Enrichment Mathematics for Grades Seven and Eight
(Part I - Numbers)

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Fifth edition

June 2, 2011
Contents

Getting Started 3
  0.1 From The Preface to the First Edition ........................................ 3
  0.2 About This Manual ........................................................................ 5
  0.3 Structuring an Enrichment Session .............................................. 6
  0.4 Ideas for the First Enrichment Session ....................................... 8

1 Number Patterns and Explanations 11
  1.1 Enrichment Activity - Recognizing Patterns ................................. 14
  1.2 Enrichment Activity - The Infinite Motel .................................... 20
  1.3 Enrichment Activity - Pattern Rules Producing the Same Sequence . 25
  1.4 Enrichment Activity - The Role of Proof ..................................... 29
  1.5 Enrichment Activity - Patterns Produced Geometrically ............... 36

2 Growth Rates of Sequences 41
  2.1 Enrichment Activity - Measuring the Growth Rate of a Sequence . . . 43
  2.2 Enrichment Activity - The Poor Soldier and the King .................... 48

3 Division 55
  3.1 Enrichment Activity - Prime Number Factors .............................. 57
  3.2 Enrichment Activity - What Prime Factors Tell You ..................... 62
  3.3 Enrichment Activity - Number Magic ........................................... 67
3.4 Enrichment Activity - How Many Primes are There? ...................... 70
3.5 Enrichment Activity - Greatest Common Factor ......................... 76
3.6 Enrichment Activity - Least Common Multiple ......................... 79
3.7 Problem Set - Practice Using Prime Factorizations .................. 82

4 Fractions .............................................. 87
4.1 Enrichment Activity - Fractions and Decimals ......................... 89
4.2 Enrichment Activity - Terminating or Repeating? ..................... 94
4.3 Problem Set - Practice with Fractions and Decimals ................. 97
4.4 Enrichment Activity - Rational and Irrational Numbers ............. 99
4.5 Enrichment Activity - Are 0.9 and 1 Different? ..................... 102

5 Remainders ............................................ 107
5.1 Enrichment Activity - Some Strange Division Problems ............. 109
5.2 Enrichment Activity - Jelly Beans .................................. 113
5.3 Enrichment Activity - Multiplication .................................. 115
5.4 Problem Set - Apples and Oranges .................................. 119
5.5 Enrichment Activity - Leftovers ..................................... 123

6 Counting ............................................... 127
6.1 Enrichment Activity - Finding a System ................................ 129
6.2 Problem Set - Counting Practice ..................................... 134
6.3 Problem Set - Coins and Dice ....................................... 139

7 Probability ............................................. 143
7.1 Enrichment Activity - Equally Likely Outcomes ....................... 145
7.2 Problem Set - Practicing Probability .................................. 150
7.3 Enrichment Activity - Should We be Surprised? ...................... 154
7.4 Problem Set - More Practice ....................................... 157
CONTENTS

7.5 Enrichment Activity - What is the Risk? .................................. 160

A Problem Sets and Templates .................................................. 167

A.1 Templates for Section 1.4 ..................................................... 168
A.2 Templates for Section 1.5 ..................................................... 171
A.3 Templates for Section 2.2 ..................................................... 174
A.4 Template for Section 3.1 ...................................................... 176
A.5 Problems for Section 3.2 ...................................................... 177
A.6 Template for Section 3.4 ...................................................... 179
A.7 Problems for Section 3.7 ...................................................... 181
A.8 Problems for Section 4.3 ...................................................... 183
A.9 Template for Section 4.5 ...................................................... 184
A.10 Problems for Section 5.4 ..................................................... 186
A.11 Problems for Section 5.5 ..................................................... 187
A.12 Templates and Problems for Section 6.1 ............................... 188
A.13 Problems for Section 6.2 ...................................................... 191
A.14 Problems for Section 6.3 ...................................................... 192
A.15 Template for Section 7.1 ...................................................... 193
A.16 Problems for Section 7.2 ...................................................... 194
A.17 Problems for Section 7.3 ...................................................... 195
A.18 Problems for Section 7.5 ...................................................... 196
Getting Started

0.1 From The Preface to the First Edition

This is the first volume in a two-volume manual for enrichment mathematics classes at the senior elementary level. The enrichment program is built around a two-year cycle, so that students can participate in it through grades 7 and 8 without repeating material. Each of the two volumes corresponds to one year in the program. The order in which they are taken up is optional. However, within each volume the material should generally be covered in the order in which it appears. There should be enough material in each volume to fill a program that runs from the beginning of October to the end of May, if the enrichment class meets once a week.

We think of the subject of mathematics as roughly divided into two domains: discrete mathematics and continuous mathematics. Of course, as a description of the field as it is explored by professional mathematicians, this is a very inadequate division, and there are many other criteria by which to classify the different parts of the discipline. Nevertheless, it is fair to say that the intuition involved in the understanding of mathematics comes in two varieties. On one side we have the intuition involved in the understanding of discrete objects such as whole numbers and finite or even countable collections. On the other we have the intuition involved in the understanding of continuous objects such as lines, planes, and surfaces, and infinite (especially non-countable) collections such as the collection of all real numbers. The first of these is associated with arithmetic, while the second is associated with geometry.

In this first volume we have collected topics that fall under the heading of arithmetic. Here you find units on patterned sequences, prime numbers and prime factors, rational and irrational numbers, modular arithmetic, counting large sets, and probability. The second volume focuses on geometry. It opens with a chapter on the Theorem of Pythagoras. From there is moves to a study of similarity transformations, regular polyhedra and the Euler number, area and volume, and geometric patterns. In both volumes there are lessons that include elementary discussions of conjectures and proofs.

We have two closely related goals in mind for this mathematics enrichment programme. In the first place, we want to give students a sense of the rich variety of the subject. The general public will typically express surprise at the fact that there is such a thing as mathematics research. “Is there more mathematics to be discovered?” is the usual response when a mathematician talks
about his or her work. Because of the carefully crafted uniformity of most mathematics curricula, students usually enter university with the perception that mathematics consists of a finite collection of tricks, so it is no surprise that people cannot imagine what mathematics research might be. We believe that the fostering of this narrow perception begins in the elementary classroom.

In the second place, we would like students to have the experience of solving a challenging problem after persistent effort. The sense of triumph experienced when a problem is solved successfully is much greater and more satisfying when the effort of finding a solution involved several attempts, and especially when the solution involved a radical change in the student’s way of looking at the problem. Too many students, when they enter university, have not had this experience. They tend to think of mathematics algorithmically: either you recognize the problem presented, and you know the algorithm that solves it, or else you cannot do the question until you have taken more mathematics. The discovery that that is not the way things are, is usually made during a student’s first year in university, and is then often an enormous shock. This discovery should be made much earlier.

Of course, a one hour a week enrichment programme spread over two years is not enough to achieve both of these goals. Especially the latter goal requires a more extended application throughout the high school years. Nevertheless, we believe that the enrichment programme described in these booklets constitutes a useful beginning.

Leo Jonker,
Kingston, July, 1999

The authors acknowledge the support of the PromoScience program of the National Science and Engineering Research Council of Canada, as well as Queen’s University’s Centre for Teaching and Learning in funding the creation of this new edition.
0.2 About This Manual

From its inception in 1999, this manual was intended for use by anyone interested in providing a mathematics enrichment programme for grade 7 and 8 students. You certainly do not need to know a lot of mathematics to use it. Much more important is that you have some curiosity about mathematics and some confidence that you can (re-) learn the material as you go along, whenever that is necessary. More recently, the manual has also been used as a text for the StepAhead program. This program combines a university mathematics course with a grade 7 and 8 enrichment program provided by the students in the university course.

The new format of this manual was designed to better structure the content of the enrichment program. The fundamental mathematical content and problems of previous editions have not changed. The manual is divided into six chapters:

1. Number Patterns and Explanations
2. Growth Rates of Sequences
3. Division
4. Fractions
5. Remainders
6. Counting
7. Probability

Within each chapter, the content is organized into a series of three to five Enrichment Activities, each of which has its own Lesson Goals, Materials, Background (if applicable), Problem Statement, Lesson Sequence, and Extensions/Modifications (if applicable) clearly stated. The new layout of the Enrichment Activities should help university students (or tutors) (1) locate a given activity quickly and (2) better prepare for their own enrichment sessions.

In addition to the Enrichment Activities, some chapters contain “Problem Sets”. In general, these are meant to supplement the Enrichment Activities by giving students practice doing certain important calculations. The university student (or tutor) has the choice of using a Problem Set to fill an entire enrichment session or to spread the problems over several weeks as “warm-up” exercises.

In terms of preparing for university lectures, it is better not to read ahead in the manual so that you (the university student) can get a sense of the initial responses and challenges that students might have when first exposed to a concept or problem. After the material has been discussed in the lectures, you should read the manual carefully to help fill in any gaps and to prepare for your own presentation of the activity.
0.3 Structuring an Enrichment Session

In general, a good enrichment session

1. engages students from the beginning

2. consists of challenging and fun mathematical problems

3. ends with something that all students can feel good about.

Beginning the Class

A popular way to engage students from the beginning is through the use of good games and/or puzzles. Such games and puzzles should not only be fun, but also focus the attention of the students on the rich mathematics that is to follow in the lesson. Much effort has been devoted to finding games and puzzles appropriate for this purpose, which are now available on the StepAhead website. For the most part, the games and puzzles are organized according to the chapters and enrichment activities within this manual, which should help university students (or instructors) find relevant puzzles and/or games quickly.

Another effective way to begin a class is to use a short problem that either introduces the topic of the current session or reviews a concept from the previous session. These questions should not take more than 3-5 minutes. You will find that having something for students to work on when they enter the classroom naturally focuses their attention. It also helps you as the tutor to assess student understanding in an informal way.

The Main Lesson

The bulk of the “challenging and fun mathematical problems” are set out in the Enrichment Activities in this manual. These activities will be the focus of your enrichment sessions. The activities are presented in a well-planned sequence and often contain modifications and/or extensions to reflect the differing abilities among classes, or even among students within a class. Some useful lesson materials, such as templates for overhead transparencies and handouts, can be found in the Appendices and are clearly referenced in the Materials section of each activity.

