essentially one-dimensional. Examples of two-dimensional objects are things like blackboards, table tops, mixing bowls and window panes. Blocks of cheddar cheese, glasses of water, and bricks are examples of three-dimensional objects. One way to describe the difference between a ball and a sphere is that the former is a three-dimensional object, while the latter is a two-dimensional object.

2. If you want to do more to catch the students’ attention, a good device is the (two-dimensional) surface called the Möbius strip. The Möbius strip is constructed by taking a long and narrow strip of paper, giving it half a twist (180 degrees) and then taping the two ends together. Bring several pre-made to class, and make one additional one in front of the class. Give one of the Möbius strips to one of the students, and ask him or her to colour one side red and the other green. The fact is that it cannot be done - the half twist ensures that if you go around the length of the Möbius strip, you end up on the back of the location you started from. The Möbius strip is a one-sided surface! As a second step in this demonstration, you can cut the Möbius strip lengthwise, once around its entire length. The students will be surprised to find that you end up, not with two loops, but with one loop of double length.
Chapter 1

The Theorem of Pythagoras

This poetry, I never know what I’m going to say.
I don’t plan it.
When I’m outside the saying of it,
I get very quiet and rarely speak at all.

Rumi, 1207-1248,
“The Essential Rumi, Translations by Coleman Barks”

Purpose of the Chapter

This chapter serves three main goals:

1. To introduce students to the Theorem of Pythagoras and give them a good understanding of what it says and how it is used.

2. To develop the students’ ability to visualize geometric relationships, especially in three dimensions

3. To give students a first exposure to the nature and role of proof in mathematics. In particular, we will discuss one of the many proofs of the Theorem of Pythagoras. I do believe that it is important to begin to include proofs at this stage, for students at the elementary school level are far too ready to accept anything stated by a teacher or a book, without critically examining the justification for the claim.

4. A subsidiary goal is to give students a better understanding of squares and square roots, both of which come up repeatedly in problems involving the Theorem of Pythagoras.

Overview of Activities

• E.A. 1.1 - Introducing the Theorem of Pythagoras
  A problem in three-dimensional geometry that sets the stage for the introduction of the Theorem of Pythagoras
• E.A. 1.2 - Investigating the Theorem
   An exercise in which students gather evidence for the theorem in order to help them
   understand what it says

• E.A. 1.3 - Using the Theorem
   A session that begins with practice exercises to ensure that students are comfortable using
   the theorem, and then uses the theorem to solve the problem posed in E.A. 1.1

• E.A. 1.4 - Proving the Theorem
   An activity devoted to the geometric proof of the theorem and a discussion of the need
   for such a proof

Some of the sections/activities contain far more material than can be covered in a single
lesson. You will have to judge whether to spread it over two sessions, or to omit some of the
more difficult material. Students vary in degree of preparation. One of the most important
tasks of a teacher is to gauge the readiness of the students for the material you present to them.
1.1 Enrichment Activity - Introducing the Theorem of Pythagoras

Lesson Goals

- To introduce the Theorem of Pythagoras via a fun problem
- To develop a facility for visualization in three-dimensional space

Materials

- concrete models of spheres and cubes (as described in item 2.)
- small light bulb (if planning to do the Spin-off activity in item 1.)

Problem Statement

Propose the following problem to the students: Harry Potter, after winning the quiddich championship match for his house (Griffindor), is given the golden snitch as a special award for being the game’s most valuable player. Since Hermione’s birthday is coming up, he wants to give the snitch to her as a birthday present. To present the gift in a nice way (it is a special birthday), he wants to have a glass cube built tightly around the ball so it does not roll around inside the cube, and then he wants to have a glass sphere constructed around the cube so that the cube cannot move inside the outer sphere. In the language of mathematics we would say that the snitch is inscribed in the glass cube, and that the cube is inscribed in the glass sphere. It means of course that the corners of the cube touch the inside of the sphere, and that the surface of the snitch is in contact with each of the six faces of the cube. Given that the snitch is a ball of diameter 6 centimeters, what will the diameter of the outer sphere be equal to?

Lesson Sequence

You could spark the discussion by raising some or all of the following questions:

1. How do you measure the size of a sphere or a ball? Of course, the measurement we are looking for is called the diameter, and to measure a diameter we could use a pair of calipers, or some equivalent instrument. If you do not have calipers, you can instead place the sphere (the ball) on a flat table, and carefully place two books or boxes on opposite sides of the ball. You can then find the diameter by measuring the distance between the books.

Spin-off Activity - Shadows

Someone may suggest that you obtain the measurement by measuring the shadow cast by the ball on the ground. You may want to raise the possibility yourself. If this option is raised, you should take some time to discuss the shapes the shadow of a ball can have. The shadow of a ball is not always in the shape of a circle! What other shape can it take, and what measurement of the shadow will give you the diameter of the ball? You could investigate the problem by bringing a light bulb (preferably small) to class. This question will take you on a tangent, and
The Theorem of Pythagoras

will not solve the problem, but it is nevertheless an excellent exploration of spatial relations. The bulb should not have a reflector behind it, so a flashlight will not work well. You want the bulb to be small because if it is large, the shadows produced will be ill-defined and difficult to measure. Most often, if you hold a ball in front of the bulb and let its shadow fall on the floor or a wall, you will get an oval shape that is referred to more precisely as an “ellipse”. If the shadow is produced by a ceiling light, or by a bulb you brought along, you haven’t a hope of getting a reliable estimate of the diameter of the ball by measuring the shadow, for the light rays are then not parallel, so the size of the shadow will always exceed that of the ball. Besides, the size of the shadow will depend on the distances between the ball, the floor and the lamp, as can be seen in the following picture.

![Diagram of light rays and shadows](image)

**Figure 1.1:** The shadow cast by a ball when the light source is a light bulb

How will the experiment change if you take the ball outside on a sunny day, and measure its shadow? If the sun is not directly overhead, you can measure the smaller diameter of the resulting ellipse. It is a good idea to discuss the difference between light coming from a ceiling lamp and light coming from the sun, and to discuss why it is that the smaller diameter of the shadow produced by sunlight is the same as the diameter of the ball. The reason is that the sun is so far away from us that for all practical purposes its rays are parallel. You can demonstrate this relationship between the light rays, the ball, and the sidewalk by getting the students to wrap a sheet of paper into a cylinder around a small ball, and then using scissors to cut the bottom of the cylinder at an angle. The students should then imagine the sun rays all parallel to the sides of the paper cylinder. Some of the rays will miss the ball, producing light on the pavement, while others run into the ball, producing the shadow. The most interesting sun rays
are those that just graze the ball, for they describe the outline of the shadow. You should be able to see on the paper model that the diameter of the ball is the same as the cross-sectional diameter of the cylinder, and that the diameter of the ball is the same as the smallest diameter of the ellipse created when the paper cylinder was cut at an angle, to represent the effect of sunlight coming in at an angle.

Figure 1.2: The shadow cast by a ball when the light source is the sun

You may find that after getting into this tangential discussion about using shadows to measure the diameter of a ball, you have used up all your lesson time. Do not take this as a failure. In fact, it probably means that the class became engaged in thinking about the way light rays run into objects and create shadows. Your first lesson has been an excellent success, for your students have exercised, and probably sharpened their three-dimensional geometric intuition. That is one of your main goals! It does mean, however, that the following material will be the substance of your second class rather than the first. Whatever you do not finish in the first lesson becomes part of the next class.

2. Getting back to the problem of Hermione’s snitch, what measurement of the cube is equal to the diameter of the snitch (that is, the inner, or “inscribed” sphere)? To get students to answer this question you could preface it by asking where the inner sphere will touch the cube. It would be very useful to have a cube with you on which you can clearly identify midpoints of opposite sides. Ideal would be a box in the shape of a cube (or very close to it), so that you can demonstrate by opening the box and looking inside or even placing a ball inside. Once it has been established that the inner sphere will touch the
cube at midpoints of opposite faces, you should then discuss why the distance between these midpoints is the same as the length of a side (or edge) of the cube. During this discussion you could take several detours. For example, it would be good to ask students how many pairs of opposite faces a cube has.

3. What measurement of the cube is equal to the diameter of the outer ("circumscribed") sphere? Again, it will help to have a cube-shaped box with you to illustrate this, on which students can identify diagonally opposite points. This time they have to imagine a sphere around the box in order to determine where the sphere makes contact with the cube. These points will of course be vertices (the plural of "verte") of the cube. If you open the box you can invite students to imagine a line that goes through the interior of the box and joins a pair of diagonally opposite vertices. We could call such a line a diagonal of the cube. How many diagonals are there? Make sure the students see that the diagonal of the cube has the same length as the diagonal of the outer sphere.

4. You are now ready to discover with the students that the original problem asked in this lesson comes down to the following question about a cube: **Given that the length of the side of a cube is 6 cm, what is the length of its diagonal?** Say to the students that in the coming few weeks we will learn how to do that calculation.

5. To finish the day’s lesson you should reflect on a two-dimensional problem that is analogous to the problem of the spheres and the cube: **Given a circle inscribed in a square inscribed in a circle, if the diameter of the inner circle is 6 cm, what is the diameter of the outer circle?** In fact you could challenge the students to formulate without your help what problem in two dimensions is analogous to the three dimensional problem being considered. The next few lessons will teach us to solve this simpler problem as well.

![Figure 1.3: The two-dimensional version of the sphere problem](image-url)
6. At the end of the lesson you could point out to the students that the procedure followed here is very typical of problem solving techniques: You reflect on the problem, and you reduce it to something simpler. If the resulting question still looks difficult, you look for an analogous problem that looks simpler, and which you think you may be able to tackle. The hope is always that eventually you will be able to return to the original question.

Possible Outcomes and Modifications

1. If the problem given in this chapter is too easy for your students (this is not likely), try reversing the information: Given the diameter of the outer sphere, what is the diameter of the inner sphere? This makes the problem considerably more difficult. Of course the Harry Potter story has to be set aside if you are doing this version of the problem.

2. It may seem that we have not made much progress solving the problem. We took a rather long detour to reflect on the shapes of various geometric objects, and then veered away from the original question to propose a two-dimensional replacement. If you can get the class engaged in this kind of extended reflection, you should count it as a significant success. Not all classes can do this. Often students are inclined or conditioned to look for a quick solution to each problem posed, and give up as quickly if they do not see it. You will have to judge, as you go along, the extent your class is ready to engage reflectively. If it does not happen immediately, look for opportunities as the course develops.

3. It may happen that some of the students will immediately see the solution to the two-dimensional version of the problem. If that happens, you should try to get those students to share their insight with the rest of the class. It will also mean that the beginning of the next lesson will become redundant, though it will certainly be appropriate to ask some of the students (not those who produced the solution initially), at the start of the next session, to remind you of the way the solution was obtained.
1.2 Enrichment Activity - Investigating the Theorem

Lesson Goals

- To investigate some of the evidence for the Theorem of Pythagoras
- To understand each of the variables in the theorem \((a, b, \text{ and } c)\) and how they are related
- To review students’ understanding of squares and square roots

Materials

- grid paper (for drawing triangles) for each student

Lesson Sequence

1. This lesson continues the discussion of the problem presented in the preceding lesson, so you should begin by repeating that problem, as well as its two-dimensional analogue. In fact, the best thing to do is to ask the students what they remember of it.

Once the problem is clear in their minds, you should go to the two dimensional version of the question, and ask them if they see any way to solve it. If no suggestions are forthcoming, draw the construction line shown in Figure 1.4.

![Figure 1.4: Two-dimensional problem with construction line](image)

Ask them whether the triangle in this figure has any particular features. For example, the triangle is right angled and isosceles. It may be necessary to review these concepts with the students. Introduce, or review as the case may be, the terms “right angle side” and “hypotenuse”.

Ask the students whether they know the lengths of the right-angle sides of the triangle in the picture. They are, of course, both equal to the length of the sides of the square (6 cm). To solve the problem, we have to calculate the length of the hypotenuse, for it is equal to the diameter of the outer circle. Review the picture with the students until it is clear to them that we will be able to solve the problem if we have a method for finding the length of the hypotenuse of a right-angled triangle, given the length of the other two sides. Ask the students whether they can think of a way of resolving this. It may be that students will remember learning the Theorem of Pythagoras, and see its relevance for this question. In that case, and especially if the whole class seems to know this theorem well, you may choose to skip this next activity.

2. An important skill in mathematics, not often identified in books, is the ability to judge whether in a particular situation involving several quantities or measurements there should be a formula to connect the measurements. There are two aspects to this judgment. In the first place, it involves determining whether, geometrically, one of the measurements is determined when the others are given; in other words, if you know all but one of the values, is there any freedom left in choosing the remaining one? In the second place, the judgment is dependent on the realization that when one quantity is determined by another, or by several others, then in principle there ought to be a way to express that determination mathematically, by means of a formula.

To develop this kind of judgment, and to apply it in a discussion of the Theorem of Pythagoras, you could bring small sticks, straws, or wooden skewers of various lengths and hold two of them up to form two sides of a triangle. Let students see that the length of the third side (the distance between the free ends of the skewers) can be given a variety of values, simply by adjusting the angle between the skewers. However, if we are told we must make that angle a right angle, it becomes physically clear that the length of the third side is uniquely determined by the other two. Point out to students that whenever there is a situation of this type, they can expect there to be a formula of some sort that will express that relationship. In this instance the formula we hope for is that given by the Theorem of Pythagoras.

3. The Theorem of Pythagoras is a very important and very clever result. Even though the proof may in fact be even older than Pythagoras, you can liven up your class by telling a little about Pythagoras. It is easy to find historical information and maps on the internet to prepare for this.

Investigating the Evidence

4. If the students remember the theorem only by name, or have only a vague memory of what the theorem claims, or do not remember studying the theorem at all, someone may suggest that we can find the length of the hypotenuse by making a careful drawing of the triangle. To do this, it is best to have squared paper on hand. One can then mark off distances of 6 cm along horizontal and vertical lines to create two sides of a right-angled triangle, and carefully measure the distance between the resulting endpoints. Ask the students to try the same thing with triangles that are larger or smaller than this one. In fact, ask the students to set up a table as follows:
The Theorem of Pythagoras

Table 1.1: Gathering Evidence for Pythagoras’ Theorem

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$a^2$</th>
<th>$b^2$</th>
<th>$c^2$</th>
<th>$a^2 + b^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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</tbody>
</table>

where $a$ and $b$ are the lengths of the right angle sides and $c$ is the length of the hypotenuse. It may be necessary to explain to some students what $a^2$ means. Encourage students to try a variety of right angled triangles, not only those that are isosceles. Help them to observe that the last two columns are equal; that is, $c^2 = a^2 + b^2$. Tell the students that in fact this is always the case and that this result is called “Pythagoras’ Theorem”.

At this point it is important to acknowledge that in the experiment the equality between the last two columns is only approximate. Ask them why they think the two did not come out exactly equal. Of course students will realize that their drawings and measurements are rather inaccurate. It is important, though, to dwell on this issue for a while. In a future lesson we will present a proof of the Theorem of Pythagoras. Since the students will have very little or no idea of what proofs are and why we should bother with them, it is important to dwell on the power of a logical argument. Ask the students how they could, even when taking the greatest care, be totally sure that $a^2 + b^2 = c^2$? There is not really any way to establish this by measurement alone. The best we could hope for is a conviction that $a^2 + b^2$ is approximately equal to $c^2$, say to an accuracy of two decimal digits.

One way to drive this home would be to suggest to the students that the Theorem of Pythagoras should say that $a^2 + b^2 = \gamma \times c^2$, where $\gamma$ is a number very close to 1, namely $\gamma = 1.00031459...$. This makes the formula look a bit more complicated than the ‘real’ Theorem of Pythagoras, but the decision which of the two versions of the theorem is correct cannot be made on the basis of the experiment. For that we need a proof.

Aside: There is a parallel with the problem of establishing that the formula for the circumference of a circle is $\pi \times d$, where $d$ is the diameter, rather than $3 \times d$. We are so fond of round numbers that if we did some experiments in which we measured the circumferences and diameters of several circles, and divided the results, we would surely conclude that the ratio of the circumference to the diameter is 3. In fact this is exactly, apparently, what happened in the time of King Solomon, for in the Bible in 2 Chronicles 4:2 we find a description of the furnishings of the temple of Solomon, part of which reads as follows:

He made the sea of cast metal, circular in shape, measuring ten cubits from rim to rim and five cubits high. It took a line of thirty cubits to measure around it.

Here the diameter is 10 and the circumference is 30 cubits. Who can blame Solomon or the royal mathematician who did the calculation for him? Why should anyone think, solely on the basis of experimental evidence that the correct ratio between circumference and diameter is 3.14159...?
That this ratio is really $\pi$ and not 3 follows from sophisticated logical reasoning. We will discuss this more fully in Chapter 3. The fact that $a^2 + b^2$ is exactly equal to $c^2$ also follows from a logical argument. However, this logical argument, as we will see in a later activity. There is of course an even more important reason for going through the exercise of drawing and measuring-right angled triangles. Doing this exercise, and tabulating the results as discussed above, helps students get a clear picture of what the Theorem of Pythagoras claims - it helps them understand what it says.

5. Once you feel you have sufficiently explained the need for a proof, tell the students that for the moment you are simply going to accept the result, and discuss with them how it can be used. Do some examples with them on the board, giving the lengths two of the sides of a right angled triangle, and asking them to calculate the length of the third side. To do these problems they will have to become familiar with the idea of the square root, so this may have to be reviewed with them as well. Begin by defining what you mean by the square root of a number $n$:

The square root of a number $n$ is that number $m$ for which $m \times m = n$.

One of my students referred to taking the square root as “unsquaring”. I told her that that was an excellent, if somewhat unconventional, name for it. To help students understand the idea, do some very simple examples with them first. For example, ask them what the square roots of 4, 25, 64 are. Then ask them to find the square root of 7 by hand. It is very important that they try it by hand first, before they start doing square roots on the calculators, so that they will know what they are getting when the calculator does it for them. There is a classical algorithm for calculating the square root by hand. However, this should not be used here. Instead, the calculation should be by trial and error. In other words, the discussion should go like this: $2 \times 2$ is less than 7 and $3 \times 3$ is more than 7, so the square root of 7 is somewhere between 2 and 3. After some experimentation the students will discover that $2.6 \times 2.6$ is less than 7, while $2.7 \times 2.7$ is more. This tells us that the square root of 7 is somewhere between 2.6 and 2.7. In principle we can continue this way and obtain as many decimal digit of $\sqrt{7}$ as we like. In practice, the calculations become more and more cumbersome as we increase the number of digits, so eventually you should point out to the students that on their calculators they can do the calculation quite easily, and find that $\sqrt{7} = 2.6457513 \cdots$.

Ask the students to type a number into their calculators, then push the square root button, and then the $x^2$ button. They should become thoroughly familiar with the fact that this will always return the number they started with. You may have to discuss with them the possibility that on some calculators the final answer will be close but not exactly equal to the original. The reason is that the square root, which is typically not a terminating decimal expression, had to be rounded off to fit the size of the calculator’s register. This means, that the calculator did not square the exact square root, but a number close to it.

You should have the students try this calculator experiment several times, and make sure that they try the experiment not only on whole numbers, but also include some decimal expressions.

Following this discussion you should do some exercises at the board involving square roots. Here are some examples:
• \( \sqrt{36} =? \) (they should be able to do this without the calculator)
• \( 5^2 =? \) (they should be able to do this without the calculator)
• \( \sqrt{5} =? \) (let them use their calculators for this)
• \( (\sqrt{5})^2 =? \) (they should be able to do this without the calculator)
• \( \sqrt{3^2} =? \) (they should be able to do this without a calculator)
• \( \left(\sqrt{\frac{2}{3}}\right)^2 =? \) (they should be able to do this without the calculator!)

6. End the lesson by doing enough problems for the students to see clearly that if the lengths of any two sides of a right angled triangle are given, then the length of the third side is determined by them. Here are some suggestions. They are presented here in the form of a table, but you can present them, or at least some of them, as pictures on the board.

<table>
<thead>
<tr>
<th>side</th>
<th>side</th>
<th>hypotenuse</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
<td>?</td>
</tr>
<tr>
<td>12</td>
<td>5</td>
<td>?</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>?</td>
</tr>
<tr>
<td>6</td>
<td>?</td>
<td>10</td>
</tr>
<tr>
<td>?</td>
<td>10</td>
<td>26</td>
</tr>
<tr>
<td>5</td>
<td>?</td>
<td>7</td>
</tr>
</tbody>
</table>

Table 1.2: Practice using Pythagoras’ Theorem
1.3 Enrichment Activity - Using the Theorem

After introducing and investigating the Theorem of Pythagoras, you have the choice of moving directly to the proof of the theorem (E.A. 1.4) or solving the original problem presented in E.A. 1.1, which is part of the present activity. Some students may be eager to solve the original problem before moving onto the proof and so you must use your discretion to decide which sequence will work better with your class.

Lesson Goals

- To consolidate the students’ understanding of the Theorem of Pythagoras, doing some further exercises to make sure that the students see how it is used
- To solve the problem of the circle inscribed in the square, which is in turn inscribed in a circle
- To solve the original 3-dimensional problem of the sphere inscribed in the cube, which is in turn inscribed in a sphere

Materials

- copies of the quadrilateral template (Appendix A.1) for each student (if doing the exercise in item 1.)
- copies of Spot Question in Appendix C.1 for each student (optional)
- models of the sphere and cube as in E.A. 1.1
- triangular inserts (made of cardboard) for the cube (see item 4.)

Lesson Sequence

1. Start this lesson by reviewing the terms “right angled triangle”, “right angle side”, “hypotenuse”, and “isosceles triangle” as necessary.

   To review the claim of the Theorem of Pythagoras you could ask the students to do the following exercise at their desks: In Section ?? there is a template for the quadrilateral (four-sided figure) shown in Figure 1.5. In that quadrilateral there is just one right angle. Ask the students how they would go about determining which one it is, without using a protractor. The idea is that to check whether an angle is a right angle, we should measure the sides adjacent to the angle, square their lengths, and sum them. This number should then be compared to the square of the distance between the ends of those two sides.

   Then ask the students what Pythagoras’ Theorem tells us about right angled triangles. Ask the students how they know that the theorem is true. The best answer at this point is that they do not know, and that they are waiting for you to present a proof.
2. If you feel the students need more practice, you may wish to give them some. This may be done in the form of a **Spot Question**, followed by a worksheet or game, depending on how much practice is needed. A Spot Question involving squares and square roots can be found in Appendix C.1. Some sample examples using Pythagoras’ Theorem are giving in the table below. Draw the corresponding triangles on the board and have the students solve them at their desks. Take them up at the board after they have worked on them for a while.

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>4</td>
<td>?</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>?</td>
</tr>
<tr>
<td>?</td>
<td>(\sqrt{3})</td>
<td>(\sqrt{6})</td>
</tr>
<tr>
<td>1</td>
<td>?</td>
<td>13</td>
</tr>
<tr>
<td>2</td>
<td>?</td>
<td>(\sqrt{12})</td>
</tr>
<tr>
<td>?</td>
<td>?</td>
<td>6</td>
</tr>
<tr>
<td>?</td>
<td>?</td>
<td>8</td>
</tr>
<tr>
<td>?</td>
<td>?</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 1.3: More Practice Using Pythagoras’ Theorem

The last three of these are intended to be isosceles right angled triangles. That is, \(a = b\), while \(c\) has the value given in the table. You can indicate this on each of the triangles you draw on the board by putting identical hatch marks on both right angle sides, to indicate that they are equal. Make sure that the solution for these last three cases becomes clear to the students: First you calculate the square of the hypotenuse. Then observe that because the other sides are equal, so are their squares. Therefore if we divide the square of the hypotenuse by two we will get the square of (either one of) the right angle sides. For example, in the problem where the hypotenuse is equal to 6, so that its square is 36, we
conclude that the square of a right angle side is 18. That is, the length of the right angle side is $\sqrt{18}$. This square root can then be computed on the calculator.

3. At this point, you should be able to apply what has been learned to solve the problem of finding the diameter of the circle inscribed by a square inscribed by a circle of diameter 6 centimeters. You have already noted that in this problem the side of the square is equal to the diameter of the smaller circle (6 cm) and that the length of the diagonal of the square is equal to the diameter of the smaller circle. If we draw one of the diagonals of the square, then the triangle below the diagonal is an isosceles right angled triangle with each side equal to 6 cm. Thus, the length of the diagonal of the square (and the diameter of the larger circle) is $\sqrt{72}$.

**Extension:** If there is time and interest, you can make the problem more difficult by asking them to suppose that the diameter of the larger circle is known, say at 10 cm. Challenge them to calculate the diameter of the inner circle. This time, they will infer that the diagonal of the square is given, at 10 cm. This gives them the length of the hypotenuse of an isosceles right-angled triangle, whose right-angle sides are equal to the diameter of the smaller circle:

$$10^2 \div 2 = 50,$$

so the diameter of the smaller circle is equal to $\sqrt{50}$.

It is also possible to make the original three-dimensional problem more difficult by giving the diameter of the outer sphere (say 30 cm) and asking for the diameter of the inner sphere.

4. Depending on time, you may wish to solve the 3-dimensional problem while the 2-dimensional solution is fresh. Alternatively, this can be left for the next lesson (either before or after the proof of the theorem).

Begin by reminding the students that we had already determined that the diameter of the snitch is equal to the side of the cube. So the side of the cube is 6 centimeters. We had also discovered that the diameter of the outer sphere is equal to the length of the diagonal of the cube. In other words, we had reduced the problem to one of calculating the length of the diagonal of a cube, given that the side of the cube has a length of 6 centimeters.

To discuss this problem it will be of helpful to have a cube available, especially one that can be opened up. The solution hinges on the construction of a right angled triangle inside the cube, whose hypotenuse is the diagonal of the cube, and one of whose right angle sides is the diagonal of one of the faces of the cube. The cube and the triangle are shown in Figure 1.6.

Draw the cube on the board, and have several cardboard cubes with you for the students to examine. Make sure they understand what measurement is known and which we are trying to find. Ask them if they can think of any way to use Pythagoras’ Theorem to solve the problem. There is a good chance they will think of themselves of the shaded triangle in the picture. If it is clear they are not going to think of it, suggest it to them. It may help them to imagine cutting the cube as shown, and studying one half of it, as in Figure 1.7.

Draw it on the board, and have a cardboard triangle ready to place in one of the (opened) cubes, so that they see exactly how it fits. Ask them if they know the lengths of the
right-angle sides of the shaded triangle. One of them is 6 cm (the length of a side of the cube); the other can be found by applying the Theorem of Pythagoras to one of the triangles created on the base of the cube when the shaded triangle is placed in the cube. The bottom side of the shaded triangle is the hypotenuse of the triangle created on the base, and so has length equal to

$$\sqrt{6^2 + 6^2} = \sqrt{72}.$$  

To find the hypotenuse of the shaded triangle, we have to square this, giving us 72, and then add to that the square, 36, of the other right-angle side of the shaded triangle, $72 + 36 = 108$, and then take the square root: $\sqrt{108} = 10.39$. Thus the diameter of the outer sphere is going to be 10.39 centimeters!
1.4 Enrichment Activity - Proving the Theorem

There are many ways to prove this Theorem. We will present what is arguably the simplest proof, adjusted so that it can appear in the form of a discovery by the students themselves.

Lesson Goals

• To help students reflect and think about the role of proof in mathematics
• To expose students to a geometric proof of the Theorem of Pythagoras

Materials

• copies of Spot Question in Appendix C.2 for each student
• 4 pre-cut triangles and 1 square for each student or pair of students (see item 4.)
• overhead transparency of a square with pre-cut triangles to fit.
• (optionally) a globe (see item 2a. in the following)

Lesson Sequence

Reflections on the Nature of Proof

1. To open this lesson, give students a quick Spot Question to review their understanding of Pythagoras’ Theorem (see Appendix C.2). Then remind the students that we are still waiting for a logical explanation for the Theorem of Pythagoras; that is, a “proof”. This proof will be the focus of this session. Engage the class in a discussion of the nature of proof. Here are some ideas:

   2. A proof is a logical argument that provides certainty about a situation without having to resort to an experiment or to rely on direct observation. A proof derives its force from the understanding that to a very high degree the world behaves logically. To illustrate the fact that a logical argument is often more powerful than a measurement, you could try one of the following scenarios on the class:

      (a) Tell your students that it will always be Monday (or whatever day you are doing your class), and that you are going to explain why it has to be so. To do this you need a globe. Ask one of the students in your class to hold up the globe, and ask one of your students to stand near the globe and to play the part of the sun. Find the (approximate) location of your town on the globe, and turn the globe so that this location faces the sun. In other words, pretend it is noon where you are. Now place your finger on the globe at the point indicating your locality, and tell the students that your finger represents a plane that you plan to fly around the world, flying West (following the sun). Now ask the student holding the globe to start turning it slowly, making sure that the globe is turned so that (from the point of view of an inhabitant of the earth) the sun moves towards the West as it should. When the globe starts turning, move your finger slightly away from the globe (the plane is taking off), but
do not move your finger from a position between the earth and the sun. In other words, ask your students to notice that you are flying at precisely the speed that will allow you to go around the earth in 24 hours. This means you are flying a very fast plane; the circumference of the earth is about 40,000 km, so you would be traveling at about 1667 km per hour, a little more than 1000 miles per hour. Now tell students that you took off at noon on Monday (or whatever day you do the class on) and you are going to check periodically, say every 10 or 15 minutes with a station directly below you what day and what time it is. Of course (since you are going around the earth at the speed at which the earth is turning) the answer will always come back "it is (close to) noon", and obviously we cannot go from Monday to Tuesday in 10 minutes, so the other part of the answer will be "Monday". Thus, the same answer comes back every time you radio down, and eventually, 24 hours later, you will arrive back at your starting point, on Monday, at noon. Obviously there is something wrong with the argument. Some part of the logic must be wrong. Certainly there is no problem assuming a plane flying that fast. Some planes do. Nor is there any problem with such a plane flying so that the sun remains directly overhead throughout the flight. This leaves just one assertion that will have to be questioned unless we are happy to agree that days do not change: The sentence "obviously we cannot go from Monday to Tuesday in 10 minutes" has to be false. And so it is. Tell the students about the existence of the international date line. Chances are it is marked on your globe.