In terms of timing, most of the enrichment activities are designed for a single session. However, depending on the depth of discussion, the number of tangents taken, and the abilities/interest of the class, the activities may take more or less time. For example, if your opening activity takes 15 minutes, then you will most likely not finish the main lesson, or will end up rushing to finish, which is not good. This brings us to the last part of the lesson.
Ending the Class

In terms of ending the session, it is important that you do not end the lesson with something difficult that is likely to frustrate some students. If there is difficult material in a lesson, try to schedule it around the middle of the session. Do not start something that will require much thought with only 5-10 minutes remaining. It is better to continue the activity at the next session and have students summarize one or two things that they have learned thus far in the activity. A short game that reviews something relatively simple can also serve as a nice ending to the session.

Of course, you should also keep in mind that there are so many variables that determine the dynamics of a lesson that not every lesson is going to go well, and certainly not every lesson is going to go exactly as you planned. The important thing is that you learn from each class and make modifications as necessary.
0.4 Ideas for the First Enrichment Session

Here is an outline of the types of things you could talk about in the first enrichment session.

1. The first session is a chance for you and the students to get to know each other, and for the students to get a sense of what you plan to do with them. You should introduce yourself to the students if they do not know you already, and write down their names so that you can get to know them. It is a good idea to bring along an ice-breaking game to help them to be comfortable with you. One game that often works well is “two truths and a lie”. In this game you write down three things about yourself, two of which are true while one is false, and ask all students to do the same. Each person, starting with you, then reads out the three claims, and the others all write down which claim they guess to be wrong. Each person then adds the correct guesses and the number of the other players who did not pick his or her false claim. The person with the greatest number of points wins. You should use the rest of the first session for a general discussion of mathematics and an outline of the enrichment program.

2. Ask the students what they do or do not like about mathematics. You can also ask them whether mathematics is mainly a mental or physical activity, whether mathematics is different in other countries, or whether mathematics is useful or whether it is merely enjoyable. Depending on the way in which the group has been selected, you should be prepared to hear at least some of the students say that they do not like mathematics at all. This is especially the case when the members of the group have been selected by the teacher for their performance in the subject rather than for their interest in it. Do not take this as a discouraging sign that you are off to a difficult start. In fact it is an excellent opportunity to tell the students that a lot of mathematics is not at all like the questions they are asked to do in mathematics class, and that along with the more familiar, the enrichment program involves types of questions that they would not have thought of as belonging to the subject. Certainly there will be enough new and surprising material in the program to give each student a fresh opportunity to come to like the subject.

3. It is a good idea to bring to this lesson some objects that can be used to illustrate the surprising connection between mathematics and other aspects of the world. Pythagoras, the Greek mathematician who lived from approximately 569 to approximately 475 BC, is thought to be the first to notice that there is a close connection between music and numbers. In particular, on a stringed instrument, say a violin or a guitar, the degree to which two notes played on the same string produce a pleasing harmony is closely related to the numerical relationship between the vibrating lengths of string that produce the notes. You could easily make a lesson around these observations if you have an instrument to bring to the enrichment session. You could demonstrate, for example, that if you play a note on an open string, and then place your finger so that precisely half the string vibrates, you will hear a note one octave higher than the first. A 2:3 ratio of lengths produces a perfect fifths; a 3:4 ratio produces a perfect fourths, and so on. If you enter Pythagoras and music on a web search engine, you will find many resources that discuss this connection between music and mathematics. To Pythagoras and his followers this connection seemed so amazing that it became the basis of a philosophy in which everything in the world was
thought to consist of numbers, much as we might say today that everything consists of atoms.

For example, at the url
you can find a discussion that includes the following quote:

“The Pythagoreans wove their musical discoveries into their mathematical cosmology to produce a hauntingly beautiful description of the Universe. The Pythagorean Universe consisted of a central, spherical earth surrounded by the heavenly objects. These were attached to crystal spheres at distances determined by the regular solids (solids which can be circumscribed by a sphere). The rotation of these spheres produced wondrous musical harmonies. The Pythagoreans explained that normal people can not hear the ”harmony of the spheres” because they have grown too accustomed to hearing it from birth (Pythagoras alone was supposed to be able to hear it). Nonetheless the quest for the mathematics behind these harmonies captivated some of the greatest minds over the next two thousand years.”

You could use this website, and others like it, to gather information about the remarkable life of Pythagoras, as well as some of the even more remarkable myths surrounding his life. Much of his thinking informed the work of the Greek philosopher Plato, and thus Pythagoras can be said to have had a profound influence on the Western world.

To our modern ears, explaining everything in terms of numbers seems a little “over the top”. Mathematics does not account for everything we see around us. And yet, there is almost nothing in science, industry, or commerce that is not touched by mathematics in significant ways.

4. For a final fun activity in the introductory session you could challenge students with some puzzles. Here are two that relate to the subject of Chapter 1, even though they are not, strictly speaking, mathematics puzzles. In each case, the challenge is to continue the pattern:

(a)  

| A | B | C | D | E | F | G | H | ⋮ |

The correct answer here is that the letters that are written without curved parts go on top of the line, while those with curves in them go at the bottom.

(b)  

OTFFSSENT ⋮

Here the letters are the first letters of the number sequence: One, Two, Three, etc.
Chapter 1

Number Patterns and Explanations

When we’ve been there ten thousand years,
bright shining as the sun,
we’ve no less days to sing God’s praise
than when we’d first begun.

(from A Collection of Sacred Ballads, 1790)

Introduction to the Chapter

As the title suggests, the focus of this chapter is on patterned sequences of natural numbers. When you start reflecting on it, you soon come to the realization that the concept of a sequence is far from clear. We can imagine all the natural numbers (even if we cannot write them all down at once):

\[ 1, 2, 3, 4, 5, 6, \cdots \]

The entire infinite collection is an example of a sequence. The same can be said about the even numbers. They, too, constitute a sequence

\[ 2, 4, 6, 8, 10, 12, \cdots \]

But now suppose we had a salt shaker that produced natural numbers at random. If we were to put the numbers produced by my “number shaker” in the order in which they fell out, and if we were to go on shaking forever, would the resulting collection be a sequence? Certainly if we were write down the first five terms of that sequence, it would be unfair to ask students to guess the next number. In other words, if this row of numbers is to be called a sequence, it is a
quite different sequence from the more usual type we will be considering in what follows. The
problem in thinking about sequences is related to the difficulties that arise when we try to think
about infinite sets. How can we really expect to picture an infinite set when all the sets we see
around us are finite? In fact we would have considerable difficulty imagining a set of one trillion.
When we try to think of it, all we manage to do is to picture the symbol for it, a one followed
by twelve zeroes. Even if we try to picture in our mind’s eye a set of 100 objects, probably the
best we can do is to imagine an arrangement of the objects, say ten rows of ten each. Similarly,
picturing the “sequence” produced by the number shaker is a hopeless task. All we can do is
imagine the activity of shaking, and the process of numbers tumbling out, but we have no hope
of picturing the finished product. It should be added that it is equally unclear what we mean
when we say that numbers come out of the salt shaker at random.

While there may be instances (and there will be such instances later on in this book) when it
makes good sense to agree that if we allow a random number generating process to go on forever
we get something infinite which we should refer to as a sequence, when we do so we are making
an assumption - extending the concept of existence, and assuming that it will continue to make
good sense. The truth is that in the examples we want to look at especially closely, though the
sequence is to be understood as the infinite final result, it presents itself in terms of a clearly
describable process that generates the numbers one by one. In other words, we will look to
understand sequences in terms of their patterns - in terms of the rules that generate them. Seen
in this light, a pattern is a function that assigns a number (the element of a sequence) to each
given natural number \( n \) (the place in the sequence). For example, the sequence 1, 4, 9, 16, . . .
assigns the number 4 to the natural number 2 (because the second number in the sequence is
4), and the number 16 to the number 4.

When we have a particular generating rule, or “pattern rule”, we can write down any given piece
of the sequence generated by it. In particular, we can ask how this sequence will continue. We
can even try to find a formula, a calculation, that will produce the \( n \)th term of the sequence
when the number \( n \) is entered in the formula. It will be appropriate to ask students what such
a formula would look like, for example, in the case of the sequence 1, 4, 9, 16, . . . . If the first few
terms of a sequence are given but the rule is not stated, we have a “guess the rule” question.
This is in fact the case in many of the instances below. In such instances the first goal is to
describe a pattern rule that fits the data, and then, if possible, to produce a formula for the \( n \)th
term, or at least a clear description of what we would do to get the next term once we have a
set of them.

Especially interesting, and central to the purpose of this chapter, are the examples where two
different pattern rules will seem to generate the same numbers. In these cases, we might want
to know whether this is a coincidence, which will be unmasked once we generate a large number
of terms, or whether this is necessarily going to be so no matter how far we proceed. The latter
is an occasion for an explanation, or proof. You could put it this way: It is impossible to really
comprehend an infinite set of objects apart from the process that generates it. The process,
unlike the sequence, allows for a finite description. In the same way it is hopeless to try to check
the equality of two sequences produced by different pattern rules, other than by some sort of
finite process of logical reasoning.

Thus a good part of this chapter is devoted to making students aware of two really big surprises:
In the first place, students should come to feel how astounding it is when two apparently unrelated rules produce the same sequence. The second surprise is that it is possible to explain logically that it must be so - that it is not necessary to check all the terms. The plan is to get students to believe that there are no coincidences in mathematics. Either there is an explanation for the equality of the outcomes, or the equality should fail at some point. By making students feel unsure about the identity of the outcomes of two pattern rules we prepare them for the proofs presented (in informal fashion) later in this chapter.

The Chapter’s Goals

This chapter serves five main goals:

1. To understand the difference between an (infinite) number sequence and a pattern rule
2. To explore some of the peculiarities of infinite sets
3. To recognize some pattern rules as functions and to introduce algebraic notation to indicate these functions
4. To notice that on occasion, as if by coincidence, two different pattern rules appear to produce the same number sequence.
5. To find geometric arguments (proofs) explaining these coincidences.