In effect, what this logical argument, this thought experiment, accomplishes is that it tells us that there must be an international date line, even if we have never heard of one! In fact, one of the authors came to this realization in precisely this way one summer Saturday when he was 13.

(b) Another illustration is based on a story sometimes used in Calculus, but perfectly understandable to a thoughtful grade 7 or 8 student. You can tell it like this:

"Suppose the police in towns A and B (pick a pair of cities near you) have their eyes on a known speed addict (not that kind of speed) who is often seen leaving A at midnight and arriving B at 12:30 am. When he leaves, the police department of town A follows him for a while, but always find him driving a respectable 95 km per hour, so they turn back to A. After all, it is midnight and even policemen like to sleep. A cruiser belonging to the B detachment picks up the speed addict’s car approaching B a little before 12:30, still going 95 km per hour. Of course, the policemen are frustrated that they did not catch him speeding this time, but they remain suspicious that he may have been speeding while they were not looking. If A and B are 80 km apart, do you think the man was speeding? Can you be sure? If the police did not see him, you certainly did not catch him in the act!"

(c) You may be able to think of an even better illustration to show that many of the things we know are known by a logical argument rather than a measurement, and that knowledge based on logic is usually more secure and always more precise than knowledge based on observation.

3. Before starting the presentation of the proof you should briefly discuss that just as in an
experiment it is very important to make precise measurements, so also in the presentation and discussion of a proof it is very important to be logically precise. However, in this case the care required takes the form of suspicion. Before we accept a proof we should ask ourselves whether the proof really does not include a logical flaw. There are many ways to illustrate the need for a skeptical attitude. Here is an example that can help, and that also fits into the geometric methods that will be used in the proof of the Theorem:

Tell the students that you are going to prove that the area of a rectangle of width 3 and height 8 is not 24, but is in fact infinite! Draw the rectangle on the board. Then cut off the top half and place it next to the lower half, creating a rectangle of width 6 and height 4. Now take the newer of these two rectangles and split it horizontally into upper and lower halves, and place the upper half next to the lower half, creating a stepped region of width 9. Keep doing this, each time splitting the newest rectangle and placing the top half of it next in the row. Ask the students to imagine that you have done this infinitely many times, creating an infinitely long figure. Since this figure is infinitely long it has an infinite area. Since area is not changed when you rearrange pieces, the original rectangle has infinite area.

Where is the mistake? You probably noticed that the mistake is in the second last line of this “proof” where it is assumed that if a region is infinitely long, its area is infinite. In fact, the argument should really be turned around: We know as part of our understanding of area that areas are preserved when pieces of a region are rearranged, and we know that the area of the original rectangle was 24. Therefore we must conclude that the area of the infinitely long figure we produced must also be 24. We have arrived at the remarkable fact that an infinite region can have a finite area, as long as it gets thin quickly enough as it gets longer and longer!

Presentation of the proof

4. As indicated earlier, the proof can be presented as a class discovery - a logical and visual experiment, as outlined next. To accomplish this you need an overhead projector, and 4 copies of the same right-angled triangle cut out of stiff paper. Assuming that the right-angle sides of the triangle are named $a$ and $b$, you should also have an overhead transparency with you on which you have drawn a square whose side is equal to the sum $a + b$ of the two right angle sides. You should also provide each student (or small group of students) with paper copies of the square and 4 triangles.

5. The first step in the proof is to get students to see how the quantities $a^2$, $b^2$, and $c^2$ can be associated with areas. This is a non-trivial point that is central to understanding the proof! You can be sure that, at this point, students think of $a^2$ as the outcome of a numerical calculation done “away” from the diagram of the right-angled triangle. In their minds it will almost certainly not connect to the geometry of the situation in any way. For this reason, you should ask the students whether, since the theorem mentions the quantities $a^2$, $b^2$, and $c^2$, there is anything we can add to the picture of the original triangle to make us think of those squares. If necessary, point out to the students that the word “squares” in the question gives a hint. Hopefully, you can guide the class to the
realization that the quantities $a^2$, $b^2$, and $c^2$ can be thought of as the areas of the squares in Figure 1.8.

6. Now place the transparency on the overhead projector. Using two of the triangles, demonstrate for the students that the square drawn on the transparency has sides equal to $a + b$, the sum of the right-angle sides of the triangle.

7. Once this is clear, invite students to come and arrange the four congruent triangles inside the square so that they do not overlap. (You may choose to have the students do this on their own first using the paper copies, and then come to the front to share their ideas.) There are many ways to do this, and some of them are not relevant to the proof of the Theorem of Pythagoras. Even so, draw students’ attention to the fact that for each arrangement, the area of the part of the square that remains blank must be the same, since in each case it is equal to the area of the original square minus the areas of the four triangles that have been placed inside it. Eventually, someone will produce some of the pictures shown in Figure 1.9, or pictures like it.

8. The important step in the process is to pay attention to the area of the space left blank. Keep reminding the students that in each case the area of the blank space is the same even though the shape is different. The reason is, of course, that in each case the area of the blank space is the area of the original square minus the total areas of the four triangles placed inside it. You should collect arrangements until you have one for which the area of the blank space is clearly equal to $c^2$ and at least one for which it is $a^2 + b^2$. The four arrangements shown are not the only ones for which this is the case. Notice that
1.4 Enrichment Activity - Proving the Theorem

in the fourth arrangement shown you need to add a line (the dotted line in the diagram) before it is obvious that the area of the blank space is equal to \( a^2 + b^2 \). You should copy the arrangements on the board as they come up. Once you have at least one example of each type, you are forced to conclude that, for a right angled triangle, \( a^2 + b^2 = c^2 \), provided \( c \) is the length of the hypotenuse, and \( a \) and \( b \) are the lengths of the right-angle sides.

9. Give enough of a summary at the end for students to realize that by comparing the areas of the blank regions in the two arrangements they have shown that the Theorem of Pythagoras is always true.
The Theorem of Pythagoras
Chapter 2

Applications of Pythagoras

Purpose of the Chapter

1. To bring students to the point where they are very comfortable with the Theorem of Pythagoras. They should not only know what the theorem says, but they should know how and when to use it.

2. To expose students to more difficult, multi-step problems involving Pythagoras.

3. To introduce students to the some fun “real life” applications of Pythagoras’ Theorem.

Overview of Activities

- Problem Set 2.1 - Getting Some Practice
  Several exercises devoted to gaining proficiency in performing the types of calculations that accompany the use of Pythagoras’ Theorem

- E.A. 2.2 - The Packaging Problem
  A “real life” application problem that requires Pythagoras’ Theorem to determine the dimensions of the smallest box to hold a given number of candy rolls

- E.A. 2.3 - The Spider and the Fly
  A fun application problem that involves both Pythagoras’ Theorem and the introduction (or review) of nets to calculate distances both on a surface and through 3-space

- E.A. 2.4 - The Crochet Needle in the Vacuum Tube
  Another fun “real life” problem that involves the use of Pythagoras’ Theorem

The application problems are particularly fun for students. Not only are the problems inherently interesting and engaging, but they allow students to see how Pythagoras’ Theorem is used in problems that at the outset, do not seem to involve right angle triangles. The Spider and the Fly problem has been particularly successful over the years. A video presentation of this session (in a university classroom) is available on the StepAhead website.
2.1 Problem Set - Getting Some Practice

Goals
The exercises in this problem set can be used in two ways: (1) all at once to fill an entire enrichment session or (2) in parts to begin several of the enrichment sessions that follow. It is your job to determine whether your class would benefit from an entire session of practice or whether it would be better to do a small amount of practice, followed by an application problem. In either case, the goals of the practice problems are:

- To give the students additional practice doing the kinds of calculations that can be performed with the use of the Theorem of Pythagoras
- To make students very clear about the fact that if you know any two sides of a right-angled triangle, then you can always use Pythagoras to calculate the third, and that if you know one of the sides and the triangle is isosceles you can find the lengths of the other two
- To give the students additional practice doing calculations involving square roots

Materials
- copies of the Problem Set in Appendix B.1 for each student

Lesson Sequence
1. To start the lesson, get the students working on the first five triangles of Question 1, reproduced in Figure 2.1. It is important that you encourage students not to turn to their calculators too quickly. For example, in the triangle with one side length equal to $\sqrt{5}$, they should know that when they square $\sqrt{5}$, the answer is 5. If they then add $2^2 = 4$ to that and take the square root, they get that the length of the hypotenuse is 3. Obviously, the numbers in the triangle on the lower left are so large that you should allow students to use their calculators.

In each of these questions the students are expected to calculate the length of the side that is marked by a question mark. By keeping the students away from their calculators
as much as possible, you force them to come to terms with the definition of the square and the square root. I have found that some students take some time to get used to these concepts. Often, students who have not used these concepts very much will confuse the square with multiplication by 2 and the square root with division by 2.

2. The students should be able to do each of the four questions in Figure 2.2 without the help of a calculator until they get to the very last calculation, and it is best if they learn to do the questions that way. The numbers involved are small enough. Of course, if they really want a value for the answer they can use the calculator at the end. For example, the hypotenuse of the second triangle comes to \( \sqrt{34} \). They can leave the answer in that form, but if they want to know precisely how big that number is, they will have to use the calculator to take that square root.

![Diagram of triangles](image)

**Figure 2.2: Question 1 continued**

3. In the two triangles of Question 2 (Figure 2.3), the two question marks on the right angle sides indicate that the unknown lengths of those two sides are equal. The students should realize that after squaring the hypotenuse they should divide the result in two and then take the square root of the answer. In both triangles they should be able to do it without the aid of a calculator, though of course they will need it if they wish to evaluate \( \sqrt{18} \) in the first.

4. Some of the exercises in Question 3 (Figure 2.4) will challenge students’ abilities to interpret and work with the square root symbol. For example, they should learn that the square of two times the square root of fourteen is equal to \( 2 \times 2 \times \sqrt{14} = 56 \). Students will probably not find it easy to see this at first. You may have to spend some time at the board discussing with them that the commutative law, the law that says that in a multiplication the order of the numbers does not matter, also applies to expressions such as \( (2 \times \sqrt{14}) \times (2 \times \sqrt{14}) \), and allows you to rewrite this as \( 2 \times 2 \times \sqrt{14} \times \sqrt{14} \). Once again, there is significant value in getting the students to try to do as much of the calculation by hand as possible.
Applications of Pythagoras

5. Once the students have done the problems on the worksheet, you can finish the lesson with a discussion of two problems that test the skills that have been learned. These problems are designed to require a number of steps, so that they cannot be done until the students are willing to plan ahead. Each problem requires a sequence of applications of the Theorem of Pythagoras. Encourage the students to decide on the sequence of calculations they will do before carrying them out. Learning to plan ahead is key to solving bigger problems.

**Problem 1**

Ask the students to carefully study the figure below (third page of Appendix B.1). You should have a large copy displayed either on an overhead screen or the blackboard. Tell the students that in a moment you will give them the lengths $a$ and $b$ and that they will be asked to find the length $x$. Before giving them the numbers, however, you want them to indicate, by numbering them, in what order they plan to calculate the various lengths on the figure. In other words, begin
2.1 Problem Set - Getting Some Practice

Figure 2.5: Find the length of $x$

by asking them what side they will be able to calculate if they know $a$ and $b$. Hopefully, they will notice the right-angled triangle on the left and decide that they will be able to compute the third side of that triangle. Indicate their answer by marking that side with the symbol \( 1 \) as shown in Figure 2.6.

Figure 2.6: Sequence of steps

Then ask them to decide what length they can calculate in the next step. Once this has been discussed to your satisfaction, mark it with the symbol \( 2 \), and so on. The figure above indicates a possible sequence. In the diagram, the third length is the length of the whole diagonal of the large square, while the fourth length is half that diagonal. The length of $x$ is then equal to the length of \( 4 \), since they are opposing sides in a square. At the end of this discussion you should tell them that $a = 3$ and $b = 2$, and ask them to calculate $x$. 
Problem 2
This problem asks students to calculate the length of the sides of a regular octagon inscribed in a circle of radius 1. Here is the diagram of the circle, the octagon, a square, and a number of radii that will help us do the problem (copy available on the fourth page of Appendix B.1).

![Diagram of circle, octagon, square, and radii]

Figure 2.7: What is the circumference of the octagon?

As in the previous problem, ask the students which lengths they know (the radii). Mark one of them with the symbol $1$ making sure that the students understand that we all the radii have the same length. The following figure shows a possible sequence for solving the problem:

![Diagram showing sequence of lengths]

Figure 2.8: The order in which lengths can be calculated

Note that the symbol $2$ represents the length of the entire vertical line passing through it, while the symbol $3$ represents the top half. The fifth measurement is obtained by subtracting the fourth from the length of the radius. Once we have the third and fifth lengths, the side
of the octagon is the hypotenuse of the small triangle of which these are the right angle sides. Once the sequence is clear, you can ask the students to do the calculations.

Here they are: \(1 = 1\); therefore \(2 = \sqrt{2} = 1.414\), using Pythagoras. By dividing in half, this gives \(3 = \sqrt{2}/2 = 0.707\). Then, using Pythagoras again, \(4 = \sqrt{1^2 - \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{1 - \frac{2}{4}} = \frac{1}{\sqrt{2}} = .707\).

From this we get \(5 = 1 - 1/\sqrt{2} = 0.293\). Finally, the side of the octagon can be calculated. It is equal to

\[
\sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(1 - \frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + 1 + \frac{1}{2} - \frac{2}{\sqrt{2}}} = \sqrt{2 - \sqrt{2}} = 0.765.
\]

Of course, to do the calculation in the manner indicated here requires a considerable knowledge of algebra. The students are more likely to use the calculator at each step. Later on, in Chapter 5, we will discuss this calculation again in our discussion of the number \(\pi\).
2.2 Enrichment Activity - A Packaging Problem

Lesson Goal

- To introduce students to a fun and practical problem that requires Pythagoras’ Theorem for its solution

Materials

- copies of Spot Question in Appendix C.3 for each student
- blank overhead transparency
- 25 pennies (or other small opaque circular items) to be used on the blank transparency
- overhead transparencies of different packing arrangements found in Appendix A.2

Problem Statement

You are asked to design a cardboard box to hold 25 rolls of candy. The instructions are that the rolls of candy should be stored vertically, and that the box should be as small as possible. What dimensions should the box have, and how should the rolls be arranged in the box? The candies are round, with diameter 2 centimeters, and the rolls are each 15 centimeters long.

Lesson Sequence

1. You may wish to begin the class with a Spot Question (see Appendix C.3) that reviews Pythagoras’ Theorem, and at the same time, gets students thinking about how to solve multi-step problems.

2. Returning to today’s problem, there are, of course, a number of possibilities. In order to discuss this, you could bring 25 pennies to class, each representing one of the rolls of candy, and ask for suggestions. Pursue each suggestion (including the simple ones) and ask for the volume of the resulting box. Use overheads to illustrate. Here are some of the simple suggestions you can expect (see overheads in Appendix A.2):

   - At one extreme, the box could be $2 \times 50 \times 15$. In this box the rolls would be arranged linearly. The volume of this box would be 1500 cubic centimeters.

   - Another arrangement would arrange the rolls in five rows of five each, requiring a box measuring $10 \times 10 \times 15$, again giving a volume of 1500 cubic centimeters. This is a more practical arrangement than the first box. But is it the smallest box? Are there other ways to arrange the rolls?

3. Encourage students to look for other, hopefully more efficient, arrangements by moving the pennies around on the overhead projector.
4. After some thought, students may suggest the “close packing” arrangement for the pennies shown in Figure 2.9.

Put up an overhead displaying this arrangement. Ask students to work in groups to see if they have any ideas about what they would have to calculate in order to find the volume of the box.

5. After some time, you may need to provide a hint by connecting the centres of the first two circles and the centre of the circle that sits on top of this pair. The resulting equilateral triangle has been included in the diagram to help you think it through. What is the height of this triangle?

6. Once the height of a single equilateral triangle has been determined using Pythagoras’ Theorem \( h = \sqrt{2^2 - 1^2} = \sqrt{3} \approx 1.73 \), you can ask students for the dimensions of the box. Of course, one dimension is fixed by the length of the candy rolls, 15 cm. The others are \( (4 \times \sqrt{3}) + 2 \text{ cm} \approx 8.93 \text{ cm} \) and \( 5.5 \times 2 \text{ cm} \approx 11 \text{ cm} \). Clearly outlining this on the overhead will help students. This gives a volume of \( V = 15 \times 8.93 \times 11 \approx 1473.45 \text{ cm}^3 \).

7. You will find that with this arrangement the volume of the box is somewhat smaller than in either of the previous cases. The difference is not great, but if a company ships lots of candy rolls, it may nevertheless be important!
2.3 Enrichment Activity - The Spider and the Fly

Lesson Goals

- To use an interesting problem that not only tests students’ understanding of the theorem of Pythagoras, but also tests their ability to think laterally; that is, their ability to look at a problem in a different way
- To introduce nets as a way of “unfolding” a three dimensional surface

Materials

- copies of Spot Question in Appendix C.4 for each student (optional)
- rectangular box made of stiff paper that can be easily cut
- scissors
- overhead transparency of a rectangular box (optional)
- overhead transparencies of various nets of the rectangular box (optional)
- copies of the handout in Appendix A.3 (of a rectangular net) for each student

Problem Statement

We imagine a rectangular room, whose floor has dimensions 2 meters by 8 meters, and whose height is 2 meters. There is a spider in one corner of the floor. A fly wants to find the spot on the walls or the ceiling of the room that is as far from the spider as possible. Of course, the spot that is farthest away in the usual sense is the spot on the ceiling in the diametrically opposite corner. However, to get around, the spider has to travel along the walls, floor and ceiling, so the ‘obvious’ answer is less obvious than it first seems. In fact, there is a place in the room that would take longer for the spider to get to than the obvious location, but you should not tell students this at the start.

Lesson Sequence

1. This is a surprisingly difficult problem. To begin, ask the students to determine how far the spider would have to travel to get to the fly if the fly sat at the diagonally opposite point. Undoubtedly the students will try several answers:
   - They might suggest that the spider should travel diagonally across the floor, and then straight up the wall in the opposite corner of the room. If this is suggested, ask the students how long that path would be.
   - Another suggestion you may get is that the spider should go along the diagonal of the narrow wall it is on, and then follow the line along the ceiling that goes from there directly to the fly. Again, the students should calculate the length of this suggestion.
2. Ask the students if there might not be some other path even shorter than these two. For example, what if the spider were to cross the floor of the room to a point on the long opposite wall that is nearer than the diagonally opposite corner, and then follow a slanted path up that wall directly to the fly? Might that not be shorter than either of the other two? It would be helpful to have a sketch of the rectangular room on the board or overhead, so that you can explain the sort of path you have in mind here. You will find students agreeing that this ought to yield a shorter path, if it is done right. But precisely what point on the opposite long wall should the spider aim for?

3. Once the students are convinced that the solution is not obvious, you should ask them to think of a way to attack this problem that will make it completely clear which way the spider should go. Here is the trick that is essential to the solution: You have to imagine cutting the room open (like a cardboard box) and then measuring the straight line that would represent the path taken by the spider. You can bring along a box and cut it open in front of the class to help students visualize the process. The result, though not drawn to scale, is shown in Figure 2.10. You should reproduce it on the board or overhead. Even better, provide each student with a handout. In the elementary school curriculum, the unfolded shape is referred to as the “net” of the original shape (the room).

4. Note: Since there is more than one way to cut open a box, there is also more than one net for the box. You may wish to explore other nets with your class before moving on. One way to do this is to use the Spot Question in Appendix C.4, which requires students to visualize a room given one net and then locate some of the room’s features on another net. Alternatively, you can present different nets of a rectangular room and have students locate the spider and the fly in each case.

5. If you draw a straight line from the spider to the fly (as they are represented in Figure 2.10) you have a possible shortest path for the spider to take. Ask the students how long that path would be, and what it would look like in the original room? The correct answer for the length of this path is $\sqrt{80} = 8.944...$ meters. Perhaps one of them should be asked to sketch this path on the picture of the actual room in three dimensions, drawn on the board or overhead earlier in the discussion.

   However, this still does not solve the question definitively, for the location of the fly can also be represented at another location in the picture. That is, there are two locations in the picture which, if the room is reassembled would come to represent the single point where the fly is located.

6. Challenge the students to find the other point representing the location of the fly. Of course, it is the point at the upper right hand corner of the far wall in Figure 2.10. Thus the straight line from the spider to that vertex represents another possible route for the spider. Ask the students to calculate its length and to describe the path, by drawing it on the original picture of the room before it was cut out. You will find that the length of this line is $\sqrt{104}$ meters, so the previous route is certainly shorter, though this might not have been obvious from the diagram.

7. But are we sure that we have found the shortest path? Are there other points on the flattened room, besides these two, that represent the location of the fly? Well, in principle
there are, for there is more than one way to cut the room open. For example, we could have opened it up so that the ceiling is not attached at the top, but rather to the wall at the bottom of the picture, or even to the right of the smaller wall on the right of the picture. Draw several of these possibilities for the students and locate the spider and the fly in each of them. After some experimentation you and the students will agree that these new possibilities will not yield shorter distances, however. The shortest path from the spider to the fly is the one suggested at first, of length 8.944 meters.

Introduce the Smart Fly

8. The problem becomes even more fun once you introduce the notion of the smart fly. So far, we have assumed that the fly is sitting in the corner diametrically opposite to the spider. Now, you should ask the class if a smart fly would choose to stay there or move to a safer location to make itself even further from the spider. Let the class think about this problem for a while. It must be clear that the location of the fly is fixed (i.e. pretend that it is taking a nap while the spider sets out to meet it.)

9. To get things started, you could suggest that the fly sit in the middle of the back wall, as
opposed to the corner. In this case there are also two possible paths for the spider. To see this you have to note again that there is more than one way to cut up a rectangular box. In the above rendering the far wall has remained attached to the floor. Another way to do it keeps the far wall attached to the wall located above the floor in the diagram. If you draw that and join the midpoint of the far wall to the spider you get a different route than you would on the diagram above. Once the students understand these two possibilities, ask them to calculate the lengths of these two paths. They will discover to their surprise that the shorter of these, the former, at a length of $\sqrt{82} = 9.055...$ meters is longer than the shortest path from the spider to the corner diagonally opposite, while the other path is longer still. A smart fly would thus choose to sit in the middle of the narrow wall opposite the spider’s rather than in the corner of that wall!

10. If the students remain interested after you have compared the lengths of these two possibilities you could ask them to see if they can find a location that is farther from the spider than either of the two considered so far. In fact, there are such points. For example, if the fly were to move to a spot 50 cm down and 50 cm horizontally onto the back wall, starting from its original location, then it will be farther from the spider than at either of the two locations considered so far. This is a surprising result!
2.4 Enrichment Activity - The Crochet Needle in the Vacuum Tube

Lesson Goals

- To give students additional instances when the Theorem of Pythagoras can be used to solve problems
- To encourage students to make interpretations and use geometric intuition to set up and solve problems

Materials

- concrete models of an “L-shape” tube (You can construct your own using clear tubing from a good hardware store.)
- several needles, straws, or small sticks (see item 1.)

Problem Statement

Suppose the central vacuum system in your house consists of tubes that are 6 cm in diameter, and that at several points there are 90-degree corners. While vacuuming, you accidentally suck up your grandmother’s crochet needle. If the needle is 16 cm long, will it get around the corners in the tubes?

This whimsical problem not only requires the Theorem of Pythagoras for its solution, but also needs a less than immediate analysis before the Theorem can be used.

Lesson Sequence

1. After some discussion and experimentation with the model, the students may eventually come up with the following picture:

![Figure 2.11: A crochet needle stuck at a corner](image-url)
2.4 Enrichment Activity - The Crochet Needle in the Vacuum Tube

The picture represents a corner in the vacuum system. The slanted line represents a needle that will get stuck. The students will realize that when a needle gets stuck, its ends will be wedged against the outside edges of the tubes and its central part (not necessarily the precise middle point!) will be wedged against the corner. To make sure that this is seen clearly, you could use your model of the corner and needle, or a picture of the corner on the board/overhead, and use needles (or straws or pencils) to show various needle positions. This will not only allow the students to see how a needle will get stuck, but also that the shortest needle that will get stuck is the one that gets wedged in such a way that the point that touches the corner is precisely the midpoint of the needle. This is the case illustrated in the picture in Figure 2.11. So the question can be reduced to the following: **How long is the needle that gets stuck with its precise center in contact with the corner?**

2. In order to proceed to a solution, it is very important that the students are very clear about the symmetries in the picture, for the solution depends on them. In particular, if you draw a line joining the inside and outside corners (at 45° to the horizontal), then the whole diagram is symmetric in this line. In other words, if you think of the line as a mirror, then the part of the picture that lies below it is a mirror image of the part that lies above it. This means as well that the triangle formed by the needle and the two outside lines is isosceles. (You may have to remind the students of the meaning of that word).

3. Ask the students how they would indicate on the picture that the point in contact with the corner is the exact center of the needle. The standard way to do it is by putting a hatch mark across each half of the needle. Alternatively, you could mark the two halves of the needle by the same letter c. This is the option chosen below.

![Figure 2.12:](image)

Now challenge the students to find a way to calculate the number c. If we are to use the Theorem of Pythagoras for this we need to add lines to the diagram in order to create a triangle whose hypotenuse is c. Give the students time to try to find such a triangle. If they can’t, you will have to help them to get the drawing shown above. However, it is best to give them time to suggest as many parts of the drawing themselves as is possible.
4. At this point it is important to make the students aware of another symmetry in the picture: Since the sides of the two pieces of tube are parallel, if you were to slide the top triangle along the needle, you would be able to have it coincide exactly with the lower triangle, for the two halves are of equal length, and the sides stay parallel as you slide. That means that both right-angle sides of the lower triangle are equal to 6. We are now in a position to apply the Theorem of Pythagoras to find that \( c = \sqrt{6^2 + 6^2} = \sqrt{72} = 8.485 \cdots \). Thus the shortest needle that would get stuck is twice as long as this, namely 16.97 cm. In other words, the needle in the question, at 16 cm, will not get stuck at a corner.

**Extension - The Turning Barge**

5. If you have time remaining, you could solve the following problem: Suppose we have a rectangular barge of width 3 meters and length 10 meters in a canal that is 11 meters wide. We want to turn the barge around. **Is the canal wide enough to do this?**

![Figure 2.13: The turning barge](image)

6. Again, you should try to get the students to analyze the problem. After a while they should come to the conclusion that if the barge is going to get stuck, it will be because opposite corners will be wedged against opposite sides of the canal, as shown below.

![Figure 2.14: The barge stuck against the sides of the canal](image)
7. Try to get the students to suggest on their own that the diagonal line be drawn in the picture. You could say that the ends of a diagonal line constitute two points on the rectangle that are farthest apart, so the question becomes one of deciding whether this line is less than the width of the canal. At this point ask the students whether they can calculate the length of the diagonal. Of course they can, by Pythagoras' Theorem. The length of the diagonal is $\sqrt{10^2 + 3^2} = 10.44$. Since this is less than 11, we conclude that the barge can be turned around in this canal.
Applications of Pythagoras
Chapter 3

Similarities

Purpose of the Chapter

1. To give students a good understanding of scale changes (known in mathematics as similarities or similarity transformations) and their effects on the length measurements of geometric figures

2. To have students combine their understanding of similarities with that of Pythagoras to perform certain important calculations and to solve fun and challenging application problems

This matter of scale change is usually not included in the elementary school curriculum. I am not sure why this is the case, for the other transformations (slides, flips, rotations) are included. In any case, the discussions and activities in this chapter will have to be done with some care if the students are to understand the effect of scale changes.

Overview of Activities

• E.A. 3.1 - Introduction to Similarity
  An investigation of scale changes and their effects on length measurements, with an application to E.A. 1.1.

• E.A. 3.2 - Equilateral Triangles and Archimedes Pool
  An application problem that uses similarity (and Pythagoras' Theorem) and introduces students to the mathematician, Archimedes

• E.A. 3.3 - The Octagon and the Hexagon
  A problem that has students construct the largest possible octagon and/or hexagon from a standard sheet of paper. The problems use Pythagoras' Theorem, but also develop a good understanding of how lengths and ratios of lengths of figures behave under similarities, which makes these problems challenging. One of these can safely be omitted.
3.1 Enrichment Activity - Introduction to Similarity

Lesson Goals

- To introduce the idea of similarity transformations
- To have students investigate the effect of scale changes on the length measurements of a geometric figure

Materials

- a computer with Adobe Acrobat to illustrate the effects of the magnification button
- models of similar geometric shapes of different sizes

Lesson Sequence

1. In order to do the problems in this chapter, students first have to obtain a good understanding of scale changes, known in mathematics as similarities or similarity transformations. We see this type of transformation a lot, especially when we use the computer. For example, when we display a document on the computer, say a pdf file presented by Adobe reader, it is often possible to zoom in or zoom out. Students have almost certainly come across this feature. What they may not realize, and should understand if they are going to succeed in this section, is that when you zoom in to a picture, all distances are increased by the same factor. For example, if you change the scale of a pdf document from 50% to 100%, all distances are doubled. If a computer is available, you can ask students to measure the width of a page on the computer screen, and compare the measurements obtained before and after the change of scale. You can do the same for the height of the page. But you can also take a particular pair of letters, and measure the distance between them before and after. The students will find that all these distances are doubled.

2. At this point, you should introduce what it means for objects to be ‘similar’. To do this, you could bring along two similar geometric objects (for example, two tetrahedra) of different sizes and ask the students if the objects are the same. The idea is to get students to think about what makes the objects similar. There are two ways to characterize similar objects, and students should be aware of both:
   - the ratios of the lengths are the same for both objects
   - to obtain one object from the other, you need to multiply all the lengths by the same factor.

3. To relate this to the Theorem of Pythagoras, we will consider similar right-angled triangles. Consider first a right-angled triangle whose right-angle sides are 3 and 4. It follows from Pythagoras’ Theorem that the hypotenuse will have length 5. On the board, draw the following triangle with the sides labelled as in Figure 3.1. Then start Table 3.1 on the board and ask the students to fill in the blanks.
4. If the students use Pythagoras’s Theorem for each line, allow them to do this. If they guess the answers by looking at the proportions between the numbers in the lines, ask them why they feel justified in doing that. This observation of a pattern is precisely the goal of this exercise. It is important that the students see the pattern in two ways. The pattern can be thought of purely numerically, in that the second line in the table is twice the first, and the third line is five times the first. While this is correct, and can be used to fill in the blank spaces, it does not explain why this pattern occurs. To see the reason for the pattern you have to think about similarity. You should help students notice that the second triangle is obtained from the first by enlarging it by a factor of 2; and the third, by enlarging it five times. In other words, the triangles are similar! This confirms what the opening activity was meant to illustrate: **when any geometric figure is enlarged, then all the linear dimensions (but not the area!) are enlarged by the same factor.** In other words, there is no need to use Pythagoras to discover that the hypotenuse of the second triangle is equal to $2 \times 5 = 10$ or that the hypotenuse of the third triangle is equal to $5 \times 5 = 25$.

Once you think the students get the point you could test it by first adding the line to the table, and asking the students to complete it.

5. If that indicates their understanding, ask the students to put the relationships between the columns into words. They should write complete sentences, that include the assumptions inherent in the problem. For example a sentence describing the relationship between the

---

**Figure 3.1:**

![Diagram of a right triangle](image)

**Table 3.1:** Fill in the blanks

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Table 3.1: Fill in the blanks

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Table 3.2: Fill in the blank

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<td>9</td>
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</table>

Table 3.2: Fill in the blank
first and the third column could be “When a triangle is similar to the right-angled triangle with sides 3, 4, and 5, then the length of the hypotenuse is always equal to the length of the shortest side multiplied by $\frac{5}{3}$. Note that the part about the triangle being similar to the right angled triangle constitutes the assumption in the table, for the sentence is not going to be true for arbitrary right-angled triangles.

6. Once the students have come up with a correct sentence to describe the relationship between two of the columns, ask them to write the sentence in the form of a mathematical equation. For example, the sentence mentioned above would translate as $c = a \times \frac{5}{3}$. Other relationships are $c = b \times \frac{4}{3}$ and $b = a \times \frac{4}{3}$. Once it is clear that the students understand the relationships as well as the formulas used to express them, add the lines shown in Table 3.3 to the table and challenge the students to complete them.

\[
\begin{array}{|c|c|}
\hline
1.5 & 2 \\
\hline
1 & \frac{4}{3} \\
\hline
\end{array}
\]

Table 3.3: Fill in the blank

Of course, the missing entries are 2.5 and $\frac{5}{3}$ respectively, since we are scaling the original triangle by factors of 1/2 and 1/3 in these two lines.

**Harry Potter’s Runaway Spell** (An Application to E.A. 1.1)

7. As an exercise to practice the effect of similarity transformations (enlargements) on the linear measurements of an object, tell the students the following story: In order to produce the glass cube and the glass sphere around the snitch he wants to give to Hermione, Harry Potter consults the appropriate book of wizardry to find the right incantation. Unfortunately, though he reads the spell that gets the process started, he forgets to look for the spell that stops it. Thus, when he recites the starting spell, the process puts a cube around the snitch, then a sphere around the cube, then another cube around the sphere, then another sphere around it, and so on. What is the diameter of the second glass sphere? How about the third? How many times can Harry let the process continue until the outer sphere is too large to get his arms around it?

8. Help the students set up a table, as in the previous exercises, to answer these questions. If time remains, be prepared to begin the next enrichment activity on equilateral triangles.
3.2 Enrichment Activity - Equilateral Triangles and Archimedes’ Pool

Lesson Goals

- To investigate similarity transformations on equilateral triangles. In particular, to have students discover the relationship between the height of an equilateral triangle and its side length.

- To apply knowledge of the height of an equilateral triangle to solve the problem of Archimedes’ Pool

Materials

- none

Lesson Sequence

1. This lesson continues to develop students’ understanding of similarity transformations. Remind students that similarity transformations can be characterized in two ways:
   - all length measurements are multiplied by the same factor
   - ratios of lengths stay the same when figures are transformed (or scaled)

2. Draw the following equilateral triangle on the board and review the term ‘equilateral’.

   ![Equilateral Triangle Diagram]

   Figure 3.2: The height of an equilateral triangle

Then start Table 3.4 on the board and ask the students to complete it.

They should notice that the left half (as well as the right half) of the equilateral triangle forms a right-angled triangle with hypotenuse $a$ and base $a/2$. Using Pythagoras’ Theorem, the values in the $b$-column will be seen to be $\sqrt{3} = 1.73$, $\sqrt{12} = 3.46$ and $\sqrt{48} = 6.93$ respectively. Above all, though, the students should be encouraged to note the pattern in the table. **The ratio between $a$ and $b$ remains the same.** You can highlight this by asking students to make a third column in their table containing the ratio $b/a$. The reason
this ratio remains the same is that the three triangles represented by the numbers in the table are obtained from the first triangle by enlarging it. They are all **similar** to each other. This affects all linear dimensions of the triangle in exactly the same way, not only the sides, but also the height \( b \). Once again, ask the students to write a sentence describing the relationship between the two measurements of these triangles; that is, between the columns of the table. One such sentence might be “The height of an equilateral triangle is obtained from the length of the side by multiplying it by \( \sqrt{3} \).” Once again, ask the students to translate this into the equation \( b = a \times .87 \).

3. To test the students’ understanding of this relationship between the side \( a \) and the height \( b \) of equilateral triangles, ask them to add to Table 3.4 the rows indicated in Table 3.5 and to fill in the missing numbers.

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Table 3.4: Height compared to side in an equilateral triangle

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Table 3.5: Four more rows to add to Table 3.4

The first line should be easy, as it is simply an extension of the previous lines. The missing number is \( 7 \times \sqrt{3} = 12.12 \). The triangle represented by the second line is half as large as the triangle belonging to the first, so the missing entry in that line is \( \sqrt{3} / 2 = 0.87 \). The last two lines are the most important as preparation for the next point in this lesson. Using the fact that the ratio of \( a \) to \( b \) is always the same in this table, the students should note that to change \( \sqrt{3} \) to 1 they have to divide it by \( \sqrt{3} \). This means that the blank entry in the penultimate line is equal to \( 2 / \sqrt{3} = 1.155 \) The last line can be gotten from that line by multiplying its numbers by 5. Thus the blank in the last line gets 5.77.

**Archimedes’ Pool**

4. We are now ready for the main exercise in this lesson. To liven up this lesson, it is a good idea to include some discussion about Archimedes, when he lived, and what he is known for. This information is readily available on the internet. Tell the students that they are going to design a swimming pool in the shape of an equilateral triangle. To make the story interesting, suggest that the famous Greek mathematician Archimedes (you may wish to digress to tell them about Archimedes’ Law) wants a pool in the shape of an
3.2 Enrichment Activity - Equilateral Triangles and Archimedes’ Pool

equilateral triangle in his rectangular backyard, and wants that pool to be as large as possible. After all, Archimedes does his best thinking in a bath or in a pool, and the latter is less likely to lead to embarrassment. The students are going to act as landscape designers for Archimedes, and design his pool.

5. It so happens that Archimedes’ backyard has width 21.59 meters and depth 28 meters, precisely 100 times the width and length of a regular sheet of paper. So give each of the students a standard letter-size piece of paper, and challenge them to draw the pool on the paper, so that it is as large as possible. They will soon realize that the triangle will have to be oriented on the page as indicated in Figure 3.3.

\[ \text{Figure 3.3: Archimedes’ backyard and triangular pool} \]

6. After some thought, the students should realize that the successful construction of the triangle hinges on their ability to locate the points \( A \) and \( B \), this is accomplished by finding the lengths \( a \) and \( c \). The students may notice that the only dimension of the triangle that is known to them is its height, for it corresponds to the width of the paper, 8.5 inches, or 21.59 centimeters. Give them a chance to make the observation themselves. If the insight is not forthcoming you may have to help them. They should notice that finding the length of \( a \) corresponds to adding a final line to the second of Tables 3.4 and 3.5 constructed earlier in this lesson. For in this case we have \( b = 21.59 \), so the new line in the table will be as shown in Table 3.6 The number \( a \) is to go into the blank space.

\[ \text{Table 3.6:} \]

\[ \begin{array}{c}
21.59 \\
\end{array} \]

Comparing this line to the second to last line included above, we see that \( a \) is obtained by multiplying 1.155 by 21.59 giving 24.94. Since \( a = 24.94 \) cm it follows that \( c \) is half that, and we have found where to mark the points \( A \) and \( B \), and are ready to construct the equilateral triangle.
### 3.3 Enrichment Activity - The Octagon and the Hexagon

**Lesson Goals**

1. To further develop students’ ability to apply their knowledge of similarity transformations to solve more difficult geometric problems
2. To introduce and discuss the properties of the octagon and hexagon

**Materials**

- blank sheets of paper

**Problem Statement**

Simply stated, the problem is to construct the largest possible octagon and/or hexagon from a standard sheet of paper.

**Lesson Sequence**

**Constructing an Octagon**

1. To start the lesson, review with the students what is meant by an octagon (an eight-sided figure). Especially ask them what is meant by a regular octagon, as opposed to an arbitrary octagon. In the discussion that will undoubtedly follow you can point out to them that a regular octagon has all its sides equal, and that it has all its angles equal.

2. The problem, after you have introduced the octagon, is this: **What is the largest octagon you can cut out of a standard sheet of paper (8.5 x 11 inches)?** This requires some discussion about the best orientation for the octagon. After some thought they will probably find that the best orientation, relative to the sides of the page is as shown in Figure 6.4.

3. We can use Pythagoras’ Theorem to calculate the length of the side of the octagon, given that the width of the sheet of paper is 8.5 inches. However, to do the calculation we have to present it very carefully, along the lines of the discussion suggested in the preceding activities (i.e. using tables). Perhaps the simplest way to do the calculation is to draw the lines shown in the right picture of Figure 6.4. Here we have taken the small triangle in the upper right hand corner of the figure, and reproduced it on the left side. This creates a new right-angled triangle, whose right angle sides have the same length as the sides of the octagon. I do not think you will have a hard time getting the students to see that the triangle is right angled, for its right angle is made up of two 45 degree angles of the smaller triangles.

4. Using the idea of similarity, you can proceed as follows: Ask the students to consider the figure drawn earlier, with distances $a$ and $b$ as indicated in Figure 3.5. Ask the students what $a + b$ is equal to. They should notice that it is the same as the width of the page. The students should notice that once they know both of these lengths they will be in a
position to construct the octagon. Ask the students if they see any other lengths that are equal to $a$. In particular, try to get them to see $b$ as the length of the hypotenuse of a triangle whose right-angle sides are both equal to $a$. Ask them what that means about the relationship between $a$ and $b$. They should be able to tell you that

$$b^2 = a^2 + a^2 = 2 \times a^2,$$

or that $a^2$ is equal to one-half $b^2$, as can be seen from Figure 3.5.

5. In order to help students recognize how this helps them solve the problem, resort to a table as in the preceding section. The start of the table is given in Table 3.6. You should
ask the students to fill in the blanks. Ask the students to write sentences describing the

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</table>

Figure 3.6: Measurements

relationship between the three columns. For example, they could say that to get \( b \) from \( a \) they have to multiply it by \( \sqrt{2} = 1.414 \). Ask them to express the relationships as equations. They may come up with \( b = \sqrt{2} \times a \), or \( a = b \div \sqrt{2} = b \times 0.707 \). However, the most important relationship for our problem is the relationship between \( b \) and \( a + b \). They should notice the ratios between the entries in each of the rows in the table are exactly the same, and that each row may be obtained from any other by multiplying it by the appropriate constant. They should notice that the reason the ratios are all the same is that we are once again considering a collection of similar figures, and that when a figure is scaled up or down (enlarged or shrunk) to another similar figure, all the linear measurements are multiplied by the same constant. They should eventually reach the all-important conclusions that \( a + b = (1 + 0.707) \times b \) and therefore \( b = (a + b) \div 1.707 \).

To test these relationships, ask the students to add some further rows to the table. For example, they could be asked to complete the rows shown in Table 3.7 below.

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<tbody>
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<td>7</td>
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<tr>
<td></td>
<td>21.59</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.7: More measurements

Since the width of a sheet of letter-sized paper is 21.59 cm, the students should realize that the last line of the table provides the information needed to complete our problem. Thus, the solution is that the maximum possible octagon that can be cut out of a standard sheet of paper has side length 3.52 inches. Complete the exercise by having the students construct the octagon.

**Alternative Solution - Using some Algebra**

6. Here is how you can proceed after step 3 if the students know some algebra: Give the name \( a \) to the length of the side of the octagon. Then the Theorem of Pythagoras gives us that the hypotenuse of the newly created triangle is equal to the square root of \( 2 \times a^2 \). You want the students to see that this simplifies to \( \sqrt{2} \times a \). To help them see this you
could first write $2 \times a^2$ as $2 \times a \times a$ and ask them whether they can think of something which if it were multiplied by itself would produce this quantity. To give them a start ask them what, if multiplied by itself, would give the answer 2, and then separately ask what, if multiplied by itself, would give the answer $a \times a$. Once each of these has been understood, they should be able to see the simplification.

Now the students should also notice that the hypotenuse of this triangle plus the top edge of the octagon are together equal to the width of the paper. That is $\sqrt{2} \times a + a = 8.5$ inches. That is, $(\sqrt{2} + 1) \times a = 8.5$. Therefore, $a$ may be found by dividing 8.5 by $\sqrt{2} + 1$.

**Constructing a Hexagon.**

7. If there is time, you could now repeat the same exercise with a hexagon. That is, the goal is to make the largest possible regular hexagon out of a standard 8.5 × 11 inch sheet of paper. You may have to remind the students that a hexagon is six sided. After some thought the students will decide that the largest hexagon is obtained if it is cut from the paper as shown:

![Figure 3.7: Fitting a hexagon on a standard page](image)

8. At first you should draw the hexagon without the three diagonal lines shown in the picture. Ask the students for suggestions as to how you might find the length of the sides of the hexagon. After some suggestions you may have to help the students by suggesting that they should try to think of adding some lines to the picture, and you may even have to suggest the diagonal lines. Once these have been included, ask the students again whether they notice anything about the triangles created by constructing those lines. They will probably notice that the six triangles interior to the hexagon are congruent to each other. One way to see this is to recall the rotational symmetry of the hexagon. If you turn it by sixty degrees, each triangle lands precisely on top of the next.
9. They should also notice that the triangles are equilateral. A triangle is equilateral when it has three equal sides. To see that these are equilateral triangles, it helps to know a little about angles. When we go in counter-clockwise direction around the rim of the hexagon, we turn by 60° at each of the six vertices. Therefore the interior angles of the hexagon are each equal to $180° - 60° = 120°$. The line segments from the centre of the hexagon divide each of these precisely in half, into two angles of 60° each. The six equal angles at the centre of the hexagon add up to 360°, so each of them must be 60°. Thus each of the six triangles has all angles equal to 60°. But if a triangle has three equal angles (and therefore each 60 degrees) then it is equilateral.

10. Now suppose we join the bottom two of the four vertices that lie along the side of the paper. This creates four small right-angled triangles, all congruent to each other. Notice that the relationship between these right-angled triangles and the equilateral triangle made up by two of them is precisely the relationship examined in the preceding lesson. You can use the conclusions drawn in that lesson, or review them, whichever is more appropriate. In any case, you can apply Pythagoras’ theorem to one of these triangles. One of the right-angle sides has length equal to half the width of the sheet of paper; that is, 4.25 inches. The hypotenuse is twice as large as the other right-angle side. Thus four times the square of the smaller right-angle side is equal to the square of the hypotenuse. If we subtract the square of the right-angle side we are left with three times the square of the smallest right-angle side. Thus three times the square of the smaller right-angle side is equal to the square of 4.25; That is, 18.06. Thus the square of the smaller right-angle side is equal to 6.02. If we take the square root of 6.02 we get 2.45. This is therefore the length of the smaller right-angle side. The side of the hexagon is twice as long as this. That is, the hexagon has sides equal to 4.9 inches.

Once again, you could end the exercise by asking the students to construct a hexagon of that size from a sheet of paper.
Chapter 4

Plato and Polyhedra

Listen to the music, how fragmented, how whole, how we can’t separate the music from the sun falling on its knees on all the greenness, from this movement, how this moment contains all the fragments of yesterday and everything we will ever know of tomorrow!

Al Zolynas, 1945-
“Love in the Classroom”

Purpose of the Chapter

• To further develop students’ abilities to visualize three dimensional objects in different ways, for example, using profiles and nets
• To introduce and explore the characteristics of the five Platonic solids (cube, tetrahedron, octahedron, icosahedron, dodecahedron)
• To give students an appreciation for some of the beauty inherent in mathematics itself, apart from its many applications

Overview of Activities

• E.A. 4.1 - The Tetrahedron and the Cube
  An introduction to the tetrahedron, together with an interesting application of Pythagoras’ Theorem
• E.A. 4.2 - The Octahedron and Duality
  An introduction to the octahedron, again with an interesting application of Pythagoras’ Theorem
• E.A. 4.3 - The Euler Number
  An introduction to the Euler number, followed by a fun counting activity that uses paper and balloons to calculate the Euler numbers of the plane and sphere
• E.A. 4.4 - The Search for Other Platonic Solids
A particularly challenging activity that stretches students to look for similarities among the 3 Platonic solids learned thus far and to develop a technique for finding others

• E.A. 4.5 - Constructing Regular Polyhedra
A nice break from the abstract tone of the first two sessions, as students make cardboard models of the five Platonic solids
4.1 Enrichment Activity - The Tetrahedron and the Cube

Lesson Goals

- To introduce two regular solids, the cube and the tetrahedron, both of which will be revisited in later activities
- To test and to stretch the students’ abilities to visualize three-dimensional objects
- To give the students additional instances of when the Theorem of Pythagoras can be used to solve problems in three dimensions

Materials

- copies of Spot Question in Appendix C.5 for each student
- cardboard, scissors or exacto knives, and tape
- models of the cube and tetrahedron (see items 1. and 7.)
- copies of the tetrahedron template in Appendix A.4, precut for each student (see item 5.)

Note: If you used a model of a cube for earlier sessions, and especially if your model is one that can be opened, it is a good idea to make the tetrahedron small enough so that it fits inside the cube with space left over.

Problem Statement

The problem concerns a regular tetrahedron whose sides are 20 centimeters long. The problem is to find the dimensions of the smallest box that will contain the tetrahedron.

Lesson Sequence

1. One way to introduce the tetrahedron is to use the Spot Question found in Appendix C.5. This question also reviews nets and is a good exercise in 3D visualization. Once the students have a good idea of what the object looks like, you can bring out a model or draw one on the board. This leads nicely into a discussion of the tetrahedron.

2. Ask the students what the figure is called. Discuss the origins of the prefix “tetra” (four) and the suffix “hedron” (base). Point out to the students that the tetrahedron is called regular when
   - all the edges have the same length
   - the number of edges around each face is the same
   - the number of edges coming out of each vertex is the same.
Thus the regular tetrahedron is an object with four identical faces, each of which could be used as a base, were the object to be placed on a table. This will help students remember what the tetrahedron looks like given only its name. Point out to the students that the plural of hedron is not "hedrons", but "hedra", in deference to the Greek origin of the word.

3. Ask the students what sort of profiles can be produced by a tetrahedron. Explain what you mean by a profile: If you have an overhead projector and a screen, hold the tetrahedron close to the screen, at the level of the overhead projector to avoid distortion. Then the profile is the shadow you get. Similarly, if you were to hang a sheet from the ceiling, and put a strong light behind it at shoulder height, and then hold the tetrahedron close to the sheet, the profile would be the shape visible on the sheet from the other side. Of course there are many possible profiles, depending how the tetrahedron is oriented, but there are two that are particularly interesting:

- The one that students see right away is the triangular profile you get if you hold the tetrahedron in such a way that one of the triangular faces is parallel to the screen.
- After that, the students will probably suggest that the other profiles tend to be diamond-shaped. Let the students think about this for a few moments, holding a model of the tetrahedron in your hands if at all possible.

4. Ask them to draw the two profiles as if the tetrahedron were transparent. In other words, ask them to draw it so that the hidden edges are visible. After some time, one of them will suggest that with certain orientations you will get a square profile.

![Figure 4.1: One of the profiles of a tetrahedron](image)

Ask the students whether they can be sure that it is really a square, and not a diamond-shaped figure that is nearly square. The square profile is obtained if the tetrahedron is positioned so that one of its edges is nearest the screen and parallel to it, while the opposite edge is furthest from the screen, and also parallel to it. By the opposite edge we mean the one and only edge that does not share a vertex with the first edge. Note that when the tetrahedron is positioned as described, then the two opposite edges look perpendicular to each other, as in Figure 4.1.

5. Before getting to the heart of this part of the lesson, you should have the students make tetrahedra out of cardboard nets. Distribute copies of the first of the templates in Ap-
4.1 Enrichment Activity - The Tetrahedron and the Cube

Appendix A.4, one for each student. Use scissors to cut out the large triangular shape before the lesson. Ask the students to place the template on a suitable piece of cardboard (or stiff paper), to mark the vertices of the inside triangle of the template on the cardboard, to draw lines on the cardboard joining these points, and then to score along these lines using the exacto knives. They can then fold the template into the shape of a tetrahedron, and secure the construction with the tape you have brought along.

6. Once the students each have their own tetrahedron, you are ready to state the problem you want the students to help you solve: What is the size of the smallest box (i.e. cube) that can contain the tetrahedron? In fact you could present the problem by telling the students that you have a company that produces tetrahedra made of silver. These tetrahedra each have edges that are 20 centimeters long. You want to market these tetrahedra as pieces of art. However, you have decided that they will sell better if each tetrahedron is sold inside a silver cube. Since silver is (say) $10 per square centimeter, you would like to minimize the cost of those cubes. You could divide the class into two groups, each group constituting a “company” that is asked to present you with a design and a price for the best cube it can come up with.

7. To get their thinking started, it is helpful if you brought one or more cubic boxes to begin with, large enough to contain the tetrahedra they made. Make sure, though, that the boxes you brought are larger than necessary, for you do not want to give the solution away. The students will undoubtedly have suggestions very quickly. For example, you will probably get the suggestion that the side of the box should be the same as the edge of the tetrahedron (20 cm) for then you can place the tetrahedron in the cube so that the base of the tetrahedron rests on the base of the cube while two of its vertices coincide with two of the vertices of the cube, as in Figure 4.2:

![Figure 4.2: One way to fit a tetrahedron into a cube](image)

8. Once this possibility is in front of the class, you could ask them whether they are quite sure that when you place the tetrahedron inside the cube in this way it will fit. In other words, why will the back vertex of the base not poke out through the back of the cube, and how do you know that the top vertex of the tetrahedron will not lie above the top of the cube? Of course this is rather easy to see intuitively, for the two slanted edges of the base of the tetrahedron have length 20 cm, but because they are slanted the distance
from the front edge of the base to the back vertex must be less than 20 cm. Similarly the
distance to the top is less than 20 cm.

9. But have we found the smallest cube that can hold the tetrahedron? Ask the students
if they can find a way to fit the tetrahedron into a smaller cube. The clue is found in
our earlier discussion of profiles. If that discussion was conducted with some success, it
is quite likely that one of the students will suggest that we should place the tetrahedron
in the cube in such a way that one of its edges is the diagonal of the bottom of the cube.
The bottom of the cube, together with that edge of the tetrahedron will then appear as
shown in the diagram below. Thus the side of the cube will have to be $\sqrt{200} = 14.14\cdots$.So, will the tetrahedron fit exactly in a cube of side equal to 14.14\cdots? You can see that
the answer is “yes” if you think it through in the following way: Once one of the edges of
the tetrahedron lies along the diagonal of the bottom face of the cube, then at each end
of that diagonal, the two tetrahedron edges attached to that vertex will each become the
diagonal of a side face of the cube.

![Diagram](image1)

Figure 4.3: The edge of the tetrahedron becomes the diagonal of a face of the cube

It then follows that the remaining edge must join the remaining two vertices, which are
now located at diagonally opposite points on the top of the cube. Figure 4.4 is the final
picture, showing the tetrahedron inside the smallest cube possible.

![Diagram](image2)

Figure 4.4: The smallest cube to fit around a tetrahedron
**Extension - The Height of the Tetrahedron**

This optional discussion should only be attempted with a very motivated group of students. You can safely omit this part if you do not have the time for it or if you feel the students will lose interest. No future lessons will depend on the calculation.

10. Go back to the first attempt, where we had the tetrahedron rest on one of its faces, with one edge of that face along a bottom edge of the cube. Clearly, if we do this the remaining vertex of the base will not reach the opposite edge of the cube’s base, nor will the top vertex of the tetrahedron reach the top face of the cube. This means that if we were happy to store our tetrahedron inside a rectangular box that is not necessarily a cube, we can make it quite a bit smaller than the cube of side 20 cm. What are the measurements of the other sides of the smallest rectangular box that will fit around the tetrahedron in that way? Doing this problem is entirely optional as far as the continuity of the enrichment programme is concerned. It will not be used later. You may skip it, and of course you may decide to postpone this discussion for a separate session.

11. To do the problem, we have to calculate exactly how far back the base of the tetrahedron goes, and just how high the tetrahedron is, for these two distances should then be picked as the remaining measurements of the rectangular box.

The first of these questions requires finding the height of the base; that is, the height of an equilateral triangle with sides equal to 20 cm. We discussed this relationship extensively in E.A. 3.2. Depending on the degree to which students understood that discussion, you may have to review it. Figure 4.5 is the diagram, with the construction line, you should try to get the students to suggest if at all possible:

![Figure 4.5: The height of the base of the tetrahedron](image)

It is now easy to see, using the Theorem of Pythagoras, that the height of the triangle is $\sqrt{300} = 17.3\ldots$, which is clearly less than 20. This confirms that the tetrahedron will not protrude through the back of the cube.

12. Finding the height of the tetrahedron is more complicated, though. Ask the students to imagine slicing vertically through the tetrahedron, starting at the top vertex, and eventually passing through one of the vertices of the base as well as the midpoint of the edge opposite to that vertex. When we separate the two halves of the tetrahedron that result from that cut, the cross-section is a triangle whose base is the height of the triangle...
we just considered. Ask the students to determine the lengths of the three sides of this triangular cross-section. In fact, one of the other two sides of that triangle is the height of one of the other faces of the original tetrahedron, so the sides of the cross-section are $\sqrt{300}$, $\sqrt{300}$, and 20, respectively. The height of this triangular cross-section will be equal to the height of the tetrahedron. However, calculating this height is more difficult, for this triangle does not have the symmetries of the faces of the tetrahedron. In particular, the perpendicular line from the top does not meet the base at its midpoint, as shown in Figure 4.6.