Overview of Activities

- E.A. 1.1 - Recognizing Patterns
  An introduction to (infinite) number sequences and the concept of the pattern rule; the terms of a sequence; introduction of function notation for sequences.

- E.A. 1.2 - The Infinite Motel
  A problem that introduces students to the peculiarities of infinite sets and allows for practice of function notation for sequences.

- E.A. 1.3 - Pattern Rules Producing the Same Sequence
  An investigation of cases where two distinct pattern rules seem to produce the same sequence; an introduction to the idea of proof.

- E.A 1.4 - The Role of Proof
  Two pattern rules that seem to produce the same sequence, but don’t in the end, are explored as a caution that we should be suspicious of claims that are not supported by good evidence.

- E.A. 1.5 - Patterns Produced Geometrically
  Problems that involve numbers based on geometric patterns.
1.1 Enrichment Activity - Recognizing Patterns

In this lesson we begin a discussion of number patterns. For the moment the purpose is only to get the students to understand a pattern as a rule or procedure, which may be given explicitly or may have to be discovered by observation, and to distinguish the rule from the (infinite) sequence of numbers that is produced if we apply the rule over and over. In effect, the pattern rule can be thought of as a machine, and the number sequence as the product produced by that machine.

Lesson Goals

- To get students to look for patterns in number sequences, and to think of these as pattern rules or instructions for continuing a number sequence.
- To get students to imagine the number sequence produced by a pattern rule as an infinite sequence, whose existence can be imagined even if we can never produce more than a small portion of it.
- To introduce notation that facilitates the description of a pattern rule.

Materials

In principle this lesson does not need any materials. However, the lesson can go particularly well if you have a computer and data projector available and have available software that produces the sequences you want to study. On the StepAhead website associated with this manual, http://www.queensu.ca/stepahead/, you will find appropriate software. Select “Numbers → Games, Puzzles & Demos → Number Patterns → What Comes Next?” to find a Java applet that provides sequences at random and allows the instructor to prepare his or her own sequences before the start of a lesson.

Lesson Sequence

1. Begin by studying the sequence

\[ 1, 3, 5, \ldots \]

You can do this by writing these first three terms on the board. Alternatively, you can make use of the Java applet ”What Comes Next?” described above. In that case you should enter the sequence beforehand as a ”New Teacher Sequence” by typing in a pattern rule that will generate it. The most obvious formula would be (choosing the third of the four templates that appear when you press ”New Teacher Sequence”)

\[ t(n) = n \times 2 + (-1). \]
1.1 Enrichment Activity - Recognizing Patterns

In either case, ask students to guess the next term in the sequence. Keep asking students for more terms until it is clear that they all implicitly understand the pattern rule that determines how the sequence goes. Ask the class to formulate a sentence that describes this rule clearly. Students will probably not come up with the pattern rule $n \times 2 - 1$, and you should not try to get them to formulate this rule at this point, for they are likely to come up with an alternative rule that will generate the same numbers. Expect something like “Each term is produced by adding two to the one before it”. Tell the students that they have formulated a pattern rule for this sequence, and invite the class to think of this sentence as describing a machine that can be used over and over to make the sequence longer and longer. Invite them to imagine the entire infinite list of numbers that would be produced if the process were to continue for ever and ever. Tell them that this infinite list of numbers (which can always only be imagined in its entirety) is the infinite sequence produced by the pattern rule.

2. Tell the students that the numbers in a sequence are known as the terms of the sequence. Thus the first term of the sequence is 1; the second term is 3; the third term is 5; and so on. Ask the students to make up a table with two columns. The second column will contain the terms of the infinite sequence, in order. The first column simply has the numbers 1, 2, 3, . . . . The left hand column gives the number or index of a term, whereas the right hand column gives the value of the term. Thus the fact that the third row of the table has entries 3 and 5 means that the third term of the sequence is (or has the value) 5.

3. Tell the students that you happen to know that the 123rd term in the sequence (if we were to continue generating numbers long enough) has the value 245. Ask them how this would be entered in the table, and then what the next row in the table would be. You should end up with something like Table 1.1.

Notice that the table is deliberately left “open” at the end. The students should try to imagine this as an infinite table with only some of its entries visible to us. Also, the open section in the middle between the third row and the 123rd row should be pictured as much larger than is shown in this finite representation of the infinite table.

4. A particularly helpful image for distinguishing the number or index of a term in a sequence from its value is the image of an infinite row of pigeon holes or mail boxes. The mail box with the number 2 on the door contains the number 3; the box with the number 3 on the

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>123</td>
<td>245</td>
</tr>
<tr>
<td>124</td>
<td>247</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Table 1.1: The numbers and the values of the sequence terms
door contains the number 5. Thus we can ask questions such as “what will we find in the 57th mailbox?”

5. Remind the students of their formulation of the pattern rule. Suggest to them that it will be convenient to invent mathematical shorthand for expressions such as “the 25th term” and “the 3456th” term, and for sentences such as “the 123rd term is 245”. Ask them to suggest a simpler way of writing “the 25th term”. Take their suggestions seriously, and if they suggest one that is reasonable you can adopt it for the remainder of this course. The one that is perhaps simplest and agrees with mathematical convention uses the symbol \( t(25) \) to indicate the 25th term. Thus the 123rd term is indicated by the symbol \( t(123) \).

There is nothing special about using the letter \( t \) here. It happens to be the first letter of the word “term”. If in a later discussion we want to talk about two different number sequences we may well denote the terms of the second sequence by the symbol \( s \) so that its 25th term is \( s(25) \). Now ask students how they would translate, in the abbreviated notation, the statement that the 3rd term of the sequence is 5. It should come out as “\( t(3) = 5 \)”. Note that the syntax of this mathematical sentence is very similar to that of the sentence ”The content of mailbox 3 is equal to 5”. Similarly, the statement that the 123rd term is 245 becomes “\( t(123) = 245 \)”.

6. Now ask students how they would use this shorter notation to re-write their sentence describing the pattern rule. Start with the translation of a specific example such as “the 745th term is two more than the 744th term”. In our choice of notation it might come out as “\( t(745) = t(744) + 2 \)”. Ask them to translate several more, such as “the 26th term is two more than the 25th”, which becomes “\( t(26) = t(25) + 2 \)”. Doing this a number of times invites a further simplification: You want to be able to say that any given term is two more than the one before it. We need a notation for the expression “any given term”, as a generalization of “the 3rd term” or “term number 123”. If no suggestion is forthcoming you should suggest that we should speak of the “\( n \)th term”. A good notation for this term is \( t(n) \). To complete the translation you also need an expression for “the term before it”. Ask for suggestions. Someone may suggest \( t(m) \) since \( m \) precedes \( n \) in the alphabet. This is really quite a good response. However, since \( n \) represents an (arbitrary) number, it makes even better sense to use \( n-1 \) rather than \( m \) for the number preceding \( n \). If we choose this, then the pattern rule becomes

\[
t(n) = t(n - 1) + 2
\]

To complete the discussion of this sequence you could add a row at the top of the table produced earlier, to indicate your choice of symbol for the \( n \)th term, to make it look like Table 1.2.

7. Now turn to the study of the sequence that begins

\[1, 4, 9, \cdots\]

Again, you can do this on the board, or you can set up a computer and data projector and produce the sequence in the Java applet ”What Comes Next?” described above. Keep asking students to guess terms until it is clear that they all see the pattern rule. Ask students to formulate the pattern rule they have discovered. At this point something interesting
may happen. You may find (it is even likely that you will) that students formulate two very different pattern rules, both apparently producing the same sequence. The two pattern rules you may get are

**Rule 1:** The sequence is made by letting the first term be $1^2$; the second, $2^2$; the third, $3^2$; and so on.

**Rule 2:** The sequence is made by letting the first term be 1; then letting the second term be 3 more than the first; the third term, 5 more than the second; the fourth term, 7 more than the third; and so on.

If you do not get both of these formulations, you should draw students’ attention to the other as well. If you get a number of superficially different formulations, they will each correspond to one of these two, and some time should be spent on getting each student to see which of the two theirs is equivalent to. Stress the fact that these two pattern rules are very different, and that at this point we are not really sure that the two pattern rules produce the same infinite sequence since no one can ever go through all the terms of the sequence to check. In other words, there seems to be no obvious reason why the difference between two successive “square numbers” should go up by two each time. The apparent identity of these two number sequences will be studied more fully in Section 1.3

8. Ask students to try to formulate the two pattern rules, Rule 1 and Rule 2 in the notation we learned earlier. In other words, we would like some sort of mathematical sentence that begins $t(34) = \cdots$ or, better still, $t(n) = \cdots$. If you wish to distinguish these pattern rules from the one we studied earlier in this lesson, you can choose new letter in place of $t$, such as $u$ for Rule 1 and $v$ for Rule 2. Students should notice that with Rule 1, this is not too difficult, for it can be expressed as

$$u(n) = n^2$$

Point out to the students that, unlike the sequence $t(n)$ studied earlier, the calculation of the $n$th term does not require that we know the term before it first. To illustrate this, ask students to calculate $u(1000)$, and then ask them if they could calculate $t(1000)$ (or $v(1000)$) as easily.
9. Now present students with the first few terms of the Fibonacci Sequence:

\[1, 2, 3, 5, 8, \cdots\]

and ask them to continue the sequence. If you are using computer and data projector, you can create this in the "What Comes Next" applet as a "New Teacher Sequence" by filling the fourth template as follows:

\[t(n) = 1 \times t(n - 1) + 1 \times t(n - 2) \text{ where } t(1) = 1 \text{ and } t(2) = 2\]

Once it is clear that students understand the pattern rule, ask them to formulate it, first as an English sentence such as “each new term is the sum of the two terms before it” and then ask them if they can formulate it as a mathematical sentence, using \(F(n)\) to denote the \(n\)th term. You should get something like

\[F(n) = F(n - 1) + F(n - 2)\]

Make sure you spend lots of time on these translations to mathematics, since this at first this will probably seem quite new to students. In particular, you could ask them first to complete sentences such as \(F(20) = \cdots\) and \(F(37) = \cdots\). Ask students whether it would be difficult or easy to calculate the 100th term of this sequence.