![Figure 4.6](image)

13. It seems hopeless to use Pythagoras’ theorem to find the height of this triangle! However, there is a neat trick that will work, if the students remember the formula for the area of a triangle. The area of a triangle is, of course, equal to one-half the product of the base and the height. If we knew the height all we would have to do is multiply it by $\frac{1}{2}\sqrt{300}$ to get the area. Ask the students how they would turn this around. That is, if we knew the area, what would we do to calculate the height. Of course, all we would have to do is divide the area by $\frac{1}{2}\sqrt{300} = 8.66$ to get the height. This seems to be a pointless, circular, argument since we know neither the area nor the height. However, you should ask the students if there is another way to calculate the triangle’s area than using the base of length $\sqrt{300}$ and the height we are trying to find. Hopefully they will notice after some thought that we can also get the area by regarding one of the other sides as the base. In particular, if we regard the side of length 20 as the base of the triangle, then because the remaining sides are equal to each other, it follows by the symmetry of the triangle that the perpendicular from the left vertex to that base of length 20 will divide that base in half. This allows us to use Pythagoras’ theorem on one of the triangles created by that perpendicular, as illustrated in Figure 4.7.

14. Draw this picture on the board, and ask the students to calculate the area of the triangle using this second suggestion. The square of the hypotenuse of one of these triangles is equal to 300, while the square of the known right angle side is 100. This implies that the length of the perpendicular is $\sqrt{200}$. But that puts us into a position to calculate the area of the large triangle, using this perpendicular as height. The area is equal to $\frac{1}{2} \times 20 \times \sqrt{200} = 10 \times \sqrt{200} = 141.42$. 


4.1 Enrichment Activity - The Tetrahedron and the Cube

15. Now, if we go back to our earlier discussion of the area of the triangle, this allows us to calculate the original height of the triangle, the one that corresponds to the height of the tetrahedron. For all we have to do is to divide the quantity $5 \times \sqrt{200}$ by $\frac{1}{2} \sqrt{300}$. The answer is $\frac{10 \times \sqrt{2}}{\sqrt{3}}$. It is probably difficult for the students to follow the calculation when so many square roots are involved, so it is better to do the calculation in decimal form on the calculator: The height of the tetrahedron is $141.42 \div 8.66 = 16.33$.

16. Summarizing our solution, we see that if we had a rectangular box of height 16.33 cm, width 20 cm and depth 17.3 cm, then we will be able to use that rectangular box to store the tetrahedron in such a way that its base rests on the bottom of the box. Note, however, that the cube we found earlier is considerably smaller than this rectangular box!
4.2 Enrichment Activity - The Octahedron and Duality

Lesson Goals

- To encourage students to visualize three dimensional objects in their minds and to predict various profiles
- To introduce and discuss the octahedron and its “duality” with the cube
- To see further applications of Pythagoras’ Theorem in three-dimensional situations

Materials

- copies of Spot Question in Appendix C.6 for each student
- copies of the octahedron template for each student (Appendix A.4)
- cardboard model of an octahedron

Lesson Sequence

1. A nice way to begin the lesson is with a question that challenges students to visualize a three dimensional object in their minds. The Spot Question in Appendix C.6 does this well using profiles, which were discussed briefly in the last session and will be discussed in greater detail today.

   Discovering the Octahedron

2. Moving on to today’s problem, ask the students to imagine joining the midpoints of the faces of a cube. Ask them to try to determine, without drawing a picture, what sort of figure they will get. It may help if you ask them to close their eyes as they try to visualize the construction. If a student notices that your question remains somewhat ambiguous, he or she may indicate this by asking whether you want them to join midpoints of opposite faces or just the midpoints of adjacent faces. If this question is raised, tell them that you want to join only the midpoints of faces that are adjacent. If the students find the question too difficult, you can draw a cube on the board, and ask them again. Challenge them to draw the cube and the mystery figure at their desks.

3. When it begins to look as if they are making some progress in the visualization exercise, ask them how many vertices the resulting figure has. There will of course be six, one for each of the faces of the cube. You could then ask them how many edges it will have. Since each edge will join the midpoints of two adjacent cube faces, that edge will be behind and perpendicular to the edge of the cube at which the two faces join. Thus each edge of the new figure can be associated with an edge of the cube, and so there will be 12 edges. Now ask about the number of faces, and what their shapes will be. The answer is that they will be triangular, and that each sits behind one of the eight vertices of the cube. The result can be seen in Figure 4.8
4. It will be helpful if you have brought a model of an octahedron to class. It is not difficult to make one from the template in Appendix A.4. Discuss with the students the meaning of the prefix “octa” (eight), and show that this is a regular octahedron: The faces are congruent (have identical shape and size); the edges all have the same length, and the number of faces and edges at each vertex is the same. Thus, the regular octahedron is a figure with eight identical faces (or bases). If desired, you could have the students construct their own octahedra out of cardboard.

**Investigating Profiles**

5. Ask the students what profiles they can get by holding the octahedron at different angles. Once again, there are many possibilities. However, there are three that should be pointed out to the students, of course only after they have been challenged to come up with them on their own. Figure 4.9 shows these more regularly shaped profiles.

Technically, the lines that are shown passing through the interiors of these figures are not part of the profile. Nevertheless, showing those lines helps the students understand the profiles. The first two profiles are a square and a regular hexagon, respectively. The third one looks like a parallelogram obtained by joining two equilateral triangles. But is it? This is something you could ask the students after showing the third picture: “Are the two triangles equilateral?” The correct answer is that they are not. The reason is that the
line drawn through the center is an edge of the original octahedron seen straight on. The
other four edges are sloped away from us, however, and will therefore look shorter in this
profile.

6. Ask the students whether they notice anything special about the three viewing directions
that give rise to the three profiles shown. What is special about them that distinguishes
them from the viewing directions that give rise to less regularly shaped profiles? The
answer is that to get the regular hexagon we have to choose a viewing direction that is
perpendicular to one of the triangular faces of the octagon. Similarly, to get the cube we
have to choose a viewing direction that faces directly one of the vertices. In this case it is
a little more difficult to say what we mean by “facing directly”. The most precise way to
say it is that the viewing direction is given by a line passing through a vertex and making
the same angle with all the faces and edges that meet that vertex. For the parallelogram
we have to choose a viewing direction that is perpendicular to one of the edges and makes
equal angles with the two faces adjacent to that edge.

You could go back at this point and ask the students what viewing directions gave the
three most regular profiles for the tetrahedron. In this case we get the same triangular
profile whether we choose a direction perpendicular to a face or emanating from a vertex.
A viewing direction perpendicular to an edge and making equal angles with the adjacent
faces produces the square profile.

If the students seem interested in these profile questions, you could also ask what profiles
are obtainable from a cube. Ask the students whether they can get a hexagonal profile
from a cube. (The answer is yes - see the Spot Question in Appendix C.6.)

Problems Using Pythagoras

7. This is probably the right time to return to questions that involve the Theorem of Pythagoras.
Ask the students to suppose we have a cube whose sides have length 2, and that we
have created an octahedron by joining the midpoints of adjacent faces of the cube. What
is the length of the sides of this octahedron?

The answer can be obtained by applying the Theorem of Pythagoras to the horizontal
triangle drawn in the figure shown on the next page. This is clearly a right angled triangle,
whose right-angle sides both have length 1, half the length of the edge of the cube. By
the Theorem of Pythagoras this means that the length of the hypotenuse is equal to $\sqrt{2}$.
Thus each of the edges of the octahedron has length equal to $\sqrt{2}$.

8. Now ask the students what the distance is between opposite vertices of the octahedron.
Some of them may go looking for additional triangles to which Pythagoras’ Theorem can
be applied. In fact the matter is very simple: In the diagram opposite vertices of the
octahedron are located at the midpoints of opposite faces of the cube. Since the edges of
the cube have length 2, this must also be the distance between these opposing vertices.

9. Now ask the students whether we can design a smaller rectangular box (not necessarily
cubic) that can contain our octahedron. The answer is clearly yes, for if we turn the
octahedron inside the cube by 45 degrees, then we can move the four vertical sides of the
cube close to each other to create a rectangular box of dimensions $\sqrt{2} \times \sqrt{2} \times 2$ that is big enough to hold the octahedron.

However, even this is not the smallest, for once we have the octahedron in this position we can start tilting the octahedron so that the top vertex moves closer to one of the sides of the box while the other moves away from that same side. To decide how far you can tilt the octahedron, and how much it allows you to reduce the size of the rectangular box is quite difficult, for it depends on the various angles between the faces of the octahedron.

**Exploring Duality of Geometric Solids**

10. An interesting next step in the exploration is to ask the students what would happen now if they were to join the vertices of the midpoints of the octahedron. The answer is that they would get another (smaller) cube. Again, the students may have to think about it for a while, and a decision has to be made about which midpoints to join and which not to join. The correct decision is to join two midpoints only if the faces they are on meet at one of the edges. Figure 4.11 gives a picture of the result:
be. However, to calculate it we would have to know the distance between the midpoint of an equilateral triangle and its vertices. This calculation would need more knowledge of geometry than is available to students at this grade level.

11. Instead you could dwell for some time on the interesting fact that if you join midpoints of faces of a cube you get an octahedron, while if you do the same to an octahedron you get a cube. Furthermore, each vertex of the new figure corresponds to a face of the old, and each face of the new figure corresponds to a vertex of the old. Edges of the new figure will correspond to edges of the old (they will look perpendicular to their corresponding edges).

One of the consequences of this “duality” between these figures is that if we let \( V \) denote the number of vertices and \( E \) the number of edges, and \( F \) the number of faces, then \( V-E+F \) is the same for both figures. Ask the students to calculate this number.

12. Now ask the students what they will get if they join the midpoints of the faces of a tetrahedron. They will soon see that this will produce another tetrahedron. You could express these relationships by saying that the octahedron and the cube are dual to each other, while the tetrahedron is dual to itself. Again ask the students to calculate the quantity \( V-E+F \). The answer will be the same as before!

13. At the end of the session you could ask the students what they will get if they join the midpoints of adjacent edges of the cube. Ask them to describe the resulting figure, and ask them whether it is regular (it is not). Also ask them to calculate the quantity \( V-E+F \), and to compare the result to the previous case. The quantity \( V-E+F \) will be studied more fully in the next chapter.
4.3 Enrichment Activity - The Euler Number

Lesson Goals

- To introduce students to the famous Swiss mathematician, Euler, and the important number named after him
- To engage students in creative activities that enable them to discover the Euler numbers of a flat plane and of a sphere

Materials

- models of the regular polyhedra discussed thus far (cube, tetrahedron, and octahedron)
- objects representing polyhedra that are not regular (e.g. triangular prism, house-shaped object)
- overheads of polyhedra and balloons (Appendix A.5)
- soccer ball, basketball
- balloons - Check to see if the teacher has any objections to the use of balloons in the classroom. Some students may be allergic to balloons.

Note: It is important for the element of surprise in the last lesson of this chapter that you not mention the icosahedron or the dodecahedron at this stage. They are meant to be “discovered” by the students during E.A. 4.4.

Background

In this lesson the students will learn about the Euler number (pronounced “Oiler number”), named after the famous Swiss mathematician Euler who lived from 1707 to 1783. This is a number produced by counting the number of faces, edges and vertices of a figure drawn on a generally “spherical” surface. (We will discuss what we mean by “spherical” later in the lesson.) It turns out that this number is the same no matter what figure you draw. Even more surprising, if you draw a figure consisting of polygonal faces, edges and vertices on another surface, such as a torus (the surface of a donut) or a double torus (the surface of a donut with two holes, if there is such a thing), then once again the number you get will not depend on the figure you draw, but it will be a different number from the number you get on a spherical surface. It seems therefore that the number you get, the “Euler number” of the surface, tells you which of these three surfaces you are on. In fact, this way of distinguishing different surfaces from each other is basic to an area of mathematics known as algebraic topology.

Lesson Sequence

1. To start the lesson, hold up for the students’ examination the three polyhedra you brought, and remind them of their names. Get the students to remind you of the meaning of the words “tetra” (Greek word for the number “four”), “octa” (Greek word for “eight”) and
“hedron” (Greek for “base”). Challenge them to think of other words that include the prefixes “tetra” or “octa”. Some of the words that come to mind are “tetrarch”, “octagon” and “octet”. A dictionary search will turn up other examples. Ask the students to guess in each case how the number comes into the concept. Ask them what another name for the cube could be, if it did not already have its more familiar name (“hexahedron”). You could challenge the students to come up with that term themselves, using the analogy with the word “hexagon”. By the way, the literal meaning of the ending “gon” is “angle”. Thus the hexagon is the figure with six angles (or vertices).

2. Next tell the students that the general term for a member of the family of figures that includes the cube, tetrahedron, and octahedron is “polyhedron”, with plural “polyhedra”. Challenge the students to think of other words that begin with the prefix “poly” and invite them to guess what it means. Of course, the prefix means “many”, and other such words are polygon, polygamy, polyglot, polynomial, Polynesia and polyphonic. Remind the students of the term **regular**:  
   - all edges are the same length  
   - the number of edges around each face is the same  
   - the number of edges coming out of each vertex is the same.

To make sure that the students are not left with the impression that all polyhedra are regular, you should now also show them the non-regular polyhedra you brought with you. Figure 4.12 shows some non-regular seven-faced polyhedra. These figures are available as an overhead in Appendix A.5.

The students should notice that when the shapes are not regular, they usually do not have special names. Certainly we would not want to refer to a house-shaped object as a “septahedron”, even though it has seven faces. It is not difficult to see that there are other seven-faced polyhedra, equally deserving to bear the name if any should.
3. Invite the students to notice that even though these polyhedra all have corners and flat faces, they are, in a general sense, “round” or “spherical”. They should try to picture them as spheres whose surfaces have been subdivided into polygonal faces. What is meant here is that if they were made of the right kind of flexible material, and if we were able to increase the air pressure inside the figures, inflating them as it were, then we would end up with objects that really look like spheres. If at all possible, though, try to get the students themselves to come up with their own way of expressing what it means to say that all these polyhedra are “spherical” in shape, before you give your description. The soccer ball is a good example of the sort of thing that is meant, for it is made up of a large number of hexagonal and pentagonal pieces of leather stitched together. Because the number is relatively large, when the ball is put under pressure it gets a very nice spherical shape, even though the ability of leather to stretch is quite limited. To illustrate other spherical shapes, you could use the “crazy balloons” overhead found in Appendix A.5. This could lead to a discussion about classification. Ask students how they might classify the different balloons. Hopefully one of the students will come up with the suggestion that we can classify the balloons according to the number of holes they have. (If not, you should suggest this yourself.) Tell the students that the mathematics they learn in this session will help them to classify such balloons.

4. You should also discuss what we might mean by dividing the surface into generally polygonal faces. Certainly the surface of each of the platonic solids is divided into very obvious polygonal faces. In these examples the edges of the faces are perfectly straight and the polygonal faces are perfectly flat. In the case of the soccer ball, and even more so in the case of the basketball, the faces are no longer flat, and their edges are no longer straight lines. The point is that this does not matter in this discussion. In this lesson we are not interested in the flatness of faces or the straightness of lines.

You might point out to the class, once they have agreed that the soccer ball and the basketball deserve to be thought of as polyhedra, that the soccer ball is not regular, even though it may look regular at first. Close examination will reveal that some of the faces are hexagons while others are pentagons. The situation with a standard basketball is more puzzling. Its edges are very curved, and its faces look more-or-less banana-shaped. However, a close look reveals that each face is in fact triangular, that the eight faces are congruent, and that four edges meet at each vertex. A basketball is a distorted octahedron.

5. You are now ready to introduce the Euler number. On the board prepare a table with one column labeled “V” for the number of vertices, one labeled “E” for the number of edges, one labeled “F” for the number of faces, and a final column under the heading “V-E+F”. Tell the students that the letters V, E and F represent the numbers of vertices, edges and faces of a polyhedron, and that you are going to ask them to count these for a large number of examples. Begin by doing some examples with the whole class. For example, hold up the cube, and ask the class how many vertices it has. Enter their answer (8) as the first entry in the “V” column on the board. Do the same for the numbers E (=12) and F (=6). Then use the entries in the V, E and F columns to calculate the quantity V-E+F, and enter the result (=2) in the last column. Go through this exercise once or twice more, with some of the other polyhedra you brought to class.

If you decide to include the soccer ball among the surfaces counted, you should assign it
to a pair of students. It is very difficult to do the count accurately. The students will have to devise a system for it. For example, they could put tick marks on the faces once they have counted them, or circles around each vertex. Alternatively, you may wish to make this a counting exercise for the whole class. For reference, this is included as a “Warm-up Activity” in the next lesson (E.A. 4.4).

**Finding the Euler Number of Spherical Objects**

6. Once it is clear to the students what it is you are trying to do, distribute the balloons. Ask the student to draw a series of vertices, edges, and faces on their balloons, given the following rules:
   - all the edges must be connected to each other
   - no face should have the shape of a race track (i.e. an inner region disconnected from the outer region)

Let students be creative in this exercise. However, you may wish to warn them that they will have to count the number of vertices, edges, and faces once they have finished. The students should be asked to enter the results in the columns on the board.

**Modification:** If you feel that your students will not be able to handle this exercise (due to maturity), or that you will not have enough time, you can blow up the balloons and draw patterns on them ahead of time.

7. You will find, assuming students do not count wrong, that the quantity $V-E+F$ equals 2 in each case. $V-E+F$ is the “Euler number” of a “spherically-shaped” surface, such as a balloon. As mentioned earlier, it is named after a German speaking Swiss mathematician, so you should pronounce it as “Oiler number”. From our experiment, it seems that no matter how you divide a spherical shape into polygons, the number $V-E+F$ always works out to 2. It is especially interesting to note that you would get a different number for a surface that has a different basic shape, such as a flat sheet of paper or a torus (the surface of a donut or inner tube). The finding of our counting experiment is summarized by saying that **the Euler number of a spherical surface is 2**.

8. Ideally you should now be able to repeat the same sort of experiment with surfaces that have the shape of a torus (the surface of a donut). Unfortunately, torus shaped balloons are not available and the surface of an actual donut does not lend itself to subdivision by a marker. Even inner tubes are hard to come by these days. However, it is not too difficult to make a “square donut” from a piece of cardboard. In any case, if we were to conduct the experiment with a surface of the general shape of a donut, we would find that for every subdivision we get $V-E+F=0$. That is, the Euler number for a torus is 0. In the even more remote possibility that we could repeat the procedure for a double torus, we would find that the Euler number for that surface is -2.

**Finding the Euler Number of Flat Objects**

9. However there is something we can do experimentally that goes beyond the case of the spherical surface. Give each person a sheet of paper and ask them to draw a series of
vertices, edges, and faces, as they had done on the balloons. (Alternatively, you could have these sheets predesigned and ready to distribute.) Ask the students to count vertices, edges and faces as they did with the balloons. Again start four columns on the blackboard, labeled as before. This time you will find that the V-E+F column consistently gets the value 1. In other words, it seems that the Euler number of a flat plane is 1.

10. This is probably a good place to end, especially if you have engaged the students in drawing their own patterns. In summary, each surface has its own Euler number. This Euler number does not change if the surface is deformed and certainly does not depend on the particular way in which it may be divided into polygonal pieces. However, the Euler number does depend on the shape of a surface if “shape” is understood in a more general sense. This more general sense is sometimes referred to as “topological shape”. Thus the surface of a football, a basketball and a cube are considered to have the same topological shape, but an inner tube has a different topological shape. On the other hand the inner tube and the surface of a coffee mug are the same (think about this). Both of these, in turn, are different from the surface of a teapot, which is the same as that of a “two-holed donut”. The coffee mug and the teapot will probably require some reflection.

Extensions

Logical argument as to WHY the Euler number of a plane is 1

11. If there is time, and the students seem sufficiently motivated, you could try to show the students by means of a logical argument why the Euler number for the plane must equal 1. In other words, you could prove to them that it is no coincidence that V-E+F=1 for every figure drawn on the page. To do this you should ask them to imagine the process of creating the figure piece by piece. The first thing you would do is to draw one polygonal face. You could distribute blank pages to the students and ask each of them to draw one polygon on his or her page. Ask the students what V-E+F comes to if they stop here. The answer is that no matter how many edges the polygon has, it has an equal number of vertices, so that V-E=0. That means that V-E+F is equal to F, and that because there is just one face at this stage, we will necessarily have V-E+F=1 if we stop here.

12. Now ask the students to suggest ways in which they might add to the figure, ensuring that whole figure remains connected after each step. The idea is to break the process into steps that are as small as possible. For example, if the students say you could attach another polygon to the first, you should object that you would do this by drawing one edge or one vertex at a time. More specifically, instead of drawing the additional polygon all at once, you would probably draw the polygonal edge first, and then mark the vertices on it. If you can show that for each small alteration of the figure the number V-E+F remains unchanged, it will follow that its value will be 1 when the figure is complete. Ask students to help create a list of steps on the board or overhead. A sample list is given here.

1. Put a vertex on an existing edge.
2. Connect two existing vertices by an edge that does not run into any other existing edges.
3. Attach a new edge, with a new vertex at the end of it, to an existing vertex.

4. Draw a new edge with a new vertex at each end, with at least one of the vertices lying on an existing edge.

13. After creating the list, ask the students if they can narrow the list down to two fundamental steps. For example, it is enough to include only steps 1 and 2. Encourage the students to experiment until they agree that every figure can be drawn by a sequence of these two steps. Be sure to explain why V-E+F does not change under steps 1 or 2 (or whichever steps you have settled on). In the case of 1, when we put a vertex on an existing edge, this increases the number of vertices by one, but also increases the number of edges by one, because an existing edge is divided into two edges. Since no new faces are created, V-E+F remains the same. In the case of 2, when we connect two existing vertices by an edge, this does not change the number of vertices, but it does add both an edge and a face, since the new edge will either divide an existing face into two faces or create a new face outside the area already contained within the existing edges. Once again, the two changes cancel out, and V-E+F remains the same.

14. Figure 4.13 provides an illustration of the process. This means, of course, that at the end of the drawing process, the quantity V-E+F will be the same as it was after the first polygon. That is, we have proved that the Euler number for a plane surface is 1.

WHY the Euler number of a spherical surface is 2

15. Ask the students if they can think of a reason why, if the Euler number is 1 for a flat figure, it should be 2 on a spherical surface. You could give them a hint by asking them how you could turn a spherical surface into a flat surface. In fact the balloon provides an excellent illustration. Suppose you take a pair of scissors and cut off the knotted opening of the balloon, creating a hole. Here is where we make use of the fact that earlier on we arranged to have the knot in the center of one of the faces. If that face is relatively large and if the balloon is quite flexible (not really very likely), we should be able to put our fingers into the hole, and stretch it out until the whole balloon can be flattened onto the table. If the balloon is not sufficiently stretchable you will have to ask the students to imagine the last part of the deformation. The picture drawn on the balloon now turns into a picture drawn on a flat page. Ask the students how it is then that on a flat page we get V-E+F = 1, while on a spherical surface we got 2. The answer is that the face that contained the balloon’s knotted opening becomes the outside of the figure once it is flattened out. In other words, regarded as part of a flat figure, that face would not be counted, changing the quantity V-E+F from 2 to 1.

The Euler number of a Cylinder

16. Here is another optional exercise: Ask them to determine, by experiment, the Euler number of a cylinder. By a cylinder we mean the surface you get when you take a sheet of paper and roll it into a tube taping the top of the page to its bottom. Now ask the students to divide the surface into polygons. There are a few things to keep in mind when they do this. One is that the lines should be drawn on one side of the paper only. In other words,
Figure 4.13: Constructing polygonal faces using only steps 1 and 2
we pretend that the paper is so thin as to be transparent. The second is that each face should have a polygonal shape. This is once again related to the fact that the lines and vertices we draw should be connected to each other. For example, if you draw two curves with vertices on them, and do this in such a way that each curve encircles the cylinder, and such that the curves are more or less parallel, then the circular “band” between them cannot be thought of as a face because it is not a polygonal region. Finally, this also means that the two open sides of the cylinder, the two circles that came from the sides of the sheet of paper, have to be subdivided themselves into edges and vertices as part of the pattern drawn on the cylindrical surface.

If the counting is done correctly, the students should find that the Euler number of a cylinder is 0. This calculation is related to the fact that the Euler number of the torus is also 0, for a torus can be produced from a cylinder (if the latter is made of stretchable material) by joining the two circles at the ends of the cylinder.
4.4 Enrichment Activity - The Search for other Platonic Solids

Lesson Goals
- To help students discover two additional regular polyhedra, the icosahedron and the dodecahedron, and to explain why there are no more than five regular polyhedra in total
- To continue to develop students’ three dimensional visualization abilities

Materials
- models of the cube, tetrahedron and octahedron
- models of the icosahedron and dodecahedron (use templates in Appendix A.6)
- soccer balls

Note: It is important to have the models of the icosahedron and the dodecahedron hidden from the class so as not to ruin the element of discovery.

Background
The surprising fact that there are only five regular polyhedra was discovered by the Greeks, and these five figures are known as the five Platonic solids. In fact, Plato associated four of them with the supposed four elements of the universe. The tetrahedron represented fire; the cube, earth; the octahedron, air; and the icosahedron, water. Plato thought the universe was made entirely of atoms in these four shapes. This left only the dodecahedron to be assigned a place, so Plato associated that with the shape of the universe itself.

Showing that the five Platonic solids are the only regular polyhedra possible normally involves solving a set of simultaneous equations in several unknowns. Since students at this level will have had little algebra, we must present the discussion in a manner that bypasses these more sophisticated methods. This can be done by setting the problem up very carefully, and turning the search for numbers satisfying the equations into a sleuthing exercise. It will be more important than usual that the discussion proceed in precisely the manner indicated in the following lesson sequence.

Lesson Sequence

Warm-up Activity - Counting V, E, and F for a Soccer Ball
1. As an exercise to prepare for the search of regular polyhedra, count the number of edges on a soccer ball by counting instead the number of faces. You may need to review that the soccer ball has two types of faces: pentagons and hexagons. The pentagons are easiest to count - there are 12. The hexagons require a little more work - there are 20. Thus, in total there are 32 faces. Now we are ready to count edges. Each pentagon has 5 edges and each hexagon has 6, so it would seem that there are $12 \times 5 + 20 \times 6 = 180$ edges. However, we have counted each edge more than once. In fact, since every edge separates two faces, we have counted each edge twice. This means there are $180 \div 2 = 90$ edges.
2. Now, use the number of edges to count the number of vertices. There are 90 edges, each having 2 vertices, which suggests a total of $90 \times 2 = 180$. However, each vertex has three edges coming out of it, so each vertex will be counted 3 times, once for each of those edges. Thus the number of vertices is $180 \div 3 = 60$. Ask students whether there would have been an easier way to get the number of vertices (Answer: use the Euler number.)

**Reviewing Relationships Between V, E, and F**

3. Begin by asking the students what they learned in the preceding lesson about the relationship between the three numbers V, E and F for a polyhedron. Make sure they are all clear on the fact that no matter what polyhedron you may construct, you always have $V-E+F=2$. Ask the students why you framed your question in terms of a *relationship* between the three numbers V, E and F. Try to get them to say in their own words that because we know that $V-E+F=2$, we can calculate one of these numbers once we have counted the other two. In particular, ask the students what they would do to work out the value of E if they knew the values of V and F. Try to steer the discussion towards the equation $E = V+F-2$. Hopefully they will come up with a sentence such as “If you add V and F and then take away 2 you will end up with E”. This sentence is equivalent to the equation $E = V+F-2$. You could say that the equation is a translation of the English sentence. In the discussion avoid algebra language that you may remember from high school, such as “subtract 2 from both sides of the equation, and add E to both sides”. That language will probably not mean anything to the students. Once this discussion is finished, clear an area of the blackboard, and enter the equation

\[ E = V + F - 2 \]

at the top of the space. This equation will be joined by two others in a moment.

**Identifying the Search Parameters**

4. Next review with the students what it means for a polyhedron to be regular. The definition was given in an earlier section:

- each face should have the same number of edges around it
- each vertex should have the same number of edges coming out of it.

Look to see that this is so for the three regular polyhedra you brought with you (cube, tetrahedron, and octahedron), and contrast this with the soccer ball, some of whose faces are pentagons while others are hexagons. (Thus, although each vertex has three edges coming out of it, each face does *not* have the same number of edges around it.)

5. Tell the students that your plan for today is to look for other regular polyhedra. Invite the students to imagine that there might be another regular polyhedron, which is unknown to us at this time. We are going to do the detective work necessary to find and apprehend this mysterious shape. One of the first things a detective does in such circumstances is to form a profile of the subject of his search, so that he will know what to look for and where to look. We are going to do precisely the same thing. We are going to form a profile of the
mystery object (in our case, a regular polyhedron) before we start our search. To make the investigation more appealing to the students, you may suggest that we give names to the mystery polyhedra, such as “Al Hedron” and “Polly Capone”. (This of course assumes that we have at least two suspects.) Figure 4.14 illustrates how you might present these suspects to the students in the form of a cartoon.