10. To continue the lesson you can discuss some of the other sequences. Again you can write them on the board, or have them produce by the applet either randomly, or after you have entered them on the applet yourself. If the students are tired or there is little time left, you can do this as an easy fun activity, by simply asking students to guess the next terms in each case. If there is time and if you feel students enjoy the attempt to formulate a pattern rule in mathematical notation, you can ask students in each case, or at least in some of the cases, to formulate the pattern rule that way. Here are some sequences you could use, together with (possible) formulations:

\[0, 7, 14, \cdots\]

This one could be described by \(a(n) = a(n - 1) + 7\). Other students may notice that the numbers in the sequence are the multiples of 7, starting with \(0 \times 7\) as the first term. A clever student might notice that this means we can also describe this pattern rule as \(a(n) = (n - 1) \times 7;\)

\[0, 1, 3, 6, 10, \cdots\]

with a pattern rule that is difficult to translate - somewhat like Rule 2 considered earlier. In English is would be something like “Start with 0; add 1 to get the next term; then add 2 to get the next term; then add 3 to get the next term; and so on.” Later we will see that there is a much simpler formulation, but for the moment it seems unavailable.

\[2, 3, 5, 7, 11, 13, \cdots\]

This is the sequence of prime numbers, in order. Recall with the students that a prime number is one that cannot be divided by any number other than 1, or itself. Unlike the other sequences so far, there is no simple pattern rule to express the \(n\)th term in this
sequence. In fact if there were a simple formula for getting even some of the prime numbers (especially large ones) it would be of great interest to mathematicians and scientists.

\[1, 2, 4, 8, \cdots\]

This time students will no doubt notice that each term is the double of the one before it. This means that the pattern rule can be described by the formulation \(b(n) = b(n-1) \times 2\).

A really clever student may suggest \(b(n) = 2^{n-1}\), which is what you get if, starting with the number 1, you keep multiplying by 2.

\[1, 2, 4, 8, \cdots\]

This sequence is produced by multiplying by 2, then by 3, then by 4, and so on. In fact, some calculators have a button for this. Look for a button that has \(n!\) on it. The pattern rule can be formulated as \(c(n) = c(n-1) \times n\). To see this, try out a few examples with small values of \(n\).

11. To close the lesson, you could challenge students to make up their own number patterns, and promise to challenge the rest of the class with these patterns at the start of the next session. Alternatively, you could explore the Fibonacci sequence geometrically, using the demo “Spirals” found on the StepAhead website under “Numbers → Games, Puzzles & Demos → Number Patterns → Spirals?”
1.2 Enrichment Activity - The Infinite Motel

Reflection on the Lesson

In the preceding lesson we asked students to think of each of the sequences as an infinite sequence. At first this can be quite difficult for students to accept, since they see nothing but finite sets in the ‘real world’. Thinking about infinite sets forces us to allow existence to something that at first looks like nothing more than a process. It is not difficult to imagine the process that produces more and more numbers according to a given pattern rule, but it is another matter to think of the process as completed right out to infinity. To do so requires a high degree of playfulness of the sort that is key to much mathematics learning.

Once we allow infinite sets, we also have to accept their strange properties. In the rather whimsical story of the infinite motel we encounter some of those peculiarities. Not only is this useful for students’ understanding of what follows, but the properties of infinite sets are so counter-intuitive that the discussion is guaranteed to be a lot of fun. One of the strange properties of infinite sets has even entered into popular usage: The quote at the top of this chapter is a verse from the familiar hymn “Amazing Grace”. You will see the thought contained in the words of the hymn presented more explicitly in the example of the infinite motel discussed below.

One of the peculiarities of infinite sets is that it is difficult to decide what we shall mean by saying that two infinite sets are the same size. The best answer is that they are the same size if their elements can be matched exactly. The strange thing is that with that definition, a set can have the same size as a set that is contained in it. As a dramatic example, note that the set of even integers has the same size as the set of all integers, for we can match each integer to its double. The connection to patterns is that there is really no way to describe a matching between infinite sets other than by means of a pattern.

Lesson Goals

- To reflect with the students on the peculiarities of infinite sets
- To practice the algebraic formulation we introduced in the preceding lesson

Materials

No special materials are required for this lesson, but if you asked students at the end of the last lesson to submit their own pattern rules, you should have these with you so you can challenge the whole class with them.
Lesson Sequence

1. Review the words "pattern rule", "sequence", and "term". Test the meaning of the words using a simple sequence, such as $2, 4, 6, 8, \cdots$ and asking students to identify the 3rd term, to describe the pattern rule (either in an English sentence or in a mathematical sentence), to give the value of the 30th term, to give a description of the whole infinite sequence (the set of all even numbers in their natural order).

2. If at the end of the preceding lesson you challenged students to invent their own pattern rules, now is the time to discuss them. For each pattern rule you should present the first few terms of the sequence that pattern rule will produce, and ask the class to guess the next few terms. After that you can ask the students to try to formulate the pattern rule in a sentence. In some cases it will be reasonable to ask them to provide a mathematical expression for the pattern rule. However, our experience has shown that students like to make up very complicated pattern rules for which a simple algebraic formulation is out of reach.

The infinite motel

3. Tell the students about a situation in which 23 travelers arrive in a motel with 20 rooms, each with a single bed. Can the motel accommodate all the travelers? Obviously not. On the other hand, if there are 20 travelers, the motel will be full and everyone will have accommodation. For two finite sets (travelers and rooms) to match they must have the same number.

4. Now tell the students of a motel with infinitely many rooms, numbered 1, 2, 3, etc. Each room has a single bed only. One day the manager gets a phone call making reservations for the travelers on an infinite bus. The seats on the bus are numbered in the same way: 1, 2, 3, etc. Can these travelers fit into the hotel? How? Since the manager does not have the time to escort each passenger to his or her room. What single instruction can the manager call out at the front of the bus to make sure that all passengers get a room and no two travelers end up in the same room? When this has been accomplished is the hotel full?
The simplest instruction the manager can give the travelers is to look at the numbers on their seats and to go to the room with that same number. In that case, the hotel will certainly be full.

5. Once the students have answered this first set of questions, and they understand the importance of issuing a single instruction that can be interpreted by all the travelers at once, you can continue the discussion by asking the following question: “Suppose that just when the manager has figured out how to instruct the bus passengers to find their rooms, a single traveler arrives, say in a Volkswagen bug, and asks for accommodation. The manager is inclined to say to the Volkswagen driver that he is sorry, but all the rooms are spoken for. If you were the manager’s assistant, what advice would you give? Can the manager change his instruction so that this traveler can also be accommodated?”

Make sure you give the students lots of time to puzzle over this. The (or at least, one) solution is to put the late arrival in the first room, and instruct the bus passengers to look at their seat numbers, to add 1 to those numbers and go to those rooms.

This problem not only provides an interesting illustration of the strangeness of infinite sets, but also is a good chance to review the use of mathematical formulas. Invite the students to express the instruction for the bus passengers as a mathematical sentence similar to the sentences used in the preceding lesson to describe pattern rules for sequences. You can suggest that $n$ might be used as a symbol to express “the number of your seat”, and $r(n)$ as the number of the room the passenger in the $n$th seat is assigned to. Eventually, possibly after some prompting, students should realize that the instruction can be expressed by the equation

$$r(n) = n + 1$$

Notice that the syntax of this mathematical sentence is very similar to "the room for the person in seat $n$ is numbered $n + 1". Once again, students get to see that mathematics
can serve as a language both more precise and more economical than ordinary English.

6. Continue the discussion telling the students that the next day the manager gets calls from the drivers of two infinite buses both of them expecting to stay the night. Can he accommodate them? Are there two simple instructions that can be announced in each of the two buses, that will ensure that the checking in will go smoothly? Again, give the students lots of time. Make sure they have paper on which they can draw representations of an infinite motel and two infinite buses. Something like the following could be helpful for their reflection:

![Motel Diagram]

They may want to experiment by drawing arrows from locations on the bus to rooms in the motel. It is important that they find a way to do this systematically, for unless something exhibits a pattern you will not be able to turn it into an instruction that will work for all the passengers. You could ask the students to complete the following sentences:

- The passengers in the first bus should be told that if they are in seat $n$ then they should go to room $2n$.
- The passengers in the second bus should be told that if they are in seat $n$ then they should go to room $2n - 1$.

There is more than one solution, but the simplest possible is to tell the passengers in one bus to double the numbers on their seats, and take those rooms, and to tell the passengers on the second bus to double their seat numbers and then subtract one. Make sure you let the students try different solutions, and to critique each other’s suggestions. Eventually, the instructions should be written both as English sentences, as above, and as a pair of formulas:

- If the number on your seat is $n$, go to room $2 \times n$;
- If the number on your seat is $n$, go to room $2 \times n - 1$.

![Figure 1.4: Two infinite buses arrive at the same time]
As a final step, invite the students to try to turn this into a pair of mathematical sentences, one for each bus. A possible choice would be

\[
\begin{align*}
r_1(n) &= 2 \times n \\
r_2(n) &= 2 \times n - 1
\end{align*}
\]

7. If you have time, and the students are doing well with the problems so far, you could ask the students how they would handle the situation if there were three infinite buses at once. This time the simplest solution is the following:

- The passengers in the first bus should multiply their seat numbers by 3; that is, they should be told “go to seat $3 \times n$ if $n$ is the number on your seat”
- The passengers in the second bus should multiply their seat numbers by 3 and then subtract 1; that is, they should be told “go to seat $3 \times n - 1$ if $n$ is the number on your seat”
- The passengers in the third bus should multiply their seat numbers by 3 and then subtract 2; that is, they should be told “go to seat $3 \times n - 2$ if $n$ is the number on your seat”

\[\infty\quad\infty\quad\infty\quad\infty\]

Figure 1.5: Three infinite buses arrive

There are other solutions, however, so give students time to examine each others’ suggestions!
1.3 Enrichment Activity - Pattern Rules Producing the Same Sequence

Reflection on the Lesson

In this lesson we continue the discussion of patterns. In particular, we begin to look more closely at pairs of pattern rules that seem to lead to the same sequence. Two things will be attempted. In the first place we will try to get the students to understand that when this happens it is surprising, and in the second place we will try to convince them that we cannot be sure by writing down a piece of the sequence that the two rules will continue to give the same sequence forever. This will lead to a discussion of proof, a finite way of showing that in some cases the two pattern rules will necessarily lead to the same result no matter how long we continue.

Lesson goals

- To reinforce the distinction between a pattern rule and the infinite sequence of numbers that results when the rule is applied for ever and ever

- To distinguish between ‘nice’ pattern rules that allow you to calculate the $n$th term for any $n$ directly, without first calculating the preceding terms, and ‘not so nice’ pattern rules for which the preceding terms must be calculated first.

- To help students see that when two different pattern rules seem to produce the same number sequence, it is not possible to be sure that they do, simply by checking a number of terms.

- To give students an example of two different pattern rules that not only seem to produce the same number sequence, for for which we can show by a geometric argument that they must.

Materials

- If you are going to use a computer and a data projector, you should go to the StepAhead website http://www.queensu.ca/stepahead/ and follow the path ”Numbers → Games,Puzzles & Demos → Number Patterns → ”What Comes Next?” to get the sequence generating applet for the course. Use the ”New Teacher Sequence” option to enter the pattern rule $t(n) = n^2 + 0$ with $n$ starting at 1.

- You should prepare an overheads of Figures 1.7 and 1.8 below, as well as an overhead copy of the first few square numbers, as in Figure 1.6
Lesson sequence

1. Remind the class of the terminology introduced in the preceding lessons: The rule that generates the numbers is called the pattern rule, and the resulting numbers constitute a sequence. Remind the students that a sequence is an infinite set— it goes on forever and ever.

2. Now, picking up a discussion started in Enrichment Activity 1.1, present the following pattern rule to the students:

   **Rule 1:** Create a sequence by letting the first term be $1^2$; the second term $2^2$; the third term $3^2$; and so on.

   Ask students what the first three terms of the sequence are going to be. Now ask students whether they are able to say what the 100th term of the sequence is going to be. It should not take them too long to realize that it is, quite simply, $100^2 = 10,000$. Once students see what to do to get the 100th term, ask them for some other terms, say the 20th or the 300th. You should finish the discussion of this pattern rule by asking them if they can complete a mathematical formulation of this pattern rule in the form

   $$r_1(n) = \cdots$$

3. Now turn to the second of the two pattern rules we want to compare:

   **Rule 2:** Create a sequence by letting the first term be 1; the second term, 3 more than the first term; the third term, 5 more than the second term; the fourth term 7 more than the third term; and so on.

   Write it on the board or on an overhead transparency.

   After examining the first few terms, ask the students if they know the 100th term of the sequence produced by this pattern rule. They will soon realize that to get to the 100th term, they first have to calculate the 99 that come before it. Draw attention to the fact that in this respect this pattern rule is very different from Rule 1. In particular, it will be very difficult to write a formula for the $n$th term in the form

   $$r_2(n) = \cdots$$

   Suggest that rules such as Rule 1 should be called ‘nice’, whereas rules for which all the terms up to a certain term have to be computed before that term can be found, should be called ‘not nice’.

4. Some students may feel that they are able to say that the $n$th term produced by Rule 2 is the same as the $n$th term produced by Rule 1, and that therefore it, too, is quite simply $n^2$. Though it would be nice if we could be sure about this, remind the students that we cannot be sure that both rules produce the same sequence forever and ever just because the first few terms are the same. The sequences are infinite; you will never have enough time to check all the terms. What is needed, in other words, is some way to see directly that the two pattern rules produce the same sequence, without checking term by term.
1.3 Enrichment Activity - Pattern Rules Producing the Same Sequence

term. What is needed is some connection between the two pattern rules, even though we were not able to see one initially. When we get to the proof that indeed they do produce the same sequence, we will see that the proof itself is essentially an observed pattern - a pattern of connections. You might say that it is a “proof rule”, a rule for generating an infinite sequence of proofs. In technical mathematical language, it would be presented as a proof by induction.

5. You should now challenge the students to find their own explanation why Rule 1 and Rule 2 will produce the same sequence, no matter how far you continue it. To get them started on this, you could ask them whether there is any way to see, apart from calculating the numbers, that to go from $3^2$ to $4^2$ you have to add 7, or why you have to add 9 to go from $4^2$ to $5^2$, and 11 to go from $5^2$ to $6^2$. To give them a hint you could ask them some leading questions. For example, you could ask them why we refer to $n^2$ as “n squared”. Or you could ask them “if you had to draw a group of 25 dots, what would be the best way to arrange them?” With any luck, one of the students will suggest a $5 \times 5$ square. Now suggest they do the same for the number 36, and to compare the results. You should give the students ample time to work on this problem. 10 or 15 minutes are not unreasonable. At the end, ask students to come to the board to present their observations to the rest of the class.

Square Numbers: The solution is found in representing the numbers produced by Rule 1 geometrically as “square numbers”:

![Square Numbers Diagram]

Figure 1.6: The first few square numbers

6. If in this sequence of pictures we compare one square with the one immediately before it, then we get something like the following diagram, where the 5th and the 6th square numbers are suggested in a single diagram by using circles rather than stars to indicate the units that belong to the 6th but not the 5th square number.

You can see that the circles form a border around (part of) the 5th square number, and that there are 6 circles along the bottom and 6 along the side, with the corner circle included in both counts. Thus the number of circles is $6 (= \text{the width}) + 6 (= \text{the height}) - 1 (= \text{the corner being counted once too often})$. In general, if you were to combine the $(n - 1)$th and $n$th square numbers in the same sort of diagram, there would be $n + n - 1$ points in the border. Thus, to get from the $(n - 1)$th square number to the $n$th you have
to add \(2 \times n - 1\). Note that this number is always necessarily odd, and that it goes up by two each time.

7. If there is time left at the end of the lesson, you could ask the students to consider the following pattern rule:

**Rule 3:** The \(n\)th term in the sequence is equal to the sum of the first \(n\) odd numbers. Thus the first term is 1, the second term is \(1 + 3\), the third term is \(1 + 3 + 5\), etc.

Ask the students to calculate the first few terms of the resulting sequence, ask them whether they notice anything, and ask them whether they can explain it.

Of course, this gives the same sequence once again, and the reason is that if you think of a square as a sequence of layers, as in the next picture, then you get the sum of the first few odd numbers:

```
  *  o  *  o  *  o  
    o  o  *  o  *  o  
  *  *  *  o  *  o  
    o  o  o  *  o  o  
  *  *  *  *  *  o  
    o  o  o  o  o  o  
```

Figure 1.8: The sum of the first few odd numbers

Thus the sum of the first \(n\) odd numbers is always a square.

As an alternative, if you have access to a computer and data projector, you can use the demo found in “Numbers → Games, Puzzles & Demos → Number Patterns → Squares” on the StepAhead web site http://www.queensu.ca/stepahead/ to discuss this pattern.
1.4 Enrichment Activity - The Role of Proof

Reflection on the Lesson

This lesson is optional. It is intended as an opportunity to reinforce the thought, presented in the preceding lesson, that to be sure that two different pattern rules produce the same number sequence you have to have an explanation (a “proof”). The fact that the first few terms are the same for each sequence is not sufficient for that. In other words, we want to convince students that they should be suspicious of general claims made without a logical explanation. We do it by exhibiting two pattern rules that produce the same sequence for a while, but then diverge. You will have to judge whether you will be able to hold students’ interest through this lesson. If you think not, you can safely skip it.

Lesson Goals

- To explore the nature of proof
- To present two pattern rules that do not produce the same sequence though they seem to do so initially
- To explore proofs for other apparent coincidences between sequences.

Materials

- Bring overheads and printed copies of the templates provided in Section A.1 (see Step 1 in the Lesson Sequence).

Lesson Sequence

1. Present the following pattern rule to the students. For now do not show your students more than the first three terms. You want them to understand what the instructions mean, but you do not want them to form any conjectures as yet about the sequence that will result:

**Rule 4:** To produce the $n$th term of the number sequence, draw a circle, and put $n$ dots on the circumference. Join each of these dots to every other dot by straight lines, and count the number of parts into which these lines divide the interior of the circle. When the number of dots is larger than five, you have to avoid situations with more than two lines going through the same point in the interior of the circle. To do this you may have to move one of the dots on the circumference a little bit.

Ask students to draw circles individually or in groups, and use these to calculate the first three terms of the sequence. Make sure they understand the pattern rule.
Figure 1.9: Pattern Rule 3
2. Ask the students to guess what the next term of the sequence will be. Hand out the first two pages (four circles) of the template. These will confirm their calculations of the first three terms. Ask the students to use the fourth circle on the handout to calculate the fourth term generated by this pattern rule.

3. Ask students if they feel ready to make a guess about the next two terms of the sequence. They will probably guess that every term is the double of the one prior to it. In other words, they will guess that the first six terms are

\[1, 2, 4, 8, 16, 32\]

Now hand out the last page of the template, and ask the students to count the number of regions in each of the last two circles. They should find that the fifth term is, indeed, 16, but that the sixth is 31, and not 32 as they supposed. Point out to them that this pattern rule can serve as a warning that even when a pattern seems to hold at the start of a sequence, it will not necessarily continue that way. If you really want to be sure that the pattern produced by one pattern rule produces the same sequence as that produced by another (simpler) pattern rule, such as “double the number each time”, then there has to be a clear explanation (a proof) of why that is so.

This discussion illustrates the difference between *inductive* and *deductive* reasoning. Inductive reasoning draws conclusions from (finite) experimental evidence, and is subject to revision when further evidence presents itself. Inductive reasoning lies at the basis of much of science, and plays an important role in mathematics in the generation of conjectures (guesses). However, at the *heart* of mathematics, more so than any other discipline, the thinking is deductive. Deductive reasoning is logical reasoning that proceeds from conditions assumed or known to hold, to necessary consequences.