6. This profile of the unknown regular polyhedron hinges on the fact that we can describe each regular polyhedron in terms of two things, which are always the same:

- the number of edges around each face (call that number $p$)
- the number of edges emanating from each vertex (call that number $q$).

The polyhedron can then be specified by the pair of numbers $\{p, q\}$. The mysterious figure shares this characteristic with the familiar tetrahedron, cube and octahedron; you might say the unknown stranger is a relative. To get the $\{p, q\}$ notation across to the students, ask them what $\{3, 3\}$ represents (it is the tetrahedron). Ask them what the notation for the cube should be ($\{4, 3\}$). Ask the same question about the octahedron ($\{3, 4\}$). Tell the students that we want to see if there are any other regular polyhedra. That is, we are going to search for other pairs of numbers $\{p, q\}$ that represent regular solids.

7. This search for other regular polyhedra hinges on a pair of counting arguments. The first of these calculates the number of vertices by counting the number of edges. Ask the students to imagine for a moment that we are thinking of a regular polyhedron of type $\{p, q\}$. This might be one of the polyhedra we already know about, or it might be one of the ones we hope to discover. For this regular polyhedron, each edge has two vertices.
This suggests that \( V = 2 \times E \). Ask the students whether this is correct. If there is no clear answer from the class, check the calculation for the case of a simple polyhedron such as the cube or the tetrahedron. Of course the formula is not correct, for if we count both vertices each time we count an edge, then those vertices will be counted many times over. Ask the students how many times each vertex will be counted. They will notice that this is related to the number of edges emanating from each vertex. For example, in the case of the cube you would end up counting each vertex three times, since each vertex has three edges attached to it. In other words, if we do this to the polyhedron \( \{p, q\} \), each vertex is counted \( q \) times. So when you count the vertices by counting two vertices for each edge, the total \( 2 \times E \) does not represent the number of vertices, but rather \( q \) times the number of vertices. That is,

\[
2 \times E = V \times q.
\]

Add this equation below the one already on the board.

8. We can use a similar calculation to get the number of faces from the number of edges. Each edge adjoins two faces, so at first we might think that \( F = 2 \times E \). Once again, though, this is not correct, for if we count the faces this way, each face will be counted several times. Ask the students how many times each face will be counted. The answer is of course that each face is counted once for each of its edges; that is, each face will be counted \( p \) times. This gives us the formula

\[
2 \times E = F \times p.
\]

This is the third equation you should write on the blackboard. Go over the three equations one more time.

**Learning the Search Tool**

9. At this point you should create a table on the blackboard. For the moment draw the headings only.

<table>
<thead>
<tr>
<th>( F )</th>
<th>( V )</th>
<th>( E = V + F - 2 )</th>
<th>( 2 \times E )</th>
<th>( p )</th>
<th>( q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>4</td>
<td>6</td>
<td>12</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>8</td>
<td>12</td>
<td>24</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>6</td>
<td>12</td>
<td>24</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>7</td>
<td>14</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

Table 4.1: Try different values of \( F \) and \( V \)

Once you have the first row of this table on the board, ask the students to take a fresh sheet of paper and copy the table on their sheets. Tell the students that this table is the tool needed to do the detective work necessary to find all other regular polyhedra there may be.
Now to show the students how the table is used, tell them that you are going to use the tetrahedron to fill in the first row. The tetrahedron has 4 faces, so you enter 4 in the first column. It also has 4 vertices, so that becomes the second entry. It is important that you set the tetrahedron aside after these two numbers have been entered, for the remaining entries are going to be obtained by calculation, and not by observation. Remind the students that V, E and F are always related to each other because the Euler number is 2, and that once we know the values of V and F we can calculate E by using the first of the three formulas we listed on the board. This formula is also reproduced at the top of the third column of the table. Thus the third column is calculated from the first two by computing $E=V+F-2$. This means that the third entry is 6. The fourth column is easy, for it is just twice the third column. We are now ready to calculate the last two columns. Here you must remind the students of the other two equations on the board: $2 \times E = F \times p$ and $2 \times E = V \times q$. Compare the fourth column and the first to see that the fifth must equal 3. Compare the fourth and the second to see that the sixth must equal 3. This seems to say that $p = q = 3$ for a tetrahedron. Check that this is so for the tetrahedron you brought along.

10. Now create the second row by going through precisely the same procedure for the cube, and then the third row by doing it for the octahedron. The resulting numbers are shown in the table shown above. Make sure that in each case you put the polyhedron aside as soon as you have entered the first two numbers, for the others must be obtained by calculation. Now tell the students that you are going to start the search for additional regular polyhedra. This will be done by trying other pairs of numbers in the first two columns in the table and then seeing what the numbers in the other columns will be. Begin by saying to the students “Let’s see what will happen if we put the numbers 4 and 5 in the two initial columns.” Point out to the students that this amounts to checking whether it is possible to have a regular polyhedron with 4 faces and 5 vertices. It is easy to fill in the third and fourth columns. The results are shown in the table on the preceding page. Point out to the students that the calculation producing the 7 in the third column shows that if there is a polyhedron with 4 faces and 5 vertices, it will have 7 edges. But now look what happens when we try to find the number to go into the fifth column. Here we are looking for a number which, if multiplied by the entry (4) in the first column will produce the entry (14) in the fifth. It is easy to see that the only number that does that is the number 3.5. However, the fifth column represents the number p, the number of edges around each face, so this should come out as a whole number! The fact that in this case the fifth column does not come out as a whole number simply means that there is no regular polyhedron that has 4 faces and 5 vertices. So as soon as we see that the answer in the fifth column is not going to be a whole number, we can stop, for it means that there does not exist a regular polyhedron with the numbers of faces and vertices indicated in the first two columns. I simply put a question mark in the fifth column to indicate this failure. The sixth column is treated in exactly the same way. Of course, as soon as we get one question mark we can abandon that pair of values for F and V and create a new row with another pair.

Let the Search Begin!
11. As soon as you are sure the students understand the procedure, ask them to start trying different combinations of numbers in the first and second column until they find a pair for which the fifth and sixth columns come out as whole numbers. As soon as this happens we have apprehended one of the unknown regular polyhedra. In fact, the pairs of numbers corresponding to these shapes are (12, 20) and (20, 12), corresponding to the dodecahedron and the icosahedron respectively. Note that once we have one pair of numbers $(F, V)$ another is found by reversing the order of the pair, for there is clearly a symmetry in the way the calculations in the table are done.

Once some members of the class announce that they have come up with a working pair of numbers, add their findings to the table on the blackboard, and bring out the icosahedron and the dodecahedron to show that these figures really exist. Check with the students that the entries in the third, fifth and sixth columns describe these shapes correctly.

12. Finally, tell the students that if they were to go on looking for even more pairs of numbers that would work out, they would fail, for there are no others. If they had had more algebra, say if they were at the end of grade nine or ten, they would be able to set up the calculations in a way that would make it clear that there cannot be other solutions.
4.5 Enrichment Activity - Constructing Regular Polyhedra

Lesson Goals

- To give students a break from the abstract discussions of the previous two sessions
- To have fun creating the newly learned platonic solids!

Materials

- models of each of the five Platonic solids
- copies of the dodecahedron and icosahedron templates (Appendix A.6)
- cardboard or heavy bristle board (enough for each student to make at least one of the two platonic solids)
- exacto knife or sharp scissors for each student
- wooden board or additional cardboard to put under students’ work (if using exacto knives)
- tape (and possibly glue)
- elastic bands for each student (make sure they are the right length - see step 4 below)

Notes:

1. You will notice that some of the templates include tabs. If you are using stiff paper or bristle board to construct the solids, it is best to include tabs, and to use them to glue the faces together. If you are using cardboard, it is better to leave the tabs off, and to use tape to construct the solids.

2. If you have not already done so, it is a good idea to make the icosahedron and the dodecahedron yourself before class, using the templates provided in Appendix A.6, so that you can anticipate the difficulties the students will have.

Lesson Sequence

1. Begin by telling the class that they are going to take a break from the abstract mathematics of the last session and have some fun constructing the final two Platonic solids. Do they remember their names? To give them an idea of how that will be done, you should have a cardboard cut-out for one of the three simpler platonic solids with you. Hold it up in front of the class, and show how by folding it you create the solid. (You may have already had the students construct these solids in earlier lessons.) Now do the same thing for the icosahedron and dodecahedron. In this case you probably have to prepare the cardboard somewhat beforehand by folding it into the icosahedral shape, or you will find it difficult to fold correctly at the time of the demonstration. The high point of the demonstration will occur when you use an elastic, as described below, to cause a dodecahedron to jump up from the table on its own. The “Oohs” and “Aahs” are worth the effort.
2. The next items outline the procedure for constructing the dodecahedron and the icosahedron. To prevent confusion, it is best to demonstrate how each of these is made before letting the students loose. Then, you can ask the students to choose which one they would like to make.

Constructing the Dodecahedron

3. Figure 4.15 shows the flower-shaped template needed to make half a dodecahedron. You can choose to make a dodecahedron by preparing two of these shapes OR you can make the dodecahedron all of one piece by cutting out the template shown in Figure 4.16 (available in Appendix A.6).

![Figure 4.15: Template for one half of a dodecahedron](image)

![Figure 4.16: Template for a dodecahedron](image)
4. After cutting this shape out of a piece of stiff cardboard, use an exacto knife to score along the lines shown. The idea is to cut through two layers of cardboard, the upper surface and the interior corrugation, but to leave the bottom layer intact. If after the scoring you start folding the cardboard, it will automatically fold into the dodecahedral shape. Actually, however, you will get a better class presentation if you make the dodecahedron in two pieces. If you cut out two of the flower shapes shown above, and score them in the same way, put them on top of each other in such a way that the scored sides point outward. Then turn one of the flowers by one-tenth of a complete rotation, so that the petals of one flower cover the spaces between the petals of the other. Figure 4.17 shows the arrangement, drawn as if the cardboard were transparent.

![Figure 4.17: Two half dodecahedra arranged for an elastic band](image)

Notice that this produces ten outward pointing "petal points". While pressing the two cardboard flowers tightly to a table surface, stretch an elastic band around the edge of the figure by alternately letting it pass under the petal points of the bottom flower and over the petal points of the top flower. Once you have done that, release the pressure that keeps the two flowers together, and ‘presto’ the dodecahedron will rise newly born out of its nest.

**Constructing the Icosahedron**

5. Making an icosahedron is less dramatic, though it is surprising that a polyhedra with twenty faces can be constructed by a single template as simple looking as shown in Figure 4.18 (see Appendix A.6). Once again, begin by showing the class a completed example.

6. Once you have given both demonstrations, ask the students to choose the platonic solid they would like to construct, and give them the necessary materials and the appropriate templates. If there is time, the students may be able to make two solids each. Apart from the case of the dodecahedron, for which they will certainly want to duplicate the elastic trick, the constructions will have to be secured with tape. You can use either scotch tape or masking tape for this.
Figure 4.18: Template for an icosahedron
Chapter 5

Perimeter, Area, and Volume: Linear Shapes

Purpose of Chapter

In this chapter we discuss area and perimeter and to some extent, volume, of linear shapes. Although this may seem standard fare (most students at the grade 7 or 8 level know very well the formulas for the calculation of the area of a rectangle, parallelogram, and triangle), student understanding of these concepts tends to be superficial. Students often do not make the connection between the value of the area of a figure and the number of unit squares that will fit into it (cut up into smaller bits if necessary to make them fit). With that said, the main purpose of the chapter is:

1. To engage students in a series of non-standard discussions intended to address some of the more conceptual issues underlying the formulas for area and perimeter

Other important goals include:

2. To give students practice calculating areas and perimeters of various linear geometric shapes, both using standard formulas and conceptual thinking

3. To further develop students’ problem solving skills by exposing them to different strategies (such as making a guess, collecting evidence via tables and graphs, and making a logical argument)

4. To have students investigate the relationships between lengths and areas (or volumes) and how changing one affects the others

Overview of Activities

- E.A. 5.1 - Areas and Vegetable Gardens
  An exercise that highlights the (partial) independence of perimeter and area, gives students practice in graphing a relationship and in using a graph to locate a maximum possible value, and reinforces students’ understanding of algebra (if their background is sufficient)
• E.A. 5.2 - Maximizing the Volume of a Paper Box
A lesson in which students explore the relationship between the height and volume of a rectangular box using graphical methods.

• E.A. 5.3 - Scale Changes and Area and Volume
An exploration of the way areas and volumes increase when figures are expanded. This lesson ties in with the discussion in Chapter 3 of similarity transformations (scaling, or zooming in and zooming out), and is essential to a good understanding of later material.

• E.A. 5.4 - Using Areas to do Algebra
An optional lesson that explores several interesting ways in which algebraic identities can be obtained by looking at rectangles and rectangular boxes. It will be more meaningful to students who have had some algebra and who would benefit from seeing a connection between algebra and geometry.

• E.A. 5.5 - The Shadow Problem
A lesson built around the problem of finding the area of a shadow. This constitutes a challenging exercise in three-dimensional visualization together with a discussion of the area of a trapezoid. This lesson also contains an optional review of the areas of parallelograms and triangles.

• Problem Sets 5.6 and 5.7
In addition to the main lessons, there are problem sets, which can either be used to fill an entire enrichment session or spread across several sessions as warm-up exercises. In any case, the problems are designed to strengthen the students’ ability to do area and perimeter calculations of rectangles, triangles, parallelograms, and composite shapes. As with the case of Pythagoras’ Theorem, having discussed the theory at a deep conceptual level does not by itself guarantee a facility with problems involving the theory. Computational practice is necessary as well!
5.1 Enrichment Activity - Areas and Vegetable Gardens

Lesson Goals

- To introduce and discuss the meaning of area
- To have students investigate the relationship between the dimensions of a rectangle and its area using tables, graphs, formulas, and logical arguments
- To develop students’ problem solving strategies

Materials

- copies of Spot Question in Appendix C.8 for each student
- copies of the graph paper templates in Appendix A.7 for each student
- overheads of the graph paper templates in Appendix A.7

Background

When it comes to calculating areas, squares and rectangles are the simplest figures. When we calculate the area of any shape in square centimeters, we are, in principle, conducting a count of the number of centimeter squares that will fit inside it. The words “square centimeter” themselves betray this principle. Of course, when the shape is at all irregular, it will not be possible to exhaust all of it by means of centimeter squares, so it may be necessary to go to millimeter squares, or squares of even smaller size. This method of filling shapes by means of small squares of standard size lies at the basis of all formulas for areas of figures.

The students know very well that the area of a rectangle is obtained by multiplying the lengths of its sides. Nevertheless, there are some interesting questions that can be asked about areas of rectangles and squares. This lesson provides several suggestions. If all of them are followed up, you will need more than a single lesson, so you may wish to spread the material over two sessions, or select certain pieces of the lesson.

Lesson Sequence

1. Before jumping into the problem, you should begin the class with a Spot Question to gauge students’ present understanding of area. A good suggestion is the Spot Question in Appendix C.8, reproduced below.

   “Consider the following rectangle, which is 4.5 units wide and 3 units high. What is its area, and explain why you think your calculation is correct. Are you simply applying a known formula or can you give a reason why it is the appropriate formula?”
After about five minutes, engage the students in a discussion of their answers to the question. How many students simply used a formula? How many thought about counting unit squares? How many combined these two (i.e. There are 3 sets of 4.5 squares, so in total there are 13.5 unit squares.) After such a discussion, explain to the class that in the next several sessions, we will be doing several interesting and fun activities involving areas and perimeters.

2. Here is the first problem:

**Stating the Problem**

Suppose you want to make a rectangular vegetable garden. Since you have to build a fence around it to keep out the raccoons, and you have just 16 meters of fencing, you want to make sure that you choose the length and the width of the garden so that the perimeter is 16 meters, while the area is as large as possible. What length and width should you make the garden?

Our intuition tells us that a square garden would give us the largest area, but can we show that this is true? Does a square garden give the best result? This may seem to be too hard a problem for the students, but in fact there are different ways to explore this question at their level. The first step involves making a guess. The next step involves gathering evidence to support that guess. This could be done by setting up a table and then making a graph, or by engaging in a relatively simple thought (logical) experiment. It is important that students understand that there are different ways to approach the problem.

**Making a Guess**

3. You could begin by asking the students what they think the answer is. It is likely that they will think that a square garden gives the smallest perimeter. Challenge them by asking how they know that a square gives the largest area. They will probably come up with specific examples, and it is important that you let them do this, for it helps them understand the question. On the other hand, it is important also to remind them that a lot of specific examples can never prove a general result.

**Gathering Evidence**

4. When it is clear to the students that there is something to explore, explain that you want them to record the area of the garden for each of several choices of width in order to study which dimensions will produce the largest area. You should begin with a table consisting of two columns, the first recording the width $w$ of the garden, and the second recording
the area. After that the results will be represented graphically. The students may not have had very much graphing up to this point in their mathematics classes. However, they will almost certainly have seen graphs in other contexts, so it should not take long for them to understand your plan. Give each of the students paper, and ask them to do the calculations and to tabulate the results for several choices of width. Here is what the beginning of the table will look like:

<table>
<thead>
<tr>
<th>width</th>
<th>area</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
</tr>
</tbody>
</table>

Table 5.1: A table to show how the area depends on the width

The first entry could be omitted at first. It represents a garden plot of width 0 and depth 8. This is clearly not a viable solution, but it does complete the mathematical picture inherent in this problem, especially once we graph the results, so \( w = 0 \) should be included eventually. The other entries are obtained by simple calculations. For example, in the third entry we let the width, \( w \), equal 2 meters. This leaves 12 meters of fencing to go along the remaining two sides of the garden. That is, the depth of the garden is then necessarily 6 meters. This implies that the area, \( A \), will be \( 2 \times 6 = 12 \). The students should notice that whenever a value of \( w \) is picked, the corresponding value of \( A \) is decided by a similar calculation. The table should be continued by the students until they get to the other absurd extreme, when the width of the garden is chosen at 8 meters, leaving no fencing for any depth and thus resulting again in area equal to 0. It makes sense to stick to integer choices for \( w \), but the students should be aware that in principle you could pick any number between 0 and 8 for the width.

A Graphical Solution

5. Though the table alone is quite convincing evidence that a square garden will have the biggest area, this is illustrated particularly well by means of a graph. Distribute graph paper to the students and begin by asking them whether they can suggest a way in which the finding summarized in the table can be recorded graphically. If necessary, guide them towards the idea of a graph where the horizontal axis records the width, \( w \), and the vertical axis records the corresponding area \( A \). The entries in the table indicate the scales you should use on the axes. The width should go from 0 to 8, and because the maximal area in the table is 16 square meters, the scale on the vertical axis should go at least that high, but not much higher. Show the students how each entry in the table should be represented by a point on the graph. Figure 5.1 shows what the graph will look like.

6. Make sure the students are aware that if we were to calculate the area for lots of the values of \( w \) between the integer values already recorded this would have the effect of connecting the points on the graph with a nice smooth curve. In this case the resulting curve would
be a parabola. Make especially sure that the students understand that our initial question of finding the largest area is answered by identifying the highest point on the graph, at \( w = 4 \), and that the area is then found by checking the value of that point on the vertical axis, namely \( A = 16 \).

**Describing the Relationship Between \( A \) and \( w \) using a Formula**

7. If you are working with a group of students who have had a lot of algebra already, or who are ready to learn some, you could do the calculations involved in the construction of the table and the graph by means of an algebraic formula, a function, that expresses the area in terms of the choice of width. To help students do this you could begin by asking students to describe the calculations they would perform once they picked the width. You should ask them to write the procedure out in full sentences. If the width is 5, the answer could be something like the following:

“One width and one length take half the fencing, so if the width is 5, the length has to be 3. I then multiplied 5 \( \times \) 3, to get the area.”

If this is the kind of answer given, challenge the students not to carry out any of the calculations in the descriptions of the process, but only to indicate how the calculations go. In particular, the number 3 is not a calculation, but the outcome of a calculation. If necessary, be very specific and suggest that instead of saying “the length has to be 3” say “the length has to be 8 \( - \) 5”, for that indicates where the number 3 came from. Having replaced 3 by 8 \( - \) 5, ask them to use this new formulation in the calculation of the area as well. Eventually, you want the students to say something like
“Since the width is 5, the length is $8 - 5$, and so the area is $5 \times (8 - 5)$.”

One of the first and most important steps in the learning of algebra is this kind of articulation of the form taken by the calculations we do. Only then does it make sense to replace numbers by letters. A meaningful sentence in algebra is always a sentence describing the structure of a calculation or a statement of the equality between the outcomes of two different calculations.

Vary the number next. In other words, ask the students to describe the calculations they would perform if the width were 6. Look for the same sentence with every occurrence of 5 replaced by 6. It should become clear from the discussion that the form of the calculation itself (as opposed to the outcome) does not depend on the width. Ask the students how they would describe the calculation if the width is indicated by the letter $w$. In this case, the area of the garden is found by the calculation $A = w \times (8 - w)$. We describe this situation by saying that “the formula for calculating the area $A$ from the width $w$ is $A = w \times (8 - w)$.”

You could complete the discussion of the algebra by inviting students to use the formula to calculate the area for a number of widths; say, $w = 3, 3.5, 3.7$. This will help students to understand the role of a formula as a function that takes an input (namely $w$), carries out a prescribed calculation, and produces an output (namely $A$).

**A Logical Solution**

8. Once you have completed the graphical solution to the problem, you should introduce the logical solution. It does not depend on any calculations or measurement, but consists entirely of a thought experiment that is typical of a certain type of argument that works in problems asking for an optimum solution. The idea is to guess at the optimum solution (we have done that already) and then to show that if you are not at the optimum, you can make a change to improve the situation. So suppose that our garden is not square, and that its width is greater than its depth.

![Figure 5.2: Pictures to illustrate the ”thought experiment”](image)
Imagine slicing a thin rectangular section from the width of the rectangle, indicated in the upper left hand corner by the dotted line. The piece of fence that went along the left side of the garden can then be moved in. The garden has a smaller area than it did initially, and we also have two small sections of fence left over, labeled as “extra pieces” on the diagram in the upper right. Now we move the fence that goes along the bottom of the garden downward by an amount that is precisely equal to the lengths of the two “extra pieces”, and then place the “extra pieces” into the resulting gaps. The addition is indicated in the bottom figure as the region below the dotted line. Since the width is greater than the depth, the area that is added at the bottom, when we do this, is greater than the area we removed earlier. In other words, by making the rectangle “more nearly square”, while keeping the perimeter the same, we increase the area. Since a rectangle that is not a square can always be made more nearly square, and since doing this always increases the area, its area is not the greatest possible. The only rectangle for which this thought experiment does not work is one for which one side is not longer than the other; that is, a square. In other words, the rectangle for a given perimeter that has the largest area is the square with that perimeter.

**Extension: A Shared Garden**

Now suppose we change the conditions of the problem a little. Imagine that you still have 16 meters of fencing, but your neighbour has already built a long straight fence between his and your properties. You realize that with your 16 meters of fencing you will be able to make a larger rectangular garden if you build it against that fence, for then you will only need to surround your garden on three sides. What dimensions (length and width) should the garden have now, if we want the area to be as large as possible?

1. As before, begin by asking the students to guess the answer. They will find it much more difficult this time. In fact it is unlikely that they will come to any sort of consensus. Once you are satisfied that they understand the question, see how the conditions are different from the preceding question, and notice that the answer is much less obvious. You should repeat the procedures used in the first problem. First have the students create a table in which the first column represents the width of the garden (the length of the side parallel to the existing fence) and the second, the area. The beginning of the table will look like this:

<table>
<thead>
<tr>
<th>width</th>
<th>area</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>14</td>
</tr>
<tr>
<td>4</td>
<td>24</td>
</tr>
</tbody>
</table>

Table 5.2: A table to show how area varies with width

Note that the table shows only even numbers in the first column. This is not necessary, of course. However, if you choose an odd number in the first column you will not get a
whole number in the second. For example, a width of 3 meters will leave 13 meters of fencing for the sides; that is, 6.5 meters on each side. The area will then be \(3 \times 6.5 = 19.5\) square meters. Of course, once again we do not even have to choose whole numbers for the width, but we have to limit the size of the table somehow.

2. Once the table has been completed, you should have the students draw a graph. A template for this is provided in Appendix A.7 This is what the graph will look like:

![Figure 5.3: The graph showing the relationship between area and width](image)

Notice that the shape of the graph is the same as in the previous problem, but that the scales are not the same.

3. Once again, if your students appear ready to do the algebra, you should challenge them to represent the relationship between the width of the garden and its area by means of a formula. You could begin, as in the preceding problem, with an examination of the sequence of calculations performed to find the area. Express it first as an English sentence and only afterwards in algebraic form. If the value of the width is \(w\) meters, then \(16 - w\) meters of fencing will left for the sides. Dividing this by two leaves \(\frac{1}{2} \times (16 - w)\) for each of the sides. Thus the area of the garden will be

\[
A = \frac{1}{2} \times (16 - w) \times w.
\]

Often, of course, multiplication signs are suppressed in algebra, when there is no danger of confusion, so the formula can be written as

\[
A = \frac{1}{2}(16 - w)w.
\]
4. Once the graphical solution has been completed, you could ask the students to try to construct a logical solution similar to the one that solved the previous question.

One way to proceed is to remove two equal thin pieces from the garden as shown in the first of the three pictures below. This produces two leftover sections of fencing, marked as “extra pieces” in the diagram. Now use these two pieces of fencing to extend the garden at the bottom. Then if the original width of the garden was $w$ and the depth $d$, then the total area we removed initially was $d \times e \times 2$ where $e$ is the length of the “extra pieces” of fencing, and the area we added at the bottom of the garden was $w \times e$.

It is not hard to see that if $w < 2 \times d$, then we will be enlarging the area of the garden, while if $w > 2 \times d$ this change will reduce the area. In other words, the area of the garden is greatest when the width is two times the height.

As an alternative, the procedure could be reversed. When we remove a small slice from the bottom of the garden, we are left with two small pieces of fencing. However, because of the neighbour’s fence we only need one of them to extend the garden sideways, unless we decide to extend it on each side (or by double the distance on one side). Try to get the students to solve the problem, so give hints only when they seem to be making no progress.

5. It is also possible to solve this problem by relating it to the preceding, though it is unlikely a student will think of it. Imagine your neighbour also has 16 meters of fencing, and is interested in building a garden of the same shape on his side of the fence. Then the circumference of your combined garden is 32 meters, and when the combined garden has the biggest possible area, then each of the two halves will also. The existing fence does not form part of the circumference of the combined garden, so the combined garden has biggest area when it is square. Thus each half has biggest area when it has the dimensions of half a square.
5.2 Enrichment Activity - Maximizing the Volume of a Paper Box

Lesson Goals

- To extend some of the ideas introduced in the preceding section to solve what appears to be an entirely different problem
- To once again investigate a “functional” relationship (namely that between the height of the box and the resulting volume) using tables and graphs

Materials

- copies of Spot Question in Appendix C.9 for each student
- square sheets of paper for each student (20cm × 20cm)
- copies of graph paper template (see Appendix A.8) for each student

Problem Statement

In this session, we will be making a rectangular box, without a top, out of a square sheet of paper, and we will investigate how we can achieve the largest volume.

Lesson Sequence

1. You may wish to begin this session with a Spot Question that reviews and expands the students’ understanding of area. The Spot Question in Appendix C.9 is a nice example involving surface area.

Making the Box - Folding Instructions

2. To begin the activity, ask the students to fold the paper along one diagonal, then to unfold it, and perform a similar fold along the other diagonal. This will result in the folds indicated by solid lines in Figure 5.5.

Next ask the students to turn the paper over on the table, and to make a fold parallel to the bottom edge of the paper, about one-fifth to one-fourth of the way up. The precise distance does not matter in the case of the first fold, as long as it remains well below halfway. In Figure 5.5 the first fold corresponds to the dotted line that runs parallel to the bottom. Now ask the students to undo the fold they have just made, and to fold parallel to the two sides of the paper in such a way that the creases produced by these folds pass through the intersections of the first fold with the two diagonal folds. Finally do a similar fold parallel to the top, again making sure that it intersects the diagonal folds where the vertical folds do. Once all six creases have been completed, create a box by turning up the four sides along the creases parallel to the sides of the paper.
Gathering Data

3. Ask the students to measure the widths of their folds, and ask them what the volume of the resulting box is. This should be entered on a table. For example, if one of the students chose 3 cm for the width of the folds parallel to the edges of the paper, then the square bottom of the box will be $20 - 3 - 3 = 14$ cm wide. Since the height is 3 cm, the volume of the box will be $3 \times 14 \times 14 = 588$ cubic centimeters. Gather all the calculations produced by the students, and enter them in Table 5.3. Once that is done, add other possible choices to the table and ask students to do the required calculations. These choices for the width of the folded margin (that is, the height of the box) should include 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. The first and the last of these will produce “boxes” of height 0. Table 5.3 shows part of the resulting table.