4. For the rest of the lesson, discuss two pattern rules that will produce the same sequence, although you will not tell them this for the time being. Consider the following:

**Rule 5:** The \(n\)th term is the sum of the first \(n\) numbers. That is, it is equal to \(1 + 2 + 3 + \cdots + n\).

**Rule 6:** The \(n\)th term is equal to \(n \times (n + 1) \div 2\).

Once again, ask the students to notice that there is no obvious connection between the two rules. Now ask them to calculate the first 6 terms. You may have to help them interpret Rule 6 if they have not seen much algebraic notation. The evidence will once again lead to the conjecture that the sequences will be the same. Ask students whether either of these two pattern rules is ‘nice’ or ‘not nice’ in the way we introduced these terms in the preceding lesson. They should notice that Rule 5 is ‘not nice’, since to calculate the 100th term, say, you first have to add from 1 to 99 (giving you the 99th term) and then add another 100 to it to get the 100th term. So, in effect, you have to calculate all the terms before it to get to the 100th term. Of course, you could add the numbers in reverse order, but that still represents the same amount of work. Rule 6 is ‘nice’ because a single calculation, namely \(100 \times 101 \div 2\), is enough to find its value.
Point out to the students that if we can really succeed in *proving* that the two rules give the same sequence, we can use the simple calculation in the second rule to find the sum of the first \( n \) whole numbers, for any \( n \).

5. Ask the students to come up with an explanation by doing something analogous to the proof that Rule 1 and Rule 2 give the same sequence; that is, by trying to represent the numbers \( 1, 1+2, 1+2+3, \ldots \) (those produced by Pattern Rule 5) geometrically. Let them work on this on their own, giving them lots of time. Eventually, one of them will suggest the following triangular shapes to represent these “triangular numbers”. You may have to give them a hint by asking what shape you can get by combining two copies of one of the triangles.

![Triangular numbers](image1.png)

**Figure 1.10: Triangular numbers**

6. Tell them that they should now look for a clever, geometric, way to get something that has \( n \times (n+1) \) stars in it, since that will allow them to make a connection between the two pattern rules. You may have to give them a hint by asking what shape you can get by combining two copies of one of, say, the last triangle.

![Two copies of the same triangle form a rectangle](image2.png)

**Figure 1.11: Two copies of the same triangle form a rectangle**

The construction we are looking for is as follows: Take the \( n \)th triangular number. Make a second copy of it, flip it over and fit it to the right of the original triangle, resulting in a rectangle of width \( n+1 \) and height \( n \). This rectangle has \( n \times (n+1) \) stars in it. Therefore, the original triangle has \( n \times (n+1) \div 2 \) stars in it. In other words,

\[
1 + 2 + 3 + \cdots + n = n \times (n+1) \div 2.
\]

7. Remind the students that this gives them an easy way to calculate the sum of the first \( n \) numbers. For example, to add the numbers 1 to 100, all you have to do is multiply 100 and 101, and divide by 2, to get 5050.
8. It is quite possible that one of the students will suggest a simpler way to obtain this formula. In our experience it is quite common to find a student who will suggest that to sum the numbers from 1 to 100, you should add 1 and 100, 2 and 99, 3 and 98, \ldots. If this is followed up, it will work, and will eventually lead to the formulas suggested above. It gets a little complicated when you reach the “middle” of the list of numbers, but if a student suggests this solution, we recommend that you follow it up. In fact the student’s suggestion can be made more transparent as follows: Write this on the board:

\[
\begin{array}{cccccccc}
1 & + & 2 & + & 3 & + & \cdots & + & 98 & + & 99 & + & 100 \\
100 & + & 99 & + & 98 & + & \cdots & + & 3 & + & 2 & + & 1 \\
101 & + & 101 & + & 101 & + & \cdots & + & 101 & + & 101 & + & 101
\end{array}
\]

Explain that each number in the third line was obtained by adding the numbers in the first two lines directly above it. Ask the students how many terms there are in the third line. They can see this by comparing the third line to the first, for it will then be clear that in the third line the number 101 is added to itself 100 times. Ask the students what the sum in the third line is equal to. They will see that it must be \(100 \times 101\). Now ask the students how the total in the third line compares to the total of the first. Since the second line is the same as the first line, though written in reverse order, it should be clear to the students that the total of the third line is twice the total of the first. Therefore,

\[
1 + 2 + 3 + \cdots + 100 = 100 \times 101 \div 2.
\]

Make sure the students see that this discussion works for any sum of successive integers, not only if it ends at the number 100.

9. This section is optional, and should be included only if the students seem to be enjoying the material on patterns. In it you will suggest two additional pattern rules, both of which produce sequences that are closely related to the sequence produced by Rule 5 and Rule 6. In addition, you will challenge the students to explain a relationship between the sequence of square numbers and the sequence of triangular numbers.

To start, give the students the following two pattern rules:

**Rule 7:** The \(n\)th term of the sequence is obtained by drawing \(n - 1\) lines on the page, making sure that every line intersects every other line (no two of them are parallel) and that no three go through the same point, and then counting the number of regions in which the page has been divided. Notice that this means that for the first term you have to draw 0 lines, so that there is just one region.

**Rule 8:** The \(n\)th number is obtained by drawing \(n\) dots on the circumference of the circle, joining each dot to every other dot by a straight line, and then counting the number of lines. Of course, this means that the first term produced by the rule is 0, for when you draw just one dot there are not going to be any lines.
10. Ask the students to find the first six terms of the sequences produced by these pattern rules, and ask them for their observations. The results will be as follows:

**Rule 7 produces:** 1, 2, 4, 7, 11, 16, \cdots

**Rule 8 produces:** 0, 1, 3, 6, 10, 15, \cdots

11. Ask students whether the number sequence produced by Rule 8 looks familiar, and whether they notice a relationship between the sequences produced by Rules 7 and 8. They should notice not only that these two rules produce sequences that differ by one, but also that if you eliminate the first term produced by Rule 8, then the remaining terms of Rule 8 are the same as those produced by pattern rules 5 and 6.

12. To explain the apparent connection between Rule 7 and Rule 5, ask the students whether they can think of a good reason why the terms produced by Rule 7 go up by 1, then by 2, then by 3, and so on. The explanation might go as follows: When you have no lines, obviously there is just one region (the whole page) so the first term is 1. To get the second term, you draw the first line, which creates two regions. When you draw the second line (to make the third term), it divides every region it passes through, creating a new region. Thus it divides the region it begins in, and when it crosses the line already on the page, it enters a new region, which it also divides. Thus it creates two new regions. That is, the third term is two more than the first term. In general, suppose we are calculating the \((n + 1)\)th term. That means there are already \(n - 1\) lines on the page, and we are going to add one more line in such a way that it crosses each of the \(n - 1\) existing lines exactly once. When we draw the new line, it starts in one of the existing regions, dividing it in two. Every time the new line crosses one of the existing lines it enters a new region, which it proceeds to divide into two parts. In total, the new line will pass through \(n\) regions, dividing each of them. Thus the new line produces \(n\) new regions. This means that the \(n\)th term is created from the \((n - 1)\)th term by adding \(n\). In other words, you can see why the terms produced by Rule 7 go up by 1, then by 2, then by 3, and so on.

13. Now this description of the effect of Rule 7 is almost exactly the same as the description given in pattern rule Rule 5. The sequence produced by Rule 5 started with the number 1, then added 2, then 3, then 4, etc. You could imagine a number 0 before the first term of the sequence produced by Rule 4, to produce the sequence

\[0, 1, 3, 6, 10, 15, \cdots\]

This produces numbers that go up exactly the same way as those produced by Rule 7, except that in Rule 7 you start with 1 rather than with 0.

14. There is a similar explanation why the sequence produced by Rule 8 goes up the way it does. To construct the \(n\)th term we draw a new dot on the circumference and connect that by \((n - 1)\) lines to each of the \((n - 1)\) dots already there and already connected to each other. Thus the \(n\)th term is obtained from the \((n - 1)\)th term by adding \((n - 1)\). This time this is exactly the same as Rule 5, except that in that case the resulting term would be called the \(n\)th term rather than the \((n - 1)\)th. That is, if we discard the first term of the sequence produced by Rule 8 we get the sequence produced by Rule 5.
15. The last part of this lesson draws attention to a relationship between the sequence of square numbers (the sequence produced by Rule 1) and the sequence of triangular numbers (the sequence produced by Rule 4). Remind the students of the sequences produced by those pattern rules by repeating the rules, and putting the first few terms of each sequence on the board:

Sequence produced by Rule 1: 1, 4, 9, 16, 25, 36, ⋯

Sequence produced by Rule 4: 1, 3, 6, 10, 15, 21, ⋯

Ask the students if they notice any connection between the two sequences. Give them time to find it, and eventually one of them will notice that each term in the first sequence is equal to the sum of the corresponding term in the second sequence and the one immediately before it. Thus 16 = 10 + 6 and 36 = 21 + 15.

16. Once again, challenge the students to decide if this will continue, pointing out that if they are going to answer this question in the affirmative, they have to find an explanation why this is so. If the students need a hint, suggest that (once again) they should think of the numbers in terms of geometric arrangements as before. After a bit they will notice that if you take two adjacent triangular numbers, turn the smaller one and place it next to the larger, you get a square number. Figure 1.12 is a picture of the 6th square number indicating how it is composed of two adjacent triangular numbers.

17. To close this enrichment activity, you could play one of the Numbers Jeopardy games posted on the StepAhead web site under “Numbers → Games, Puzzles & Demos → Number Patterns”. In fact, it is easy to use Power Point to make your own Jeopardy games.
1.5 Enrichment Activity - Patterns Produced Geometrically

Lesson Goals

• To present students with some interesting pattern rules generated geometrically
• To practice writing a general term using mathematical notation
• To turn a pattern rule into one that yields a single formula (that is, a ‘function’ expressing the nth term, or a ‘nice’ pattern rule).

Materials

• If you have access to a data projector and a computer, you should consider making a power point presentation showing the geometric patterns used below. If you do not have access to this technology, you could prepare overhead transparencies of some of the geometric patterns included in this Power Point file. These are included in the Appendix to this manual in Section A.2

Lesson Sequence

1. Ask students to consider the sequence

\[ 0, 3, 8, 15, 24, 35, 48, \ldots \]

Ask them whether they notice any pattern. In view of their experience with the sequence of square numbers, some may notice that these numbers are each one less than the corresponding square number. Others may notice that the numbers go up by 3, then by 5, then by 7, and so on. However, this time our goal is to look for a ‘nice’ pattern rule. This means that we want to make use of the first option, and describe the nth term as

\[ t(n) = n^2 - 1 \]

or

\[ t(n) = n \times n - 1 \]

2. To confirm students’ understanding of the rule, ask them for, say, the 20th term, or the 100th term.

3. Now show students the powerpoint slide, or the overhead picture, of the geometric pattern consisting of chairs around a longer and longer table, as in Figure 1.13. A larger copy is available in the Appendix in Section A.2. Notice that the number of each term is represented in the number of tables: The first term has one table, the second term has two tables, and so on. The purpose of this exercise is two-fold. On the one hand, the
students should look for a pattern in the number of chairs at each table. This can take several forms. Some students may say that every time a table is added, it should be added somewhere in the middle, with one chair on each side, so the number of chairs goes up by two at each step. This is, of course, correct. They have found a pattern rule, but it is not a ‘nice’ rule in the manner suggested earlier in this chapter. It is not a rule that allows them easily to calculate the 100th term, or to find a general formula for the $n$th term.

This brings us to the second purpose of the exercise, for when this is suggested as the pattern rule, students should be challenged to see if there is a ‘nice’ pattern rule that will describe the number of chairs. A good answer (and one that students may have suggested early on) is that when there are $n$ tables, then each table has two chairs on either side of it, for a total of $2 \times n$ chairs, except that at the two ends of the row of tables there are two extra chairs, for a total of $2 \times n + 2$ chairs. So a good formula for the $n$th term in this number sequence is

$$c(n) = 2 \times n + 2$$

4. Now show students the powerpoint slide, or the overhead picture, of the geometric pattern of white flagstones around a row of blue flagstones. A copy of this picture is seen in figure 1.14, and a larger copy is available in the Appendix in Section A.2.

As in the problem with the chairs around tables, the number of the term is clearly represented in the geometric arrangement: The first term involves one blue flagstone; the second term involves two blue flagstones, and so on. Once again, students may notice that the number of white flagstones goes up by two each time. However, this way of describing the pattern rule does not allow students to calculate the 100th term easily, nor will it allow them to find a formula for the general term. On the other hand, if they notice that each
blue flagstone has a white one above it and a white one below it, and that in addition to this there are three white flagstones on the left and three on the right, then they can conclude that the \( n \)th term is given by

\[
f(n) = 2 \times n + 6
\]

and thus that the 100th is \( 2 \times 100 + 6 = 206 \).

5. The problem presented in the next powerpoint slide, or in the second flagstone picture, Figure 1.15 below, is a little more complicated. As always, a copy of the figure is available in the appendix, in Section A.2. This time, the number of the term is not represented geometrically. In fact, the number of blue terms in the 2nd term is \( 2^2 \). Similarly, the number of blue terms in the 3rd term is \( 3^2 \), and so on. See if the students can tell you what the number of blue terms will be in the 20th term, or in the 100th, or in the \( n \)th.

However, the question asks about the number of white flagstones. Again, students may start by comparing these numbers and notice that they go up by four. However, a more helpful analysis involves noticing that the 5th term (for example) has a \( 5 \times 5 \) square of blue flagstones, and that adjacent to the edges of that square there are 5 flagstones to the left, 5 to the right, 5 at the top, and 5 at the bottom. This makes for a total of \( 4 \times 5 \). However, then we still have to include the corners, for an additional 4 flagstones. In other words, the total number of white flagstones in the 5th little patio is \( 4 \times 5 + 4 \). For the \( n \)th term it is

\[
g(n) = 4 \times n + 4
\]

by exactly the same analysis.
6. You can close the lesson by challenging the students to find general formulas for the $n$th terms of the sequences

- $3, 5, 7, 9, 11, \cdots$
- $2, 5, 8, 11, 14, \cdots$
- $2, 8, 18, 32, 50, \cdots$

The patterns are not difficult to discern. In the third sequence, the terms are the doubles of the square numbers. The challenge, for the students, is to find a general formula (a ‘nice’ pattern rule) in each case.

There are several ways to do this. In the first sequence, for example, you could argue that to get to the 5th term, you have to add 2 four times to the initial term 3; or to get to the 20th term you have to add 2 nineteen times to the initial term 3. In general, this will yield the formula

$$t(n) = (n - 1) \times 2 + 3$$

Another student may put it differently: You can extend the pattern backwards, and regard the first term as calculated by adding 2 once to the number 1; and the second term, by adding 2 twice to the number 1, and so on. If it is analyzed that way, the formula comes out as

$$t(n) = n \times 2 + 1$$

Of course, as anyone with a little algebra will see right away, these two formulas are equal. By a similar argument, the second sequence has the pattern rule

$$s(n) = (n - 1) \times 3 + 2$$
or, equivalently,

\[ s(n) = n \times 3 - 1 \]

The last sequence produces

\[ u(n) = 2 \times n \times n = 2 \times n^2 \]

7. You might draw students’ attention to the fact that the first two sequences go up evenly, while the third one goes up faster and faster as you add terms. Point out to them that this has something to do with the fact that the third term has \( n \times n \) in it, rather than just \( n \). Doing this will anticipate nicely some of the ideas discussed in the next section.

8. As an alternative closing activity, if you have access to a computer and a data projector, you can discuss the sums of successive even numbers using the demo “Rectangles” found under “Numbers → Games, Puzzles & Demos → Number Patterns” on the StepAhead website. Notice that this connects (again) to the earlier discussion of the sum of successive numbers, for the even numbers are doubles of the natural numbers.
Chapter 2

Growth Rates of Sequences

Tiger, tiger, burning bright
In the forest of the night,
What immortal hand or eye
Framed thy fearful symmetry?

William Blake, 1757-1827
(Songs of Innocence and of Experience)

This chapter continues the discussion of number sequences, but from a different point of view. This time the emphasis is not on detecting and coding the pattern rule, as was the case in the preceding chapter. Instead we will focus on the fact that some sequences grow much more rapidly than others.

The Chapter’s Goals

• To observe the fact that different sequences grow at very different rates
• To explore what it is about some sequences that makes them grow very fast
• To study the sums of successive powers of 2 in the context of an old fable
• To become familiar with exponential notation for powers higher than squares.

Overview of Activities

• E.A. 2.1 - Measuring the Growth Rate of a Sequence
  An investigation of growth rates of various number sequences.
• E.A. 2.2 - The Poor Soldier and the King
An investigation into the sums of successive powers of 2, an illustration of the power of exponential growth, and of measuring the “size” of a number by counting its digits.
2.1 Enrichment Activity - Measuring the Growth Rate of a Sequence

In this lesson we return to some of the sequences discussed in the first lesson. However, this time we will look at a different kind of behaviour, namely the rate at which the numbers in the sequences get larger and larger. There is enough material in this lesson to spread it over two one-hour sessions.

Lesson Goals

- To observe that number sequences grow at very different rates
- To form conjectures about the kind of number sequence that grows faster than others
- To explore the use of the length of a number as a measure of its size.

Materials

- The students should have calculators for this lesson.

Lesson Sequence

1. We will begin with a story that contrasts a sequence that grows “linearly” (at a constant rate) with one that grows exponentially (at a rate that remains proportional to the magnitude of the term). We will then look at several sequences in more detail, exhibiting linear growth, quadratic growth, exponential growth and growth that is more than exponential.

Tell the students the following story:

A merchant owes a large amount of tax. The tax collector has warned him that if he does not pay up in 30 days, all his possessions will be sold at auction. Filled with despondency, the merchant takes a walk in the woods at the edge of town. There he meets a little man by an old oak tree, who asks him why he looks so sad. The merchant tells his story. The little man tells the merchant that he is moved by the merchant’s story, and that the merchant may choose one of the little man’s magic money boxes. The first of these boxes is white. It is empty, but every day the amount of money in the box will go up magically by $1001. The other box is black. It has $1 in it, but every day the amount of money in the box will double. The little man also tells the merchant that he must not open the box until the day he needs the money, for once the box is opened the amount of money in it will stop increasing. Which box should the merchant choose?
Get the students to think about this question, using calculators where they can. At first it will seem attractive to choose the white box. In fact, if the merchant needs the money as soon as, say, the 10th day, then the white box is the better choice. However, after 30 days the white box will have roughly $30 \times 1000 = 30,000$ dollars in it. By contrast, the black box will have $2^{10}$ dollars after 10 days, which is equal to 1024 and can be thought of as approximately 1000 dollars, far less than the white box at that stage. However, in 30 days the total will be $2^{30} = 2^{10} \times 2^{10} \times 2^{10}$ dollars, which is approximately $1000 \times 1000 \times 1000 = 1,000,000,000$ dollars. This is far more than the white box!

The story illustrates that asking which of two sequences grows faster is a relative term. One sequence may grow faster at first even if the other may start growing faster once we have let enough terms pass. Give the students lots of time to explore this. In this enrichment activity we are mainly interested in knowing how fast (relatively speaking) a sequence grows in the long term, but this intention should not be announced until students have had time to get a feel for what “in the long term” means.

To help the students think the problem through, you could ask them which box the merchant should pick if he needs his money in ten days, or after how many days the black box begins to be the better choice. Once they see that the black box is the better choice at the end of 30 days, you can ask them whether they think the white box will ever become the better choice again, say after 100 days or 1000 days. Ask them to give reasons for their answers.

The growth in the white box is called linear. That is, the rate of growth remains the same from day to day. Every day adds the same to the total. The growth in the black box is exponential, which means that every day the total is multiplied by the same factor. In the long run, exponential growth always outstrips linear growth.

2. For the second stage of this lesson you will look at some other sequences in order to examine their rates of growth. Here are some sequences you could look at, together with the formulas that express the pattern rules producing the sequences. We recommend giving the first three of these only at first, in order to help students focus on the question without feeling overwhelmed.