<table>
<thead>
<tr>
<th>height</th>
<th>volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>324</td>
</tr>
<tr>
<td>2</td>
<td>588</td>
</tr>
</tbody>
</table>

Table 5.3: A table showing the relationship between volume and height

A Graphical Solution

Once the table has been completed, it should be represented as a graph. The horizontal axis, as always, represents the choice we made (the height of the box), and the vertical axis indicates the resulting volume. We already noted that the only possible heights are between 0 and 10. This indicates the scale we should use on the horizontal axis. To determine the scale of the vertical axis we should look at the numbers appearing in the second column, and choose something a little larger. You will see that 600 is a good choice.
Ask the students to graph the table entries and to try and find the height that gives the largest volume. The graph will not be a parabola this time. The result is shown in Figure 5.6.

This time, because of the lack of symmetry of the graph, it is not clear where exactly the maximum value is reached. In other words, it looks believable that somewhere between \( h = 3 \) and \( h = 4 \) an even larger volume is achieved than at \( h = 3 \). You could challenge the students to find an even larger value by choosing such values for the height. For example, they could try \( h = 3.5 \) or \( h = 3.3 \). In fact, it can be shown by more sophisticated methods that the precise maximum occurs when \( h = \frac{10}{3} \).

**Describing the Relationship using a Formula**

Once again, if you are dealing with a group of students who are ready for the algebra, you can do some of the calculations involved in this problem, especially those at the end, by developing a formula for the volume \( V \) of the box in terms of the height \( h \). Try to get them to find the formula themselves, using the thought processes discussed in connection with the two fenced garden problems. Since the bottom of the box is square with width equal to \( 20 - 2 \times h \), therefore the volume is given by the formula

\[
V = h \times (20 - 2 \times h)^2 \text{ or } V = h(20 - 2h)^2
\]
5.3 Enrichment Activity - Scale Changes and Areas and Volumes

Lesson Goals

- To discuss how enlarging a two-dimensional or three-dimensional figure affects its area and volume. The discussion can be thought of as a continuation of the discussion of similarity transformations in E.A. 3.1 and reinforced in E.A. 3.3.

- To help students obtain a good grasp of the general formulas for lengths, areas and volumes, as well as their units. For example, why are there 1,000,000 square meters in a square kilometer, when there are only 1,000 meters in a kilometer?

Materials

- copies of the graph paper templates in Appendix A.9 for each student

Problem Statement

This lesson has two parts. The first is an exploration of the relationship between length measurements and area measurements. There is a very important principle to be discovered here: When a two dimensional figure is “enlarged” by a factor $r$, by which we mean that all length measurements on the figure are multiplied by $r$, then the effect of this on the area of the figure is that it is multiplied by $r^2$. Similarly, we will see in the second part of the lesson that if we enlarge a three-dimensional object by a factor of $r$, then any area measurement of the figure (say the surface area) is multiplied by $r^2$, while the volume is multiplied by $r^3$.

Lesson Sequence

Length and Area

1. To explore this with the class, you should begin by asking the students to construct a table with four columns. The first column will indicate the area calculation of a square or a rectangle. For example, the entry $2 \times 2 = 4$ indicates a square of side 2, and area 4. The second entry indicates the factor by which (each length measurements of) the object in the first column is enlarged. Thus an entry of $\times 3$ indicates that the object in the first column is enlarged by the factor 3. Make sure the students understand what you mean by enlarging the object in this way. It is analogous to putting it under a microscope or to getting closer to the object so that it looks larger. Point out to students that when an object such as a square or a rectangle is enlarged $r$ times, it is not just the lengths of the sides that are multiplied by the factor $r$. Also the length of a diagonal and the perimeter are multiplied by the same factor. The entry in the third column represents the new area calculation. For example, if the entries in the first two columns are the ones used in the preceding as illustrations then this entry will be $6 \times 6 = 36$. The final column of the table will indicate the effect the enlargement has had on the area. In the case illustrated the last entry will be $\times 9$. Table 5.4 shows some possible entries.
5.3 Enrichment Activity - Scale Changes and Areas and Volumes

<table>
<thead>
<tr>
<th>Old area</th>
<th>Magnification</th>
<th>New area</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 × 2 = 4</td>
<td>÷ 3</td>
<td>6 × 6 = 36</td>
<td>÷ 9</td>
</tr>
<tr>
<td>5 × 3 = 15</td>
<td>÷ 2</td>
<td>10 × 6 = 60</td>
<td>÷ 4</td>
</tr>
<tr>
<td>4 × 7 = 28</td>
<td>÷ 2</td>
<td>8 × 14 = 112</td>
<td>÷ 4</td>
</tr>
<tr>
<td>3.1 × 1.4 = 4.34</td>
<td>÷ 0.3</td>
<td>0.93 × 0.42</td>
<td>÷ 0.09</td>
</tr>
</tbody>
</table>

Table 5.4: How area varies with size

You may want to stick to whole numbers initially, and include decimal expressions only gradually.

2. Once the students have entered sufficiently many calculations, ask them if they see a pattern. The pattern they should observe is that number in the fourth column is the square of the number in the second column. You could ask the students to make up an English sentence that expresses their observations. Something like “If the lengths are multiplied by a certain factor, then the area is multiplied by the square of that factor”. Challenge the students to express this algebraically. You could begin by adding a last entry to the table as shown in Table 5.5

<table>
<thead>
<tr>
<th>Old area</th>
<th>Magnification</th>
<th>New area</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 × 3 = 4</td>
<td>÷ r</td>
<td>2 × r × 3 × r = 6 × r²</td>
<td>÷ r²</td>
</tr>
</tbody>
</table>

Table 5.5:

Suggest the first two of these entries to the students, and then ask them to come up with the other two. In particular, help them to understand what it means to say that $2 \times r \times 3 \times r = 6 \times r^2$. They should understand that the letter $r$ in this equation is a variable used here to indicate that no matter what number $r$ is equal to, the equation is valid because order is not important when you are multiplying (multiplication is commutative).

3. To complete this discussion of the relationship between length and area you should ask the students to produce a graph representing the relationship between the second and fourth columns in the table. In order to do this you should make sure that you have included in the table at least a few values of $r$ less than one, for example $r = 0.5$ and $r = 0.7$, as well as some values between 1 and 2, such as $r = 1.5$ and $r = 1.2$. This will allow us to choose a scale for the graph on which the parabolic shape of the resulting curve will be most apparent. This has already been done on the graph templates of Appendix A.9. The resulting graph is shown in Figure 5.7.

4. If students plot the entries correctly, the points should be such that if they were connected by a smooth curve, the resulting graph would be a parabola as shown in the figure. You may want to discuss the parabola with the students, and relate the concept to parabolic reflectors in head-lamps and flashlights, as well as the parabolic shape of a satellite dish.
5. If the students complain that most of their data points will not fit on the graph because the numbers are too large, use this as an opportunity to show them how the appearance of a graph is altered by a change in the scale. Ask them to do a second graph, using the second of the templates provided for in Appendix A.9. If you ask the students to enter their data points on this graph you should get something like the set of points shown in Figure 5.8, though probably not as many.

**Length and Volume**

6. As a follow-up to the discussion of the relationship between length and area when a figure is enlarged, you can organize a similar discussion of the relationship of length and volume when a three-dimensional object is enlarged. In fact, to really make this a rich discussion, you can discuss the behaviour of length, area and volume at once. Whether to do all three immediately or to do length and volume alone at least initially should depend on the extent to which the students have understood the explorations so far.

7. This time we use cubes and rectangular boxes. If the students are stretched it is probably best to work with cubes only. If a discussion of area is to be included, the nicest area to choose for the purpose would then be the total surface area of the cube. In the following discussion we assume that surface area is not included in the exploration. Otherwise you should include three more columns under the headings “Old surface area”, “New surface area”, and “Area factor”. Alternatively, you can separately discuss the relationship between surface area and length once you have completed the discussion of volume.
8. Ask the students to start a table with four columns, as follows:

<table>
<thead>
<tr>
<th>Old volume</th>
<th>Length factor</th>
<th>New volume</th>
<th>Volume factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \times 1 \times 1 = 1$</td>
<td>$\times 2$</td>
<td>$2 \times 2 \times 2 = 8$</td>
<td>$\times 8$</td>
</tr>
<tr>
<td>$2 \times 2 \times 2 = 8$</td>
<td>$\times 3$</td>
<td>$6 \times 6 \times 6 = 216$</td>
<td>$\times 27$</td>
</tr>
<tr>
<td>$3 \times 3 \times 3 = 27$</td>
<td>$\times 1.5$</td>
<td>$4.5 \times 4.5 \times 4.5 = 91.125$</td>
<td>$\times 3.375$</td>
</tr>
</tbody>
</table>

Table 5.6:

Encourage them to select their own sizes for the cubes and their own enlarging (or shrinking) factors. Once again, after they have a fairly substantial table, encourage them to compose an English sentence describing the relationship between the second and fourth columns. Something like “When you enlarge a three-dimensional object by a factor, then the volume of that object is multiplied by the cube of that factor”. Once again invite them to translate this sentence into mathematics by adding a last row to the table like the one indicated next. You should supply the first two entries. They should be encouraged to fill in the last two. Of course, the numbers 2 and 8 can be replaced by any other number with its third power.
Perimeter, Area, and Volume: Linear Shapes

<table>
<thead>
<tr>
<th>Old volume</th>
<th>Length factor</th>
<th>New volume</th>
<th>Volume factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2 \times 2 \times 2 = 8$</td>
<td>$\times r$</td>
<td>$2 \times r \times 2 \times r \times 2 \times r = 8 \times r^3$</td>
<td>$\times r^3$</td>
</tr>
</tbody>
</table>

Table 5.7:

9. Once again, the results can be tabulated. Give them a copy of the third template provided in Appendix A.9 and ask them to tabulate the second and final columns. You will get a graph whose shape superficially resembles the graph found for the area of a two dimensional figure. In fact, however, this graph is flatter near 0 and is steeper past 1. To see the contrast, Figure 5.9 shows some of the points on these two graphs on the same scale. The solid dots represent the enlargement of area (e.g. surface area) while the circles represent volume.

Figure 5.9: The growth of volume and area contrasted
5.4 Enrichment Activity (Optional) - Using Areas to do Algebra

This lesson is optional. The material will be more interesting to students who have seen a fair bit of algebra than to those who have little or no introduction to it.

Lesson Goals

- To use areas to introduce the students to some basic algebraic formulas
- To get students to see certain geometric relationships, and to express these in a convenient manner
- To get the students to discover themselves how to use letters to say general things about how numbers behave. That the results sometimes turn out to be important algebraic facts is a convenient by-product.

Materials

- cardboard cubes as described in item 6. (could also use wood or styrofoam)

Lesson Sequence

This lesson is divided into five short problems:

Making a Patio

1. We begin with a brief review of the preceding lesson. This could be done in the form of a Spot Question or as an open class discussion. Here is the question:

   “Suppose you have a patio that is five meters long and four meters wide, and you decide to double both the length and the width. What effect will that have on the area of the patio? That is, how will the new area compare to the old? What if the original dimensions of the patio were 3 and 7 before you doubled? Or 6 and 4?”

   Try to get the students to compose an English language sentence that summarizes our findings. It should be something like “If you double the length and the width of a rectangle, then the area of that rectangle is multiplied by 4”. Now ask what would happen to the area if you tripled both the length and the width? What if you multiplied both the length and the width by one-half? Try to get the students to formulate the rule that if the dimensions are multiplied by \( m \) then the area gets multiplied by \( m^2 \).

2. Suppose the patio is \( a \) meters long and \( b \) meters wide, what is the area equal to? Try to get the students to suggest that the area should then be expressed as \( a \times b \). Now ask the students how the area should be expressed if the patio is \( b \) meters long and \( a \) meters wide? Notice that if the first question has the answer \( a \times b \) then the answer to the second question should be \( b \times a \). Undoubtedly the students know without seeing the question put this way that \( a \times b = b \times a \). However, when the result of the two calculations is a
geometric quantity that clearly is not affected by the order, the answer is intuitively clear to a degree not possible if the equality is seen as expressing a purely algebraic fact.

Point out to the students that, when we use letters to represent numbers, then we often write $ab$ in place of $a \times b$. Thus we have a rule that says $ab = ba$. In the same way, making a patio $m$ times as long and $m$ times as wide changes $ab$ to $m^2ab$. That is, $(ma)(mb) = m^2ab$.

**Volume of a Block**

3. Suppose we have a rectangular block of width $a$ meters, length $b$ meters and height $c$ meters. What will the volume be equal to? How will the volume change if the dimensions of the rectangle are doubled? What if the dimensions were tripled? Once again, try to get the students to formulate a general rule. It should come out something like “if you multiply the length, width, and height each by $m$ then the volume is multiplied by $m^3$.”

As an added challenge you can ask the students to describe the effect on the surface area. The answer should be something like “If you multiply the three measurements by $m$ then the surface area gets multiplied by a factor of $m^2$.” Notice how areas are associated with second powers, even if the areas are measured on three-dimensional objects.

**The Square of a Sum**

4. Suppose we have a square whose length and width are both equal to $a + b$. What formula would give the area of the square? As always, try to get the students to give the answer. The more they are allowed to come up with the formulations themselves, the more the formulas will seem natural and intuitive to them. The answer is, of course, $(a + b)^2$. Now suppose we remove from this square a smaller square of side $a$ and another of side $b$, as shown in Figure 5.10

![Figure 5.10: A diagram showing the formula for the square of the sum of two numbers](image)

5. What is the area of the pieces that are left? It is easy to see that they are each $ab$ square units. Ask the students to formulate our discovery in a sentence. Something like this would be acceptable: “If you have a square of area $(a + b)^2$ then that square (and thus its area) can be broken up into four pieces, namely a square of area $a^2$, a square of area $b^2$ and
two rectangles of area $ab$ each.” Challenge the students to turn this sentence about areas into an equation (that is, a mathematical sentence). The equation we get is the following:

$$ (a + b)^2 = a^2 + 2ab + b^2. $$

Though this formula is obtained by means of geometry, it is valid even if the numbers have no reference to geometric quantities at all. Do some examples with the students. For example, let $a = 3$ and $b = 5$, or let $a = 1.5$ and $b = -4$. In each case compare the two sides of the formula.

A Three Dimensional Version

6. If you feel ambitious you could challenge the students to come up with a three dimensional version of this formula. The idea is that they should think of a cube whose sides are each of length $a + b$. On the one hand, the volume of the cube will then be $(a + b)^3$. On the other hand, if they place a cube of length $a$ in one corner (say a bottom corner) of the large cube and one of size $b$ in the opposite corner, what is the volume of the rest of the cube? This will be a really good exercise in geometric imagination. Ask the students to draw the large cube with the smaller ones inside it, and then to draw the remaining shape(s) and to indicate their dimensions. If you have the time and the resources, you could prepare yourself for this lesson by making, say out of cardboard, a large cube and two smaller cubes of different side lengths, say $a$ and $b$, whose edges add up to the edge of the large cube, $a + b$. You could also make various rectangular boxes of sizes $a \times b \times b$ and $a \times a \times b$ that can be combined with the smaller cubes to form the larger cube (see the next point). However, even if you do this, I would not bring the models out until the students have tried to visualize the situation through drawings.

7. Suppose the smaller cubes are placed inside the large cube in opposite corners as suggested earlier. Then we can think of the interior of the large cube as made up of two layers, the bottom layer of height $a$ containing the cube of volume $a^3$ in one corner, and the top layer of height $b$ containing the cube of volume $b^3$ in the other corner. Then there will be a rectangular space in the bottom layer directly below the cube of volume $b^3$. This space will have width and length $b$ and height $a$. That is, its volume is $ab^2$. The other two spaces that make up the bottom layer are also rectangular, and both have width and height equal to $a$ and length equal to $b$. Thus these two spaces each have volume $a^2b$. If we do the same sort of thing to the top layer, we find that in addition to the cube of side $b^3$ it contains two rectangular spaces of volume $ab^2$ and one space of volume $a^2b$. If you add all those pieces together, you should get the volume of the large cube, so you should come up with the formula

$$ (a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3. $$

For example, to calculate $13^3$ we could think of it as $(10 + 3)^3$ and use the formula. Try the formula with several choices of $a$ and $b$. 
Difference of Squares

8. In the next exercise the students will be asked to work out a formula for

\[ a^2 - b^2. \]

You could challenge the students to find another way to express this difference, by combining two squares. Hopefully, the students will eventually come to the conclusion that they should draw a square of length \( b \) inside a square of side \( a \). For the moment at least, the students should assume that \( a \) is the larger of the two numbers. The challenge is to find a reasonably simple expression for the area left over when the smaller square has been subtracted from the larger. Figure 5.11 shows a picture that suggests a good way to resolve the issue: The rectangle on the upper left of the diagram is moved to the right and turned to attach to the bottom rectangle. Note that it will fit precisely. Get the students to work out the dimensions of the rectangle created in the second picture, and discover the formula

\[ a^2 - b^2 = (a + b)(a - b). \]
5.5 Enrichment Activity - The Shadow Problem

Lesson Goals

- To challenge students to visualize shadows from different perspectives and to think about the behaviour of light
- To further develop students’ problem solving abilities and application of known formulas
- To review the formulas for the areas of parallelograms and triangles, and to see that the areas do not change when the top of a figure is moved in a direction parallel to the base while keeping the base fixed

Materials

- copies of the Spot Question in either Appendix C.10 or C.11 for each student
- several cardboard boxes together with rectangular pieces of cardboard fashioned so that they can stand on their sides, one set for each group of 3-5 students
- flashlight
- meter sticks or other long thin sticks to simulate light rays

Problem Statement

The owner of a rectangular building set 5 meters back from the edge of the sidewalk, is concerned about graffiti on a wooden fence, 5 meters wide and 2 meters high, set along the edge of the sidewalk. His building is 4 meters high. He wants to attach a bright light to the front wall of his building so that as much of the sidewalk in front of his fence is lit as possible. Where should he locate the light, and what is the area of the shadow produced in front of the fence? Figure 5.12 shows a picture of the building and the fence.

Lesson Sequence

1. A nice way to begin this session is with a Spot Question that requires students to calculate the area of a triangle (either by using a formula or by fitting triangles into unit squares) because this will be needed to solve today’s problem. The Spot Questions in Appendix C.10 and C.11 are both good examples. (If using the latter, students should be encouraged not to use any formulas.)

2. In addition to the area of a triangle, students will also need to know how to calculate the area of a parallelogram. They will almost certainly have seen these formulas. However, you should be prepared to review the formulas with them if that becomes necessary. Keep in mind, though, that the main goal of the lesson is to teach the students how to use the formulas in a problem, and generally to get the students to think about shapes. If you feel a review is necessary, you will find a suggested discussion with exercises at the end of this lesson.
3. Divide the class into small groups (3-5 students) and distribute cardboard models of the building and fence to each group. Make sure the students are well equipped with pencils and paper, and let them discuss the problem as a group, or in groups, with the help of the cardboard models. The students will soon offer the opinion that to make the area of the shadow as small as possible the light should be placed as high as possible. This is, of course, correct. At this point you should challenge them to sketch or describe the shape of the shadow that will be produced. This is a really good exercise in three-dimensional visualization, and in thinking about optics. You will probably find that some of the students are not totally clear on the behaviour of light. You will find that while they will all agree that lights travels in straight lines, some of them will assert, paradoxically, that if you increase the power of the light bulb, the shadow will become smaller. Almost certainly, most of them will find it difficult at first to connect the straight light rays with the edges of the fence and the shape of the shadow. It is best to persist as long as possible in having the students rely only on the model, their paper and their minds. Keep the flashlight hidden, for if you demonstrate with a flashlight what the shape will be, the students will simply accept the evidence.

4. Identify a location on the upper edge of the front wall of the building, say directly behind the centre of the fence, where the light bulb is located. Tell the students that light travels in straight lines, and that they should imagine light rays issuing from the bulb in all directions. Some of these shine into space, others hit the back of the wall, while others just miss the wall and illuminate the sidewalk and the street. Try to get the students to see that the light rays that matter in determining the shape of the shadow are the ones that just miss the side- and top edges of the fence. These are the lines that determine the outline of the shadow. Eventually, the students should come up with the picture shown in Figure 5.13. If they continue to have difficulties, use the flashlight. In any case, use the flashlight to confirm the picture once it has been understood. In the end they should be able to identify the shape of the shadow as a trapezoid, with the fence as the smaller of
two parallel sides.

5. The next stage of the problem involves finding the area of the shadow. Remind the students of the measurements of the fence and the building, and the distance between the building and the fence. If students ask how wide the building is, tell them that they should think of it as “quite long”. In the end we will find that the location of the bulb, and therefore the length of the building, does not affect the area of the shadow. Ask the students to produce a side view, a top view, and a front view (that is, facing the fence from the street), in that order. Ask the students to enter as many measurements as they can on their diagrams.

6. Students will probably discover that they should think of the light rays that graze the top of the fence. Figure 5.14 shows the side view.

![Diagram](image)

Figure 5.14: A side view of the light rays, the building and the fence

Students will notice that the light rays that graze the top of the fence have slanted down (from an initial height of 4 meters to a height of 2 meters) over a horizontal distance of
5 meters (the distance between the building and the fence). These same light rays will therefore need an additional 5 meters to get down to street level. This means that the distance between the fence and the outer edge of the shadow is also 5 meters.

**Note:** This can also be seen by thinking about the side view in terms of similar triangles, which is nice way to revisit and apply the material of Chapter 3. If the little triangle (whose base is the shadow) is enlarged by an appropriate factor, we get the larger triangle (comprising the entire blank region to the left of the rectangle representing the building). What is this ‘appropriate’ factor? We can answer this question by comparing the two vertical sides of the triangles. The larger, at 4 meters, is 2 times the smaller. This tells us that the enlargement factor is 2. Therefore all linear measurements of the larger triangle must be 2 times that of the smaller. By comparing their sizes, we conclude that the smaller triangle must have a base of 5 meters and the larger a base of 10 meters.

7. Seen from the top the light rays form a cone, as shown in Figure 5.15.

![Figure 5.15: The top view of the light rays and the fence](image)

To determine the relative sizes of the shaded part of the cone and the top part, the students should look at the side view again. From that they can see that the distance between the two parallel edges of the shadow is 5 meters. It remains to find the measurement of the length of the longer of the two parallel sides of the trapezoid. Once again the students will be aided by their sense of things happening in proportion. From the light bulb to the fence, over a distance of 5 meters, the cone widens out from a point to a width of 5 meters (the width of the fence). Since the distance from the fence to the outer edge of the shadow is now known to equal 5 meters as well, they will conclude that by that time the cone will have widened by another 5 meters. That is, the outer edge of the shadow is 10 meters long. Again, a discussion of similar triangles can be used to obtain this result.

8. Thus the problem of finding the area of the shadow has been reduced to one of finding the area of a trapezoid of height 5 meters, and parallel sides of lengths 5 meters and 10 meters. The students will probably not know a formula for the area of a trapezoid. In fact it is better that they do not. Challenge them to find a way to calculate the area. They will soon notice that the trapezoid can be split up either into two triangles and a rectangle or (the better alternative) a parallelogram and a triangle. Figure 5.16 shows the pictures. Either way, using the formulas for areas of triangles, parallelograms and rectangles, they will discover that the total area of the shadow is 37.5 square meters.
9. To finish the lesson you should ask the students how the shape and the area of the shadow will change if the light bulb is moved (horizontally) along the top edge of the building. The cone of light rays (seen from the top) that graze the top edge of the fence will become distorted, as shown in Figure 5.17.

However, the rate at which these light rays slope down is the same as before (2 meters over a distance of 5 meters measured perpendicular to the street. This means that they will reach street level at a distance of 5 meters beyond the fence, forming the outer edge of the shadow. This also means that by that time the cone of light rays has reached a width of 10 meters (twice the width of the fence. In other words, the shadow is a trapezoid of height 5 meters and parallel sides of lengths 10 meters and 5 meters. This time the trapezoid does not have the symmetry it had when the light bulb was located behind the center of the fence. In fact, the new shadow is related to the symmetric shadow by a shear, as in the following Figure 5.18.
This review is optional, and is included mainly for completeness. It is generally better to begin with the shadow problem and review and discuss the area formulas as they are needed to solve the problem. You will need to use your own judgement.

Parallelograms

1. Ask the students how they would define a parallelogram. Review with them the fact that the opposite sides of a parallelogram are parallel (hence the name), and that they are then also automatically equal in length. Review with the students how parallel lines are indicated on a diagram and how equal line segments are indicated. Assuming that the students remember that the area of a parallelogram is given by the formula $\text{area} = \text{base} \times \text{height}$, ask them what they mean by the height and make sure they distinguish the height from the length of the second pair of sides. For example, in the parallelogram in Figure 5.20, the height is the length $b$ of the dotted line.

2. Ask the student how they know that the area of a parallelogram is given by the formula $\text{area} = \text{base} \times \text{height}$. Ask them to remember how the teacher taught them this formula. The secret behind the formula for the area of a parallelogram is to convert the parallelogram into a rectangle. This is best shown by means of the diagram shown in Figure 5.20.

In this diagram, one of the sides of the parallelogram is designated the base. Suppose its length is $a$ units. For the construction of the rectangle we drop a perpendicular from the upper left hand corner to the base. You could discuss the expression “dropping a perpendicular” with the students. As mentioned already, the length of that perpendicular is what we mean by the “height” of the parallelogram. The height could also be described...
as the perpendicular distance between the base and the top side of the parallelogram. Make sure the student notice that the height is not the same as the length of the other side! The two are equal only when the parallelogram happens to be a rectangle. Now we take the triangle to the left of the perpendicular, and move it to the opposite side of the parallelogram, as shown in the Figure 5.21.

![Figure 5.21:](image)

The students will see that the resulting rectangle is $a$ units wide and $b$ units high. Therefore the area of the rectangle is $ab$ square units. But that means that the area of the parallelogram is also $ab$ square units. In this way we have obtained a formula for the area of a parallelogram:

> If a parallelogram has a base that is $a$ units, and a height that is $b$ units, then the area of the parallelogram is $a \times b$ square units.

3. Make sure that the students notice that this formula does not say that the area is equal to the product of the lengths of the sides of the parallelogram! Note as well, that at the start we could have chosen to put the parallelogram on its side and treated the other side as the base. Then we would have a different height as well. Nevertheless, it is a consequence of the theory that the product of these two numbers will give the same answer as the product of the first pair.

4. This discussion points to another very interesting fact about areas of parallelograms: Since parallelograms with the same base and the same height have the same area, therefore if you “shear” a parallelogram in a direction parallel to the base, you will not change its area. The areas of the three parallelograms shown in Figure 5.22 are all the same.

![Figure 5.22: All three parallelograms have the same area](image)
This picture indicates what we mean by a shear. The students probably know about translations and rotations as transformations that do not change shapes. The same can be said about reflections, except that they change the ‘orientation’ of a figure. A shear changes the shape of a figure, for example, a rectangle becomes a parallelogram, but a shear does not change areas. While the effect of a shear can be pictured by imagining a cardboard box from which the top and bottom have been removed, and which is then put on its side and pushed at one of the corners, this picture is not totally correct, for when you do this to a cardboard box, the height of the rectangle decreases when you shear it. A ‘shear’ in the mathematical sense we use here, does not do that.

5. Here is another picture illustrating the shear transformation. Each of the four parallelograms shown in the large letter W shown in Figure 5.23 can be obtained from the others by a succession of shears.

![Figure 5.23: Each part of this letter W has the same area](image)

To help the students see clearly that a shear does not change the area of a parallelogram, you could draw Figure 5.24 on the board, and ask them to tell you which of the two parallelograms has the larger area.

![Figure 5.24: Which has the larger area?](image)

The answer if of course that the areas are the same, as the slanted parallelogram is obtained from the rectangle by a shear. Another way to make this clear is by pointing out that the two parallelograms have the same base and also have the same height. And, yes, it is correct to refer to a rectangle as a parallelogram, for a rectangle is a parallelogram with special properties (right angles).
5.5 Enrichment Activity - The Shadow Problem

Triangles

6. The students will undoubtedly know that the area of a triangle is given by the following formula:

The area of a triangle is equal to one half the product of the base and the height.

Thus the area of the triangle shown below is equal to $a \times b \div 2$.

![Figure 5.25: The area of a triangle](image1)

7. Ask the students again to explain why this is the correct formula for the area of a triangle. They will almost certainly be able to tell you. However, if they have forgotten, Figures 5.25 and 5.26 show the simplest way to do this:

![Figure 5.26: Explaining the formula for the area of a triangle](image2)

Take two copies of the triangle in Figure 5.25, invert the second one and place it against the first. The result is the parallelogram of base $a$ and height $b$ shown in Figure 5.26. The area of the parallelogram is $a \times b$, according to the preceding discussion. Therefore the area of the triangle is half that.
5.6 Problem Set - Exercises Involving Area

Goals

- To consolidate some of the things students have learned about the areas of parallelograms and triangles
- To challenge students to use their basic understanding of area and ratios to calculate the areas of various shaded regions
- To see how Pythagoras’ Theorem is needed as a first step in many of the problems

Note that, while this problem set can be used to make up a separate lesson, the problems can also be used to supplement the other enrichment activities in this chapter.

Introductory Exercises

You could start by doing some exercises with the students at the blackboard, to get them ready to do a set of problems at their desks. Ask the students to find the areas of the shaded regions in the following pictures.

1. Beginning with the first triangle in Figure 5.27, the students simply have to notice that the height of the triangle is 8, and that they can use Pythagoras’ theorem to calculate the base. In fact, the base has length $\sqrt{10^2 - 8^2} = \sqrt{36} = 6$. Thus the area of this triangle is $6 \times 8 \div 2 = 24$.

2. For the second triangle, the students have to notice that it is easier to analyze if it is first rotated, so that the side of length 5 becomes the base. The other right angle side then becomes the height, so once again we must use Pythagoras’ theorem to find its length. This will give us $\sqrt{8^2 - 5^2} = \sqrt{39} = 6.245$. Thus the area of this triangle is $5 \times 6.245 \div 2 = 15.612$.

2. In the first triangle of Figure 5.28, we need to first notice that the shaded triangle has the same area as the other triangle whose base is one half of the base of the large triangle. The reason is that the base of the large triangle has been divided into two equal pieces, so that the bases of these two triangles have equal length. Furthermore, because they share the
same apex, their heights are also automatically equal. We can calculate the base of the large triangle by means of Pythagoras: It comes to \( \sqrt{13^2 - 12^2} = 5 \). Therefore the base of the shaded triangle has length 2.5. Its height is 6, so the area of the shaded triangle is equal to \( 2.5 \times 6 \div 2 = 7.5 \).

![Figure 5.28:](image)

The second triangle of Figure 5.28 presents more of a challenge. It is easy to see that the area of the large triangle is equal to \( 3 \times 4 \div 2 = 6 \), by rotating the triangle to make the side of length 3 its base. We would be able to finish the problem if we knew the ratio between the area of the shaded triangle and the total area. Since (with the large triangle rotated back to its original position) those two triangles share the same height, the ratio of their areas is equal to the ratio of their bases. Thus, if we knew the length of the base of the large triangle, we could finish the question. So let us use Pythagoras’ theorem to calculate this third side of the large triangle. It is easy to see that the answer is \( \sqrt{3^2 + 4^2} = 5 \). Thus the ratio of the area of the shaded triangle to the entire triangle is \( 2 : 5 \). In other words, the area of the shaded triangle is equal to \( (2/5) \times 6 = 2.4 \).

3. Next ask the students to determine the ratio between the area of the whole triangle and the shaded portion in triangle of Figure 5.29.

![Figure 5.29:](image)

This question is entirely about ratios of areas. First of all, you note that since the base of the triangle is divided into two equal parts, the area of the shaded triangle is equal to...
the area of the triangle adjacent to it. On the other hand, if we rotate the picture so that the side on the right becomes the base of the large triangle, we see that the two smaller triangles together make up one half of the area of the whole. Thus the whole triangle is four times as large as the shaded part.

After discussing the preceding problems with the class at the board, it is time to give them some to do themselves. Here is a suggested list of questions, with solutions. The list without comments is available in the Appendix as Section ??.

◊◊◊

Solutions to Problem Set

Next, we present the solutions to the exercises of the problem set found in Appendix B.2.

1. Find the areas of the shaded regions:

Figure 5.30: Question 1
In the first figure, the shaded area is a parallelogram with base 1 and height 2, so its area is 2.

In the second figure, the shaded area consists of two parallelograms which, if you think of them on their sides, both have base equal to 1. We do not know their heights, but we do know that the heights add up to 3. Since the base is equal to 1 in each case, each parallelogram has area equal to its height. Thus the sum of the two area is equal to the sum of the two heights, giving a total value of 3.

In the third figure we see three parallelograms which each have a base of length 1 and also have the same height (=3). You can think of the second parallelogram as having been produced from the first (the rectangle) by a shear, and the third produced from the second by another shear. In any case, the area of the shaded region is the same as the area of the rectangle, namely 3 square units.

The last figure can be analyzed in several different ways. A very simple analysis goes as follows: If you shear the shaded parallelogram along the higher slanted line, you can deform it into a parallelogram with base 2 and height 2. Thus the area is 4 in this case.

2. Find the areas of these triangles:

![Figure 5.31: Question 2 - first three triangles](image)

- The first triangle is equilateral. To find the area we need to calculate the height of the triangle, so draw a perpendicular from the bottom vertex to the horizontal line at the top, which we will think of as the base. This divides the triangle into two right-angled triangles. Pick one of these, and note that its hypotenuses has length 2 and that one of its right-angle sides has length 1. By Pythagoras’ theorem we find that the other right-angle side (which is the height we are trying to find) is equal to
  \[ \sqrt{2^2 - 1^2} = \sqrt{3}. \]

Thus the area of the equilateral triangle is \((1/2) \times 2 \times \sqrt{3} = 1.732.\)
• The second triangle is a right-angled triangle. If we think of the left side as the base, then the area of the triangle is \((1/2) \times 3 \times 4 = 6\).

• The third triangle is isosceles. We use the same method as in the first of these three triangles. Draw a perpendicular from the vertex at the top to the middle of the base (at the bottom of the picture this time). This divides the triangle into two halves, each a right-angled triangle with hypotenuse equal to 3 and one of the right-angle sides equal to 1. The other right-angle side, which is also the height of the original triangle, is therefore equal to

\[
\sqrt{3^2 - 1^2} = \sqrt{8}.
\]

Thus the area of the original triangle is equal to \((1/2) \times 2 \times \sqrt{8}\).

3. Find the shaded areas in Figure 5.32

\[
\text{Figure 5.32: Question 2 - the last two triangles}
\]

• In the first figure, the area of the large triangle is found by thinking of the side of length 3 as the base. The side of length 4 is then the height. Thus its area is equal to \((1/2) \times 3 \times 4 = 6\).

Now the shaded triangle and the remaining blank triangle have equal base lengths, and have the same height, since they share the top vertex. Thus these two triangle have the same area. That means that the shaded triangle has half the area of the whole, so the area of the shaded triangle is equal to 3.

• In the second figure, the area of the whole triangle is equal to \((1/2) \times 3 \times 5 = 7.5\). Now if we compare the shaded triangle and the remaining triangle, we note that they have the same height, but that the base of the shaded triangle is twice as long as the base of the blank triangle. Therefore its area must also be twice a large. In other words, the area of the shaded triangle is two-thirds of the area of the entire triangle, and is therefore equal to 5 square units.
4. Calculate the surface area of an octahedron whose edges are 2 cm long.

In the first part of question 2 we discovered that an equilateral triangle of side 2 cm has area equal to $\sqrt{3} = 1.732\ldots$. The surface of the octahedron consists of eight such triangles, so its area is equal to $8 \times \sqrt{3} = 13.856\ldots\text{cm}^2$.

Extension

As an extension, you could review again the effect of enlargements on areas. Discuss with the students what will happen to a triangle if all its dimensions (i.e. length measurements) are enlarged by a factor of 2. Point out to the students that because both the base and the height will be multiplied by 2, the area will be multiplied by a factor of 4.

As we saw in an earlier lesson, it is a property of areas of any sort, as opposed to volumes, that they will be enlarged by a factor of $m^2$ if the length measurements are enlarged by a factor $m$. One way to see this is to remember that areas are always a count of how many squares of unit measurement will fit into the region. If you imagine these squares (and, where necessary, pieces of squares) drawn in, then enlarging the figure by a factor of $m$ will enlarge the area of each of the little squares by a factor of $m^2$. That is, each of the enlarged squares will now contain $m^2$ unit squares. Thus the number of unit squares required is multiplied by $m^2$. This property of all areas to grow by the square of the factor by which the figure is enlarged is central to the discussion of circles in the next chapter.
5.7 Problem Set - More Exercises Involving Area

Goals

- To further consolidate what has been learned about area
- To test the students’ understanding of Pythagoras’ Theorem, as well as the formulas for the areas of parallelograms and triangles

Once again, doing this problem set as a separate session is optional. You may feel that the students understand the material well enough, or you may want to use some of the questions as part of the other enrichment activities in this chapter.

Solutions to Problem Set

A copy of the problem set, without solutions, is found in the Appendix B.3. Here we outline the solutions.

1. Find the areas of the following figures:

![Figure 5.33](image)

- The first of these figures represents a parallelogram of base 7 and height 3, so its area is 21 square units.
- The parallelogram in the second figure has base 4 and height 5, so its area is equal to 20 square units.
- The triangle in the third figure has area \((1/2) \times 5 \times 8 = 20\) square units.
• The first figure represents a triangle with area equal to 3, as can be seen by rotating it through 108 degrees first.

• The second triangle has base 1 and height 3, so it has area equal to 1.5.

• The parallelogram has base 7. Its height can be calculated by means of Pythagoras’ theorem applied to the triangle contained in the parallelogram. It has the value $\sqrt{5^2 - 3^2} = 4$. Thus the area of the parallelogram is equal to 28.

• These two figures are trapezoids, four sided figures that have two sides parallel. The area of a trapezoid is found by drawing a diagonal line through the figure. This divides it into two triangles, whose bases lie along the two parallel sides, and whose height is the distance between the parallel sides.

• In the first trapezoid this creates a triangle of base 5 and height 4 together with a triangle of base 3 and height 4. Thus the area of the trapezoid is \[\frac{1}{2} \times 5 \times 4 + \frac{1}{2} \times 3 \times 4 = 16\] square units. Note that the observation made in this problem can also be expressed by
saying that the area of the trapezoid is equal to the average of the lengths of its two parallel sides multiplied by the distance between the parallel sides.

- In the second trapezoid, the parallel sides are the vertical sides, so we get two triangles with heights 8 and basis lengths 4 and 8 respectively. Thus the area is \[ \left\lfloor \frac{1}{2} \times 4 \times 8 \right\rfloor + \left\lfloor 2 \times 8 \times 8 \right\rfloor = 48 \text{ square units.} \]
2. Find the areas of the shaded regions:

- In the first of these figures the shaded triangle and the rectangle share the left side, which can be regarded as the base of both figures. Then the heights are also the same. This implies that the area of the triangle is one half the area of the rectangle. That is, the area of the triangle is 6.

- In the second figure, the shaded parallelogram has base equal to 1 (put the figure on its side to see this) and height equal to 4. Therefore its area is equal to 4.

- In the first figure in the second row, the big triangle divides into three smaller triangles, all of the same height. To calculate this height we need to calculate the remaining right-angle side of the large triangle. This is done by Pythagoras’ theorem, and is equal to $\sqrt{5^2 - 4^2} = 3$. Thus the shaded triangle has height 3 and base 2. Its area must then be 3 square units.

- In the second figure in the bottom row, the rectangle has area $3 \times 4 = 12$. Therefore the large triangle above the diagonal has area equal to 6. If we regard the diagonal line as the base of this triangle, we see it divided into three triangles with equal bases and a common vertex, one of which is the shaded triangle. Thus the area of the shaded triangle is $6 \div 3 = 2$. 

Figure 5.36:
Chapter 6

Perimeter and Area: Circles

He made the sea of cast metal, circular in shape, measuring ten cubits from rim to rim and five cubits high. It took a line of thirty cubits to measure around it.

2 Chronicles 4:2,
The Bible, New International Version

Purpose of Chapter

In this chapter, we continue our discussion of area and perimeter, this time focusing specifically on circles. As with the areas of rectangles and triangles, we expect that most students will know the formulas for the area and the circumference of a circle. However, their ability to connect the value of the area given by its formula and the number of unit squares that will fit into it is especially lacking in the case of the circle and other non-linear shapes. Students will almost certainly not know why the area of a circle depends on the square of the radius while the circumference depends only on the first power of the radius. In fact, students may be unclear about the relationship between circumference and area. With that said, the main goals of this chapter are:

1. To engage students in non-standard discussions regarding the formulas for area and circumference of a circle, in particular, to the presence of $\pi$ in these formulas
2. To stimulate interest in the number $\pi$ itself and its origin
3. To give students practice calculating areas and perimeters of circular objects, both using standard formulas and conceptual thinking
4. To challenge students to use their knowledge and understanding of area and circumference to solve interesting problems.

Overview of Activities

- E.A. 6.1 - $\pi$ and the Area of a Circle
  A challenging topic that introduces students to the number $\pi$ (defined as the area of a
circle of radius one) and how it arises in the formula for the area of a circle.

- **E.A. 6.2 - $\pi$ and the Circumference of a Circle**
  This lesson introduces the formula for the circumference of a circle and connects this to the area of a circle in two ways. Both of the methods anticipate the concept of a limit, as well as other ideas of differential calculus, but do this in a manner that is accessible to good students at this stage of their schooling.

- **E.A. 6.4 - Measuring the Thickness of Toilet Paper**
  A fun problem for students that challenges them to use their understanding of area and circumference to measure the thickness of a given roll of toilet paper!

- **Problem Sets 6.3 and 6.5**
  In addition to the main lessons, there are two problem sets that can be spread across several weeks. These are designed to strengthen the students’ ability to do calculations involving areas and circumferences of circles. The second of these sets is devoted entirely to word problems.
6.1 Enrichment Activity - $\pi$ and the Area of a Circle

Lesson Goals

- To introduce the number $\pi$ as a number representing the area of a circle of radius 1
- To estimate the value of $\pi$ using squares, hexagons, and other n-gons
- To use knowledge of the effects of scale changes on area to understand the presence of $r^2$ in the formula for the area of a circle of radius $r$

Materials

- copies of Spot Question in Appendix C.14 for each student
- computer with internet access (see item 10.)

Problem Statement

Simply stated, the problem is to calculate the area of a circle of radius $r$ by first calculating the area of a circle of radius 1.

Background

This is a challenging topic. Students will have learned that the area of a circle is equal to $\pi r^2$, without making the connection between this formula and the exercise of fitting squares or other simple shapes into the circle and counting how many it takes. In particular, the number $\pi$ exists only as received and unmotivated wisdom. It is probably not even clear to them that $\pi$ is a number actually representing a particular magnitude. In a sense, this is the least satisfying of the lessons on area, for even though we do engage in some really good discussion about the formula for the area of a circle, it is unfortunate that in a course that means to rouse students’ skepticism and tries to persuade them of the importance of proofs, we have to concede that a complete derivation of the area of a circle is far beyond the level of a grade 8 student. In fact, few graduating university mathematics students are able to give an adequate account of the origins of the number $\pi$. We will investigate the value of $\pi$ to the point of obtaining a very rough estimate, but in the end we will have to set aside our lofty purposes and ask the students to accept the answer. However, we can at least make sure that the students realize that there is something to check, and that they can see how, in principle, this can be achieved.

Lesson Sequence

1. This lesson starts nicely with a Spot Question asking students to estimate the area of some circular (or non-linear) shape superimposed on a grid. The Spot Question in Appendix C.14 is a nice example. The idea is to get a sense of whether the students will jump to the use of some formula or whether they will use their understanding of estimating area using unit squares.
2. This question leads nicely into a discussion about the area of a circle. Unlike the parallelogram and the triangle, the circle is an awkward figure when it comes to working out its area. If you give it a try, you will soon realize that no matter how much cutting and reassembling you do to a circle, you will never get it to look like a rectangle, parallelogram or triangle. This makes it difficult to find the area of a circle in terms of the figures we have studied so far.

3. However, there is one observation that simplifies things for us. Suppose that somehow we were able to work out the value for the area of a circle whose radius is equal to one unit. Since this area must be the same no matter where the circle is located, we could start by giving a name to the number we are looking for. Let us agree to call that number $\pi$. Explain to the students, if they do not know this already, that this is a Greek letter, and is pronounced “pie.” Since the students may not have much experience with letters used to denote numbers, it would be good to dwell on the nature of what is happening here. We know the area of a circle of radius one has a certain definite value, but we don’t know what that value is. Nevertheless, in anticipation of finding out more about it, we have given the number the name “pie”. In fact, even once we find out something about the value of the number, it will be convenient to keep the Greek letter, for the decimal expression for the number is very complicated. Note that it is not correct to speak of $\pi$ as a variable. It is the name we give to a specific number we are trying to find out about.

4. Suppose, then, that the area of a circle of radius one is $\pi$ square units. Now suppose we were to enlarge the circle by a factor of $r$. What would that do to the radius? The students will probably agree that this changes the radius from 1 to $r$. But when length measurements are multiplied by a factor $r$, then the area is multiplied by $r^2$. We saw this on several occasions in the discussions in previous lessons. Thus, a circle of radius $r$ automatically has its area equal to $\pi r^2$. This is a good thing to know, for it means that the whole problem of finding a useful formula for the areas of circles, boils down to finding the area of a circle of radius one; in other words, to the problem of finding the value of our mystery number $\pi$.

**Estimating $\pi$**

So how do we calculate $\pi$? This turns out to be difficult, and a complete answer is certainly beyond a treatment at the level of this course. However, there are some things we can say. The next sections explore how $\pi$ can be estimated using inscribed and circumscribed polygons. There will likely not be enough time to cover the four examples that follow. However, you should at least cover the case of the using squares, as well as discuss the process of getting better and better estimates.

**Using Squares**

5. Consider the diagram in Figure 6.1. In it we see a circle of radius one, together with a circumscribed square and an inscribed square. Ask the students to calculate the areas of the two squares.
By using the Theorem of Pythagoras to find the area of the inner square, the students should find that it has area equal to 2 square units, while the outer square has 4 square units. Ask the students what that tells them about the number $\pi$. You may have to help them remember that they should think of the number $\pi$ as the area of the circle. Since the smaller square is inside the circle, and since the larger square contains the circle, we have discovered that the mystery number $\pi$ is somewhere between 2 and 4. It would be nice if it were exactly equal to 3, but alas this is not the case.

Using Hexagons

6. We can improve our estimate by using hexagons instead of squares, as shown in Figure 6.2. In this diagram, we see that the inscribed hexagon is made up of six equilateral triangles. The fact that the triangles are equilateral is a consequence of the fact that the sum of the angles in a triangle is always $180^\circ$. (Students will almost certainly be aware of this already, so we can assume that they know this.) Ask the students what the length of the sides of these equilateral triangles is. Of course this length is equal to the radius of the circle, and is therefore one unit. Now challenge the students to figure out a way to determine the area of one of these equilateral triangles, using Pythagoras’ Theorem.
The best way to do this is to draw a line segment representing the height of the triangle, as in the next picture:

![Figure 6.3:](image)

If we then apply Pythagoras’ theorem to either of the two smaller triangles, we will see that the height of the triangle is equal to $\sqrt{3}/2 = 0.866$. Some students may remember from E.A. 3.2 that the height of an equilateral triangle is always $\sqrt{3}/2 (\approx 0.866)$ times the side length. Now that we know both the height and the base, we can calculate the small triangle’s area: $(1/2) \times (1/2) \times 0.866 = 0.2165$. Therefore, the area of the inscribed hexagon is 12 times this amount, $0.2165 \times 12 = 2.598$. Thus we have discovered that the value of $\pi$ is greater than 2.598.

7. By calculating the area of the circumscribed hexagon next, we will find a number somewhat larger than $\pi$. The circumscribed hexagon consists of six congruent equilateral triangles whose height is one unit. Again, some students may remember from E.A. 3.2 that the side length is $2/\sqrt{3} (\approx 1.155)$ times the height. Since the height is one unit, therefore the side length is simply 1.155.

Thus the area of each of the six triangles is equal to $(1/2) \times 1.155 \times 1 = 0.5775$. Therefore the area of the circumscribed hexagon is $6 \times 0.5775 = 3.465...$ Thus,

$$2.598... < \pi < 3.465...$$

Using Octagons

8. If you and your students did Problem 2 of Problem Set 2.1 (which involved calculating the length of the side of a regular octagon inscribed in a circle of radius one), you can go a step further in the calculation of $\pi$. Figure 6.4 shows the circle with an inscribed octagon.

In Problem 2 of Problem Set 2.1, we found that the length of the side of the octagon was $\sqrt{2} - \sqrt{2}$. We can use that number to find the area of that octagon. The octagon consists of 8 congruent isosceles triangles each with base $\sqrt{2} - \sqrt{2}$ and two sides of length 1. If we can find the area of this triangle, we can then multiply that by 8 to get the area of the octagon. So consider one of those triangles. If we draw the line representing the height of the triangle, then the left side of the figure is a right-angled triangle with hypotenuse 1 and base $(1/2)\sqrt{2} - \sqrt{2}$, as shown in Figure 6.5.
6.1 Enrichment Activity - $\pi$ and the Area of a Circle

Figure 6.4: The area of a circle compared to the area of an inscribed octagon

![Figure 6.4: The area of a circle compared to the area of an inscribed octagon](image)

Figure 6.5: A single isosceles triangle

![Figure 6.5: A single isosceles triangle](image)

By the Theorem of Pythagoras, the other right-angle side (the height of our original isosceles triangle) is equal to

$$
\sqrt{1^2 - \frac{1}{4} \times \left(2 - \sqrt{2}\right)^2} = \sqrt{1 - \frac{1}{4} \times (2 - \sqrt{2})} = \sqrt{\frac{1}{2} + \frac{1}{4} \times \sqrt{2}}.
$$

If we multiply this by half the length of the base we get the area of one of the eight triangles in the octagon. Taking 8 of these amounts to multiplying the height by 4 times the base:

$$
4 \times \sqrt{2 - \sqrt{2}} \times \sqrt{\frac{1}{2} + \frac{1}{4} \times \sqrt{2}} = 2 \times \sqrt{2 - \sqrt{2}} \times \sqrt{2 + \sqrt{2}} = 2 \times \sqrt{2}.
$$

This calculation is simplified, of course, if at each step the calculator is used to get a decimal expression for the length. Presented in the form given here, it assumes greater understanding of algebra than students at this level can be expected to have. Nevertheless,
the formula using square roots is interesting, for it is part of a patterned sequence of formulas for regular polygons of 4, 8, 16, 32, and generally $2^n$ sides.

9. The area of a regular polygon of $2^n$ sides inscribed in a circle of radius one is given in Table 6.1. For the rest of the polygons of $2^n$ sides the pattern continues. Try these numbers,

<table>
<thead>
<tr>
<th>Number of sides</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^2$</td>
<td>$2$</td>
</tr>
<tr>
<td>$2^3$</td>
<td>$2\sqrt{2}$</td>
</tr>
<tr>
<td>$2^4$</td>
<td>$2^2\sqrt{2} - \sqrt{2}$</td>
</tr>
<tr>
<td>$2^5$</td>
<td>$2^3\sqrt{2} - \sqrt{2} + \sqrt{2}$</td>
</tr>
<tr>
<td>$2^6$</td>
<td>$2^4\sqrt{2} - \sqrt{2} + \sqrt{2} + \sqrt{2}$</td>
</tr>
</tbody>
</table>

Table 6.1: The areas of inscribed polygons with $2^n$ sides

and see if you do not get closer and closer to the value of $\pi$.

**Using n-gons**

10. If we had the required knowledge of trigonometry, we could use the idea inherent in the previous three examples (using squares, hexagons, and octagons) to get estimates for the value of $\pi$ to an arbitrarily high degree of accuracy. All we have to do is to circumscribe and inscribe n-gons in place of squares, where $n$ is very large. In Figure 6.6, we see that for a value of $n = 12$, the sides of the 12-gons are almost indistinguishable from the circumference of the circle. Certainly, you would expect the area of either of these polygons to be very close to the area, $\pi$, of the circle. On the other hand, the n-gons are each made up of $n$ identical isosceles triangles. In principle, we should be able to use Pythagoras’ Theorem to calculate the areas of these isosceles triangles, and so get a more accurate estimate for the number $\pi$.

![Figure 6.6: The area of a circle compared to the area of an n-gon](image-url)
This method for finding the value of \( \pi \) is actually very old, dating back to the work of Archimedes (287-212 BC). There is a wonderful website illustrating this method at

http://www.math.utah.edu/~alfeld/Archimedes/Archimedes.html

This website, created by Peter Alfeld, allows students to choose the numbers of edges of the inner and outer polygons, and calculates their areas. It also calculates their perimeters, but that is the topic of a later section in this book.

11. The actual mathematical calculations involved are too complicated to present at this level. In fact, of course, \( \pi = 3.14159... \), and the students should be told this. Point out to them that the decimal expression for \( \pi \) is neither terminating nor repeating. In other words, \( \pi \) is an irrational number. At the conclusion of the lesson, you should review with the students what we learned: \( \pi = 3.14159... \), and the area of a circle with radius \( r \) is equal to \( \pi r^2 \).
6.2 Enrichment Activity - $\pi$ and the Circumference of a Circle

Lesson Goals

- To discuss with students the formula for the circumference of a circle and to relate this to area via a thought experiment
- To help students use their knowledge of how scale changes affect length measurements to understand the presence of $r$ (compared to $r^2$) in the formula for circumference
- To give students practice performing area and perimeter calculations involving circles

Materials

- copies of Problem Set 6.3 in Appendix B.4 for each student

Problem Statement

The problem is to derive the formula for the circumference of a circle of radius $r$ using knowledge of area and scale changes.

Lesson Sequence

1. Begin by asking the students to imagine on the page several pieces of line and curve. Now ask the students to imagine the same picture seen under a magnifying glass that enlarges everything by a factor of two. Discuss with the students what effect this will have on the lengths of lines. For vertical and horizontal lines the lengths are multiplied by a factor of two, for that is what we mean when we say that the magnifying glass magnifies 2. But the lengths of a skew line and a piece of curve will also become twice as long. The students will almost certainly consider this evident for lines. If they seem uncertain why this should also be the case for curves, you can suggest that they should think of a curve as made up of many very small (perhaps infinitely small) line segments. When you enlarge, each of these line segments is multiplied by the same factor, so the length of the curve is multiplied by that same factor.

2. Having established this, tell the students that you want to find a formula for calculating the circumference of a circle, and that you will do this first for a circle of radius 1. In fact, agree with the students that you will use the symbol $c$ to represent the length of the circumference of a circle of radius 1, and ask them what, in view of the pictures above, will be the length of the circumference of a circle of radius 2. Try to get the students to notice that circumference is a length measurement, and that enlarging the radius by a factor of two enlarges everything by two, so that the circumference will be $2c$. In other words, if we know what $c$ is equal to then we can deduce from that what the circumference of a circle of radius 2 is. Ask the students to give expressions for the circumferences of circles of radius 3, 1/2, 6, 1/10. The answers will be $3c$, $(1/2)c$, $6c$, and $(1/10)c$ respectively. Finally ask the student what the formula will be for the circumference of a circle of radius $r$. At this
6.2 Enrichment Activity - $\pi$ and the Circumference of a Circle

point it should be clear to the student that it is essential for us to find out what the value of $c$ is, so that is what we will do next.

There are two very different ways to accomplish this. We will present the more geometric method here. The second method is left as an extension for a strong and motivated class.

**First method for finding the formula for Circumference**

3. We begin with a circle of radius 1. We know from the preceding section that the area of this circle is equal to $\pi$. We have also agreed that temporarily we will refer to the circumference of the circle as $c$. The goal is to express the value of $c$ in terms of $\pi$. To do this we take our circle with radius 1, split it up into a number of equal sectors, and reassemble them in a different way. Figures 6.8 and 6.9 illustrate the process.

4. Ask the students what the figure on the right begins to look like as we increase the number of sectors into which we divide the circle. Some of them will suggest that it begins to look like a parallelogram, for they will recognize that as the little arcs that make up the top and bottom of the “parallelogram” get shorter they also become less curved. However, if they concentrate on the sides of the “parallelogram” they should also notice that these sides become more and more nearly vertical. In other words, the figure begins to resemble a rectangle! Once the students agree that the figures will begin to resemble a rectangle,
ask them what the area of that rectangle will be. Here the students have to remember that the figures were constructed by reassembling pieces of a circle of area $\pi$, so the area of the rectangle must also be equal to $\pi$.

5. Next ask the class what the height of the rectangle is. Once again, the answer is found by comparing the figures to the circle they came from: The “sides” of the figure are radii of the original circle, and must therefore have length 1. So the figures will converge to a rectangle of height 1. So how wide is this rectangle? To help the students determine this, ask them how the width of the rectangle relates to the circle we started with. They will see quite easily, especially in the first picture, that the bumpy curve at the top of the figure makes up precisely one half of the circumference of the circle. Thus the width of the rectangle is equal to $\frac{1}{2}c$, and therefore its area is equal to $1 \times \frac{1}{2}c = \frac{1}{2}c$. But we noted earlier that the area is equal to $\pi$. This can only mean that $\frac{1}{2}c = \pi$. That is, $c = 2 \times \pi$. Finally, in view of our earlier discussion of the behaviour of the circumference under enlargement of the circle, we come to the conclusion that the circumference of a circle of radius $r$ is equal to $2 \times \pi \times r$, or $2\pi r$!

**Comparing Area and Circumference**

6. Review with the students the formula for the area of a circle so that they can see the area and circumference formulas side by side. Ask the student this question: If a circle has its circumference (in units) equal to its area (in square units), what is the radius of that circle? Ask them also to compare the area and the circumference of circles of radius 100 and 1000. The idea here is to get them to notice that the area grows much more rapidly than the circumference when $r$ increases. The reason for this is that the formula for the area involves the square of the radius, while the circumference involves only the first power of the radius. The power of $r$ needed in the formula indicates the dimension of the measurement. Circumference is a one dimensional concept, while area is a concept that belongs to two dimensional objects. The circumference measures the outer edge of the circle, which is one dimensional, being a curve. The area measures the size of the region inside the circle, and this region is a two dimensional object.

7. You should end the lesson by giving students some questions that test their understanding of the formulas for area and circumference, and allow them to get some practice using the two formulas. We recommend Problem Set 6.3. You should note that the terms
circumference and perimeter are sometimes used interchangeably. This makes perfectly good sense, for a circumference is just the perimeter of a region that happens to be circular.

### Extensions

#### Second method for finding the formula for the circumference

1. Of the two explanations, this second method is the more difficult, and should be considered optional. We begin with a discussion about multiplication. Ask the students to calculate \((1.0001)^2\). You should insist that they not use their calculators to do this. It is essential to what follows that the calculation be done by hand. The answer they get will be 1.00020001. Now ask students to do the same calculation using first 1.00001 and then 1.000001 in place of 1.0001. Ask the students whether there is a pattern. Once the students have observed the pattern ask them what will happen if we square 1.(n zeros)1. Undoubtedly they will guess that the answer is 1.(n zeros)2(n zeros)1. Ask the students to explain why this is the right answer. If they cannot come up with a satisfactory explanation, you can point out to them that the answer is a result of the way long multiplication works: When we multiply 1.(n zeros)1 by itself, we first write down the number itself, the result of multiplying by 1, the last decimal digit in the second number. We will then enter one dot for the digit we just took care of and then n more dots as we skip past the zeros, and then multiply the first digit in the second number into the second number. This will give rise to the following arrangement:

\[
\begin{array}{c}
1. (n \text{ zeros}) & 1 \\
1. (n \text{ zeros}) & 1 \\
\hline
1 \times
\end{array}
\]

\[
\begin{array}{c}
1 \text{ (n zeros)} & 1 \\
1 \text{ (n dots)} & \cdot \\
\hline
1 \text{ (n zeros)} & 2 \text{ (n zeros)} & 1
\end{array}
\]

Table 6.2:

Finally you would put a decimal place behind the first digit, and conclude that \((1.(n \text{ zeros})1)^2 = 1.(n \text{ zeros})2(n \text{ zeros})1\).

2. Now discuss the following questions with the students: If we were to say that \((1.(n \text{ zeros})1)^2 = 1.(n \text{ zeros})2\), would we be far off? How much of an error would we be making? After some discussion they will notice that if the number of zero digits between the 1’s is large (that is, if n is large) the error will be very very slight, so for large n we can say that, “for practical purposes”,

\[
(1.(n \text{ zeros})1)^2 = 1.(n \text{ zeros})2.
\]
To indicate that this inequality is only approximately correct, but with an error so small that “for all practical purposes” it can be ignored, mathematicians often use a wavy equal sign, as in the equation
\[(1.(n \text{ zeros})1)^2 \approx 1.(n \text{ zeros})2.\]

3. There is another, more geometric, way to make this observation about numbers. Suppose we start with a square whose sides have length 1. Now suppose we make the square slightly larger, say \(1 + a\) where \(a = 0.(n \text{ zeros})1\). Then, of course, \(1 + a = 1.(n \text{ zeros})1\). The relationship between the first square and the enlarged square is illustrated in the following diagram:

![Figure 6.10:](image)

To make sure the students know what you are thinking of when you talk about the number \(a\) you may have give some examples, such as

\[a = 0.000001 = \frac{1}{100000}.\]

Of course, in the picture \(a\) is drawn much larger than this example, simply because if we were to let \(a = 0.000001\) in the picture, we would not be able to see the difference between the smaller and the larger square. The original square has area 1, and the larger square consists of the original square and three extra pieces: the two thin shaded rectangles and the tiny white square at the top right hand corner. The two shaded rectangles each have area \(1 \times a\), so their combined area is \(2 \times a\). The tiny square in the corner is so small (especially if \(a\) is made really small) compared to the other areas that “for all practical purposes” we can ignore its area. In other words, the area of the enlarged square is approximately \(1 + 2 \times a\). Since the area of the enlarged square can also be computed directly, we see that when \(a\) is very small

\[(1 + a)^2 \approx 1 + 2 \times a\]

. This is simply another way of saying

\[(1.(n \text{ zeros})1)^2 \approx 1.(n \text{ zeros})2.\]

4. We now turn to the circle again. Ask the student to imagine a circle of radius 1 (and of circumference \(c\)) and a slightly larger circle of radius 1.000001 around it. Ask the students
if they can figure out the area $A$ of the thin region between the two circles, in terms of the number $c$ if necessary.

Here is a picture for this discussion, where the shaded ring is supposed to have thickness 0.000001, though for obvious reasons it is drawn much thicker.

Figure 6.11:

5. The idea is to try to get the students to obtain the answer by two different methods. On the one hand, after some attempts some students may decide that they should use the formula for the area of a circle, learned a little earlier. If necessary, you can remind the students that there must be some way to use that formula here. Using this method, the area of the thin shaded ring is equal to the area of the larger circle minus the area of the smaller. That is,

$$A = \pi \times (1.000001)^2 - \pi \times (1)^2 = \pi \times 0.000002000001 \approx \pi \times 0.000002.$$

6. On the other hand, some of the students may solve the problem by the following reasoning: Since the ring is very thin, if we were to cut it somewhere, and unroll the ring, we will essentially get a thin rectangle of width 0.000001 and length $c$. At each step try to get the students to come up with the answer. In other words, we have the following alternative formula for the area of the ring:

$$A \approx c \times 0.000001.$$

Now these two calculations should give the same answer, so we have

$$c \times 0.000001 \approx \pi \times 0.000002.$$

If we multiply these numbers by one million we get

$$c \approx \pi \times 2.$$

If we were to do the same calculations again, with a ring of thickness 0.0000000000000001, we would come to the same conclusion, but in this case, the calculations would be even more accurate. Since we can do these calculations as accurately as we like, we must come to the conclusion that the circumference of a circle of radius 1 is equal to $2 \times \pi$. If we combine this with the discussion done earlier, we obtain:

**The circumference of a circle of radius $r$ is equal to $2 \times \pi \times r$.**
6.3 Problem Set - Areas and Perimeters of Circles

Goals

• To give students practice using the formulas for area and circumference of a circle
• To further consolidate students’ understanding of area and perimeter using composite shapes and more complicated inner regions of shapes

Materials

• copies of the problem set in Appendix B.4 for each student

Solutions to Problem Set

1. Find the areas of the shaded regions.

![Figure 6.12: The first three figures of Question 1](image)

- The first of these figures is a circle of radius 3, so its area is equal to $\pi \times 3^2 = 28.27\ldots$ square units.
- The second is a quarter of a circle of radius 2, so its area is equal to $(1/4) \times \pi \times 2^2 = \pi = 3.14\ldots$
- The third figure consists of two pieces. The piece on the left is a right-angled triangle whose second right-angle side is calculated by Pythagoras’ theorem, giving the answer $\sqrt{5^2 - 3^2} = 4$. This means that the other part of the figure is a quarter of a circle of radius 4. The total area is then equal to $[(1/2) \times 3 \times 4] + [(1/4) \times \pi \times 4^2] = 6 + (4 \times \pi) = 18.566\ldots$
6.3 Problem Set - Areas and Perimeters of Circles

- In the first figure, the area of the shaded region is found by calculating the area of the square and subtracting the area of the circle. The area of the square is 9, and the area of the circle is \( \pi \times (1.5)^2 = 7.0685 \ldots \). Thus the shaded area is equal to 1.9314.

- The second figure consists of circles of radii 1 and 2 respectively. Their areas are therefore equal to \( \pi \) and \( \pi \times 2^2 \) respectively. We must subtract the former from the latter to get the area of the shaded region. This comes to \( 3 \times \pi = 9.42 \ldots \) square units.

- The third figure can be done in several ways. The large rectangle has an area of 48 square units. The shaded part is surrounded by four triangles each of area equal to 6. Thus the shaded region has area equal to \( 48 - 24 = 24 \).

2. Find the areas and the perimeters of the shaded regions:

- The first of these figures constitutes half a circle of radius 2, so the area of the shaded region is \( \pi \times 2^2 \div 2 = 6.28 \ldots \). The perimeter is equal to \( 4 + \pi \times 2 = 10.28 \ldots \).

- The second figure is a quarter of circle of radius 3, so its area is \( \pi \times 3^2 \div 4 = 7.06858 \ldots \). The perimeter of the figure is \( 3 + 3 + 2 \times \pi \times 3 \div 4 = 10.712 \ldots \).
• The first of these figures consists of a rectangle of area 6 together with two halves of a circle of radius 1. The total area of the two half circles is $\pi$, so the area of the shaded region is $6 + \pi = 9.14 \cdots$. The perimeter of the figure is equal to 6 plus the circumference of the circle. In other words it, too, comes to a total of $6 + \pi = 9.14 \cdots$.

• The shaded part of the second figure consists of the area of the smaller triangle, less the three wedges of circle enclosed in it. Each of the three wedges has angle 60°, so together they make up 180°, or half a circle. That is, the total area of the three wedges is equal to $\pi \times \frac{2^2}{2} = 6.283 \cdots$.

• To find the area of the smaller triangle, we note that it is equilateral and that its sides are each 4 units. Using Pythagoras’ theorem we can deduce that the height of this triangle is $\sqrt{4^2 - 2^2} = \sqrt{12}$. This means that the area of this triangle is $2 \times \sqrt{12} = 6.928 \cdots$. To get the area of the shaded region, we subtract: $6.928 - 3.283 = 0.645$ square units.

• The circumference of the shaded region is made up of three arcs of circle, each making up one-sixth of the circumference of the circle it belongs to. That is, the three together have the same length as one half the circumference of the circle. That is, the perimeter of the shaded region is equal to $\pi \times 2 = 6.28 \cdots$. 

Figure 6.15: Last two figures of Question 2
6.4 Enrichment Activity - Measuring the Thickness of Toilet Paper

Lesson Goals

- To present students with a very nice and fun problem that involves the calculation of areas and circumferences of circles
- To challenge students to think hard about how they can apply what they have learned to this non-standard problem

Materials

- copies of Spot Question in Appendix C.15 for each student
- rulers and calculators for each student
- rolls of toilet paper for each group of 3-5 students

Problem Statement

Given a roll of toilet paper with 200 sheets measure the thickness of the paper. [Note: The rolls you bring may have more or less than 200 sheets - this information is usually given on the plastic wrap in which the toilet paper is sold. Use the actual number to make the problem real for the students.]

Lesson Sequence

1. A nice way to begin this session is with a Spot Question that gets students thinking about the area and/or circumference of a circle. A good example is the Spot Question in Appendix C.15, which asks students to calculate the amount of rope required to wrap around a cylinder a given number of times.

2. After the Spot Question (or other opening activity), divide the class into groups of 3-5 students and give each group a roll of toilet paper. State the problem above, being sure to give them the number of sheets. Here, we will suppose that the toilet paper rolls have 200 sheets. The students will need rulers and calculators. Allow the students to unroll a bit of the paper so that they can measure the length of one sheet of toilet paper. Suppose it comes to 11.1 centimeters.

3. Allow the students to struggle with the problem for as long as possible. If they get frustrated, give them some hints. There are at least two ways to solve the problem: one using area and the other, circumference. Some students may even choose to look at volume (this method is similar to that of area). You will have to decide which method will work best for each group, depending on the work they have done already. Some groups may be completely stuck, in which case, you might suggest that they get started by computing some areas, as outlined in the area method. Try not to give away the solution all at once though. The students need to feel as if they are the ones solving the problem.
4. Two possible solutions are given next. They are presented in sequential steps, which can be used to guide students along. The calculations are based on an actual roll of toilet paper.

Solution using Areas

- We can solve the problem by first calculating the area of the side of one sheet of toilet paper. To help visualize what we mean by the side, imagine the single sheet of toilet paper, embedded among all the sheets rolled up. If we grossly exaggerate the thickness, it will look like this:

![Figure 6.16: Side view of a single sheet of toilet paper](image)

- The area of this single sheet is 11.1 times the thickness. Since there are 200 sheets in the roll, the total side area of the toilet paper is $200 \times 11.1 \times$ the thickness or 2200 times the thickness.

- On the other hand, if we view the entire roll of toilet paper on its side, as shown in Figure 6.17, we can compute the total side area by subtracting the area of the circle described by the inner radius (of just the roll itself) from the area described by the outer radius (of the roll with toilet paper):

  \[
  \pi \times 5^2 - \pi \times 2^2 = 65.97 \text{ square centimeters}.
  \]

- Thus, we have that $2200 \times$ thickness is equal to 65.97 square centimeters, which means that the thickness is equal to $65.97 \div 2200 = 0.03$ centimeters!

Alternate Solutions:

- In the above steps, we found expressions for the total side (or cross-sectional) area in two different ways and then compared them to find the thickness. Alternatively, students may choose to compare expressions for the side area of a single sheet of toilet paper. In this case, they would compute the side area of a single sheet as $65.97 \div 200 = 0.33$ square centimeters. Since the length of one sheet is 11.1 centimeters, the thickness is equal to $0.33 \div 11.1 = 0.03$ centimeters. (Note that this is actually the same calculation.)

- The same result will be found if students choose to use the volume of toilet paper (either of a single sheet or the whole roll). The only difference is that the height of the toilet paper will enter into their expressions (although it will also “cancel out” when the expressions are compared).
6.4 Enrichment Activity - Measuring the Thickness of Toilet Paper

Solution using Circumference

- In this solution, students measure the thickness of the roll, from the central opening to the outer circumference. Suppose it comes to 3 centimeters, as illustrated in Figure 6.17.

- They then need to know how many paper thicknesses there are in that thickness. For this, they will need to know how many sheets it takes to go once around the roll. The students will soon realize that this number is smaller near the center of the roll than near the outer circumference. Suppose the radius of the hole in the center of the roll is measured as 2 centimeters, and the radius of the outer perimeter as 5 centimeters. Then the outer circumference is $10 \times \pi = 31.41$ centimeters, and the inner circumference is $4 \times \pi = 12.57$ centimeters. Thus the outer circumference needs $31.41 \div 11.1 = 2.83$ sheets to get around, whereas the inner circle needs $12.57 \div 11.1 = 1.13$ sheets. The students will want to average these two numbers to get the number of sheets needed “on average” to go around the roll once. Though it is perhaps not totally obvious that this is the right thing to do (why should the inner and outer circumferences receive equal weight in the averaging process?) it is in fact correct. The reason it is correct is that the formula for the circumference involves the radius to the power one. That is, it goes up evenly (“linearly”) as the radius is increased. In any case, if we average $(2.83 + 1.13) \div 2 = 1.98$ we see that on average it takes 1.98 sheets of paper to go around the roll once. Since there are 200 sheets of paper, therefore the thickness of the roll consists of $200 \div 1.98 = 101$ sheets.

- Thus the thickness of one sheet of paper is $3 \div 101 = 0.03$ centimeters, or 0.3 millimeters.
6.5 Problem Set - Word Problems Involving Circles

Goals

- To give students practice solving word problems that challenge them to apply their knowledge of the areas and circumferences of circles in practical situations

Materials

- copies of the problem set in Appendix B.5 for each student

Solutions to Problem Set

1. A farmer has a circular pond. The diameter of the pond is 10 meters. How many meters of fence does the farmer have to buy if he wants to put a fence around the pond?

The radius of the pond is 5 meters, so the circumference is \(2 \times \pi \times 5 = 31.4159\).

2. A circular track has a radius of 400 meters. A car is driving 50 km per hour along the track. How long does it take to do one lap?

The circumference of the track is \(2 \times \pi \times 400 = 2513\) meters, and the car is moving at 50,000 meters per hour, so it will take \(2513 \div 50,000 = 0.05026\) hours, or \(0.05026 \times 60 = 3.01\) minutes to go around the track once.

3. A small bulldozer has a caterpillar track around two wheels of radius 30 cm, whose axles are 2 meters apart. How long is one caterpillar track?

To solve this problem it helps to draw a picture. The caterpillar track goes half way around the left wheel and half way around the right wheel shown, and includes each of the two straight line pieces. Since the straight line pieces are 200 cm each, and the circumference of one wheel is \(2 \times \pi \times 30 = 188.5\). Thus the total length of the caterpillar track is \(200 + 200 + 188.5 = 588.5\) cm.

![Figure 6.18: Caterpillar track](image-url)
4. Two wheels are touching so that if one rotates, it makes the other one rotate in the opposite direction. If the larger wheel has a radius of 2 cm, and the smaller one a radius of 1 cm, which wheel rotates faster? How many times as fast?

When the two wheels roll (without slipping), the circumference of each wheel rolls past the point of contact at exactly the same rate. For example, if 23 cm of one wheel have rolled past the point of contact, then the same is true for the other wheel. However, the larger wheel has a circumference of $2 \times \pi \times 2 = 4 \times \pi$, so when it rotates once, the length of its circumference that goes by the contact point is $4 \times \pi$. On the other hand, the circumference of the smaller wheel is $2 \times \pi$, half as much as the larger. Therefore the smaller wheel has to rotate twice for every rotation of the larger. That is, it rotates twice as fast.

5. A car has wheels of diameter equal to 40 cm. When the wheels rotate exactly once, how far has the car moved? How many times does a wheel rotate when when the car travels 20 km?

When the wheels rotate once, the distance traveled is exactly the same as the circumference of the wheel, that is $\pi \times 40 = 125.664$ cm, which is equal to 0.00125664 kilometers. To travel 20 km, the wheels have to rotate $20 \div 0.00125664 = 15915.49$, or approximately 15915 times.

6. A tractor has wheels with a radius of 30 cm. While the tractor is moving, the wheels turn two revolutions every second. What is the speed of the tractor? Give the answer in centimeters per second, and in kilometers per hour.

The circumference of a wheel is $2 \times \pi \times 30 = 188.5$ cm. Therefore, two revolutions correspond to $2 \times 188.5 = 377$ cm. That is, the tractor travels at 377 centimeters per second. To change this to kilometers per hour, we have to multiply by 3600 and divide by 100,000. This gives us $377 \times 3600 \div 100,000 = 13.57$ kilometers per hour.

7. A circular track has a radius of 100 meters. A runner takes three minutes to go once around the track. How fast is she running? (Answer in km per hour).

The circumference of the track is $2 \times \pi \times 100 = 628$ meters. Thus she is running at (approximately) 209 meters per minute, which comes to $209 \times 60 \div 1000 = 12.5$ kilometers per hour, approximately.
8. Two runners start off in opposite directions at the same point on a circular track. One of them runs 10 meters per second, while the other covers 5 meters per second. If the track is 3000 meters in diameter, how long will it take before they meet again?

Solving this question requires a trick in the form of a change of viewpoint. Instead of concentrating on the distance covered by each runner separately, think of the distance they cover together, or “between them”. Between them they cover 15 meters per second. This means that as they move away from each other the distance separating them increases at that rate, and as they approach each other it decreases at that rate. Between two meetings their combined distances equal the circumference of the track, which is $3000 \times \pi = 9424.8$ meters. This will take $9424.8 \div 15 = 628$ seconds.
Appendix A

Templates for Enrichment Activities
A.1 Quadrilateral Handout (E.A. 1.3)
A.2 Packaging Problem Overheads (E.A. 2.2)
A.3 Net Handout/Overhead (E.A. 2.3)
A.4 Tetrahedron and Octahedron Templates (E.A. 4.1 and 4.2)
A.5 Polyhedra Overheads (E.A. 4.3)
A.6 Dodecahedron and Icosahedron Templates (E.A.s 4.4 and 4.5)
A.7 Graph Paper Handouts (E.A. 5.1)
A.8 Graph Paper Handout (E.A. 5.2)
A.9 Graph Paper Handouts (E.A. 5.3)
$r^2$ and $r^3$
Appendix B

Problem Sets
1. Using the Theorem of Pythagoras, calculate the lengths of the sides labeled with a question mark.
2. In each of the next two triangles there are two sides that are labeled with a question mark. The fact that they are both marked the same way is meant to indicate that the two sides have the same length. Find that common length for each of the two triangles.

3. The next few triangles are more difficult to do. To solve them you have to notice that because $\sqrt{5}$ is defined to be the number which, if you multiply it by itself, gives the answer 5, therefore you have $\sqrt{5} \times \sqrt{5} = 5$. Notice that this also means that if you multiply $3 \times \sqrt{5}$ by itself you will get $3 \times \sqrt{5} \times 3 \times \sqrt{5} = 3 \times 3 \times \sqrt{5} \times \sqrt{5} = 9 \times 5 = 45$. In other words whenever an expression involving a square root is multiplied by itself, you can get the answer without using the calculator.
Problem 1: Determine the sequence of steps needed to find $x$ if you know $a$ and $b$. Then, find $x$. 
Problem 2: Determine the sequence of steps needed to calculate the length of the sides of a regular octagon inscribed in a circle of radius 1, as shown below. Then, calculate the length.
1. Find the areas of the shaded regions:
2. Find the areas of these triangles:

3. Find the shaded areas:

4. Calculate the surface area of an octahedron whose edges are 2 cm long.
B.3 Problem Set 5.7

1. Find the areas of the following figures:
2. Find the areas of the shaded regions:
1. Find the areas of the shaded regions:

2. Find the areas and the perimeters of the shaded regions:
1. A farmer has a circular pond. The diameter of the pond is 10 meters. How many meters of fence does the farmer have to buy if he wants to put a fence around the pond?

2. A circular track has a radius of 400 meters. A car is driving 50 km per hour along the track. How long does it take to do one lap?

3. A small bulldozer has a caterpillar track around two wheels of radius 30 cm, whose axles are 2 meters apart. How long is one caterpillar track?

1. Two wheels are touching so that if one rotates, it makes the other one rotate in the opposite direction. If the larger wheel has a radius of 2 cm, and the smaller one a radius of 1 cm, which wheel rotates faster? How many times as fast?

2. A car has wheels of diameter equal to 40 cm. When the wheels rotate exactly once, how far has the car moved? How many times does a wheel rotate when the car travels 20 km?

3. A car has wheels with a radius of 30 cm. While the car is moving, the wheels turn two revolutions every second. What is the speed of the car? Give the answer in centimeters per second, and in kilometers per hour.

4. A circular track has a radius of 100 meters. A runner takes three minutes to go once around the track. How fast is she running? (Answer in km per hour).

5. Two runners start off in opposite directions at the same point on a circular track. One of them runs 10 meters per second, while the other covers 5 meters per second. If the track is 3000 meters in diameter, how long will it take before they meet again?
Appendix C

Spot Questions
C.1 Squares and Square Roots (E.A. 1.3)

NAME:

Spot Question (take about 3 minutes, put name on sheet and hand in)
What is \((\sqrt{7} \times 5)^2\) equal to, and why?
C.2 Pythagoras Calculation (E.A. 1.4)

NAME:

Spot Question (take about 3 minutes, put name on sheet and hand in)
Calculate the length of the third side:

\[ \sqrt{21} \]

NAME:

Spot Question (take about 3 minutes, put name on sheet and hand in)
Calculate the length of the third side:

\[ \sqrt{21} \]
C.3 Multi-step Pythagoras (E.A. 2.2)

NAME:

Spot Question (take about 3 minutes, put name on sheet and hand in)

Consider the following diagram. Are there any other distances in the diagram that can be calculated? Mark them (1), (2), etc in the order in which they can be calculated. O is the centre of the circle. You do not need to do the calculations.
C.4 3D Visualization and Nets (E.A. 2.3)

Note: This spot question uses an overhead transparency of a different net, provided on the next page. Before using this question, you need to draw a door and painting on the overhead.

NAME:

Spot Question (take about 3 minutes, put name on sheet and hand in)

Draw the door and the painting on this net (refer to the net on the overhead):
C.5  Nets and the Tetrahedron (E.A. 4.1)

NAME:

Spot Question (take about 3 minutes, put name on sheet and hand in)
Circle the two dots that will be closest to each other once this net has been folded together.

NAME:

Spot Question (take about 3 minutes, put name on sheet and hand in)
Circle the two dots that will be closest to each other once this net has been folded together.
C.6 3D Visualization using Profiles (E.A. 4.2)

NAME:

Spot Question (take about 3 minutes, put name on sheet and hand in)

A certain object produces the following profiles. Describe the object, perhaps by giving it a name or by sketching it.

From the front: From the side: from the top:

[Diagram of a hexagonal prism]

NAME:

Spot Question (take about 3 minutes, put name on sheet and hand in)

A certain object produces the following profiles. Describe the object, perhaps by giving it a name or by sketching it.

From the front: From the side: from the top:

[Diagram of a hexagonal prism]
C.7  Euler Number (E.A. 4.4)

NAME:

Spot Question (take about 3 minutes, put name on sheet and hand in)

What is the Euler number of this (not sphere-like) polyhedral surface (a rectangular box with diagonals drawn on the front and back faces, and then a cubic hole through the centre?)

NAME:

Spot Question (take about 3 minutes, put name on sheet and hand in)

What is the Euler number of this (not sphere-like) polyhedral surface (a rectangular box with diagonals drawn on the front and back faces, and then a cubic hole through the centre?)
C.8 Area of a Rectangle (E.A. 5.1)

NAME:

Spot Question (take about 3 minutes, put name on sheet and hand in)

Consider the following rectangle, which is 4.5 units wide and 3 units high. What is its area, and explain why you think your calculation is correct. Are you simply applying a known formula or can you give a reason why it is the appropriate formula?
C.9 Surface Area (E.A. 5.2)

NAME:

Spot Question (take about 3 minutes, put name on sheet and hand in)

The solid shape shown is constructed by gluing together 30 cubes whose sides are 1 cm in length. What is the surface area (the total area of the outer surface) of the resulting shape?
C.10 Areas of Triangles (E.A.s 5.2 - 5.4)

NAME:

Spot Question (take about 3 minutes, put name on sheet and hand in)

Assuming that the grid lines are 1 cm apart, the area of the triangle A can be found by first finding the area that is outside A. The same can be done for B. What are the areas of these triangles?

NAME:

Spot Question (take about 3 minutes, put name on sheet and hand in)

Assuming that the grid lines are 1 cm apart, the area of the triangle A can be found by first finding the area that is outside A. The same can be done for B. What are the areas of these triangles?
C.11 Area of a “Dog” (E.A.s 5.2 - 5.4)

NAME:

Spot Question (take about 3 minutes, put name on sheet and hand in)
If the lines in the grid below are one cm apart, what is the area of the horse?

[Diagram of a grid with a shaded shape resembling a dog]

NAME:

Spot Question (take about 3 minutes, put name on sheet and hand in)
If the lines in the grid below are one cm apart, what is the area of the horse?

[Diagram of a grid with a shaded shape resembling a dog]
C.12 Scaling and Area & Volume (after E.A. 5.3)

NAME:

Spot Question (take about 3 minutes, put name on sheet and hand in)
When the sides of a regular octahedron are each 1 cm, then it can be shown that the total surface area of the octahedron (that is, the sum of the areas of all eight triangular faces) is $2\sqrt{3} = 3.46$ square centimeters. It can also be shown that the volume of (the interior of) the octahedron is then $\frac{\sqrt{2}}{3} = 0.47$ cubic centimeters. What are the surface area and the volume of a regular octahedron whose side are each 2 cm?

NAME:

Spot Question (take about 3 minutes, put name on sheet and hand in)
When the sides of a regular octahedron are each 1 cm, then it can be shown that the total surface area of the octahedron (that is, the sum of the areas of all eight triangular faces) is $2\sqrt{3} = 3.46$ square centimeters. It can also be shown that the volume of (the interior of) the octahedron is then $\frac{\sqrt{2}}{3} = 0.47$ cubic centimeters. What are the surface area and the volume of a regular octahedron whose side are each 2 cm?
C.13 Area of a Trapezoid (after E.A. 5.5)

NAME:

Spot Question (take about 3 minutes, put name on sheet and hand in)

We have learned formulas for calculating the areas of a rectangle, of a triangle, and of a parallelogram. In the picture below you see a trapezoid. Draw a congruent copy of this trapezoid, placed in such a way that the two together form one of the three figures whose areas we know how to calculate. Can we use the resulting shape to obtain a general formula for calculating the area of a trapezoid?
C.14  Area of an Ellipse (E.A. 6.1)

NAME:

Spot Question (take about 3 minutes, put name on sheet and hand in)
What is the area of the interior of the following oval shape? Select what you think is the best answer.

A. \(\pi \times 3^2\)
B. \(\pi \times 2^2\)
C. 19 square units
D. 12 square units
C.15 Cylinder and Rope (E.A. 6.4)

NAME:

Spot Question (take about 6 minutes, put name on sheet and hand in)
A string is wrapped around a cylinder of diameter 5 cm and length 20 cm. If the string wraps around exactly 3 times as shown in the figure, how long is it?