   \( a(n) = n \times 7 \)  
   \( b(n) = n \times (n + 1) \div 2 \)  
   \( c(n) = c(n - 1) + c(n - 2) \)  
   \( d(n) = 2^n \)  
   \( e(n) = 1 \times 2 \times 3 \times \cdots \times n \)  
   \( f(n) = n^2 \)

You could frame this part of the lesson in terms of a story as well: The little man could offer a range of boxes, one for each of the sequences, and you could ask the students to consider different times for opening the box.

3. Ask students, preferably in small groups, to put the first three sequences in order from the one that grows most slowly in the long run to the one that grows fastest in the long run. This is a somewhat vague question, but precisely for that reason, it is one that can lead to very interesting discussions between students about what it means and how it can be decided. Give them lots of time to think about it.
2.1 Enrichment Activity - Measuring the Growth Rate of a Sequence

4. When they get stuck, or when you think additional questions will help their reflection, ask the students to say about how long it takes for each sequence to get to size 1000 (or 2000, or 10,000).

This question, if posed for the first sequence, will require dividing 7 into 1000, which of course does not give a whole number. However, one could say that it will take about 140 or 150 terms in the sequence before the sequence reaches a number close to 1000. Once the students agree on that, you could ask them how long, approximately, it will take for that same sequence to reach 2000. The answer is that it will take about twice as long. You can check this by dividing 2000 by 7, but you can also say that this sequence “goes up by the same amount each time”, or that it “does not speed up or slow down”, so reaching 2000 should take about twice as long as reaching 1000. Similarly, reaching 1,0000 should take about 10 times as long as reaching 1000 did. This is an example of what in mathematics is referred to as a sequence that increases ‘linearly’; its rate of increase stays the same as you go along.

When you ask the students how long it takes the sequence in example 2 you will have to let them use their calculators. They will find that it will take 45 terms in the sequence before the number 1000 is reached. If they recognize the sequence as the one studied earlier as the sequence of triangular numbers, they could use the formula provided by Rule 5. in Section 1.4 to calculate how long it will take to reach 1000, for that amounts to finding the first whole number \( n \) for which \( \frac{n \times (n + 1)}{2} \) is greater than 1000; in other words, the number \( n \) for which \( n \times (n + 1) \) is more than 2000. They will see that this number is 45. Now ask them to see how long it takes to reach 2000. They will discover that this happens after 63 terms. In other words, unlike the preceding sequence, it will not take twice as long to reach 2000.

You can ask the students whether there is something about the sequence of example 2 that might have led them to expect that this sequence “speeds up”. They will probably suggest something like “when you are further along in the sequence you are adding larger numbers to get to the next number, so you should expect it to go up faster than before”. Do not try to get too precise about it. All that matters is that the students reflect on it a little and see that there is a relationship between the pattern rule by which the sequence is created and the rate at which it goes up.

Now turn to the Fibonacci sequence (example 3) and ask how many terms are needed to reach 1000 or 2000. The students will soon notice that this sequence goes up much faster than the ones considered before this. In fact, once you have reached 1000 (after 17 terms), the next term is larger by the size of the preceding term, which may be somewhat less than 1000, but the one after that will go up by at least 1000, so it is not surprising that it grows so fast.

In fact for sequences that grow as fast as this one, we should really be looking for a different measure of number size. Rather than asking about the size of a number in the usual sense, suggest to the students that they could measure the size of a number by counting the number of digits in it. You could then ask the question of the rate of growth of the Fibonacci sequence as follows: How many term does it take before you get a two digit number? (7) Or, How many terms to get to three digits? or four? They will find
that using this very different way of measuring the size of numbers the “size” of the terms (i.e. the number of digits required to write the terms down) goes up at a more or less even rate. In fact the first terms that have two, three, four, five, six and seven digits are, respectively, the 7th, the 12th, the 17th, the 21st, the 26th and the 31st. As you can see the number or digits increases about every fifth (or sometimes fourth) term.

At the end of the discussion so far, it should be clear to the students that, of the first three sequences, the third grows most rapidly (in the long run), followed by the second, with the first sequence growing most slowly.

5. At this point you should show students the last three sequences and ask them to discuss, again in groups at first, how the growth rates of these sequences compare to the first three. They will soon notice that the sequences in examples 4 and 5 are also very fast growing sequences, so you should suggest that they measure these numbers, too, by counting the digits required to write them down rather than by measuring size in the conventional sense.

In the case of the sequence of example 4 you should review the exponential notation, for the grade 7 students may not have seen it before. As always, the symbol $2^3$ is shorthand for $2 \times 2 \times 2$, $2^5$ is short for $2 \times 2 \times 2 \times 2 \times 2$, and so on.

By convention, $2^0 = 1$. This convention is by no means arbitrary! There are several good ways to justify this convention. Perhaps the simplest is to observe that in the sequence of powers of 2,

\[
2, \quad 2^2 = 4, \quad 2^3 = 8, \quad 2^4 = 16, \quad 2^5 = 32, \quad 2^6 = 64, \ldots
\]

going from right to left involves dividing by 2, and involves decreasing the exponent (the superscript that indicates the power) by one. Thus, if we wanted to go to the left starting from $2^2 = 4$, we see that we ought to refer to 2 as $2^1$, and that $2^0$ should be what you get if you divide 2 by 2, namely $2^0 = 1$! This convention about exponents is analogous to the convention that $0 \times 2 = 0$, or for that matter that zero times any number is zero.

In any case, the sequence in example 4 can be written in the form

\[
2^0, 2^1, 2^2, 2^3, \ldots
\]

It is important as a preparation for things we want to say later about this sequence, to write down at least the first 11 of these powers of 2, and to refer to them often enough that students will start memorizing them, especially the last of these terms:

\[
2^0 = 1, \quad 2^1 = 2, \quad 2^2 = 8, \quad 2^4 = 16, \quad 2^5 = 32, \ldots \quad 2^{10} = 1024
\]

Ask the students what the answer will be if they multiply $32 \times 32$. The goal is to get them to recognize that when you multiply $2^5 \times 2^5$ you get $2^{10}$. That is, exponents are added. This follows directly from the fact that $2^5$ is shorthand for $2 \times 2 \times 2 \times 2 \times 2$, for if you multiply this list of factors by another copy of the same list you end up with ten factors.

Challenge the students to find out on each of their calculators the greatest power of 2 the calculator can display in full (without going into scientific mode), and in the process to write down how long it takes for the terms of the sequence to grow to two, three, four,
five . . . digits. This should not take very long. They will find that, as in the Fibonacci sequence, the length of the terms grow at a more or less equal rate. (In fact, the number of digits will grow at a rate of one digit for every three or four terms in the sequence - this has to do with the fact that 10 lies between \(2^3\) and \(2^4\).)

The fifth sequence grows even faster. This sequence is created by multiplying successive whole numbers. The mathematical notation for that is \(n!\), which is pronounced “\(n\) factorial”. For example, \(5! = 1 \times 2 \times 3 \times 4 \times 5 = 120\). A fancy calculator may have a button for the factorial function. Here are the first eleven terms:

\[
1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, 39916800
\]

It takes four terms to reach two digit numbers, but by the time you reach the ninth term the number of digits goes up at each step. This is not surprising in view of the kind of sequence this is. After the ninth term you will be multiplying by numbers at least as big as 10, and multiplying by 10 already adds a digit, so multiplying by a larger number will certainly add at least one digit to the number. Note as well that by the same principal, once you get past the 99th term, the number of digits will go up by at least two at every step, and so on.

The last sequence grows much more modestly. It does not grow as fast as an exponential sequence, but faster than a linear sequence. This sequence is produced by squaring numbers; its growth is known as quadratic growth. Its rate of growth increases, but (remember our discussion of Rules 1 and 2 in Section 1.3) the rate at which the terms go up increases linearly.

6. As a closing exercise you could ask students each to make up a pattern rule they think will grow very rapidly. If you have access to appropriate computer software, or a good programmable calculator, you could offer to collect their suggestions, and to program these for a demonstration at the start of the next lesson.

Alternatively, you could play “Race to the Finish” or “Pick the Sequence” with the class. You will find these games on the StepAhead web site under “Numbers → Games, Puzzles & Demos → Growth Rates of Sequences”.

2.2 Enrichment Activity - The Poor Soldier and the King

We will end the chapter on the growth of number sequences by telling the story of the poor soldier and the king. To solve this problem we need not only to estimate the exponential growth of the sequence of powers of 2, but also to observe the growth of the sum of this sequence.

Lesson Goals

• To explore sums of powers of two
• To practice using exponential notation
• To reflect on the number of digits of a number as a way to measure its size.

Materials

• Make a copy for each student of the 'chess boards' needed for Section 2.2 and available in the Appendix in Section A.3

• If you plan to do the last steps (Steps 7 and 8) of this enrichment class, bring a lot of pennies, or poker chips, and prepare to use the demo “Bouncing Boxes” available on the StepAhead web site under “Numbers → Games, Puzzles & Demos → Growth Rates of Sequences”.

Lesson Sequence

1. In this section, you will tell the story of the poor soldier and the king. This story very nicely combines recognition of pattern and growth rate, though it can be done even if you skipped the first section of this chapter.

The poor soldier and the king  The story goes as follows: A rich king wanted to reward a common soldier who had saved the king’s life. The soldier objected at first, saying that he had just done his duty. But the king insisted, so the soldier asked the king for a chess board. When a chess board was brought in the soldier asked if the king would ask the finance minister to put one penny on the first square of the chess board, two on the next, four on the next, etc., doubling the number of pennies each time, and then to let these pennies be the reward. Was the soldier asking for a big reward? If you challenge the students to work out the total, they will quickly discover that the calculation is far beyond the capacity of most calculators.

2. The goal of the lesson is to estimate the amount of money the soldier received. Begin by reviewing the powers of 2. In fact, you should write the first 11 powers of 2 on the board or on an overhead slide:
2.2 Enrichment Activity - The Poor Soldier and the King

\[ 2^0 = 1, \quad 2^1 = 2, \quad 2^2 = 4, \quad 2^3 = 8, \quad 2^4 = 16, \quad 2^5 = 32, \quad 2^6 = 64, \]
\[ 2^7 = 128, \quad 2^8 = 256, \quad 2^9 = 512, \quad 2^{10} = 1024. \]

It is important to continue the list at least as far as \( 2^{10} = 1024 \), for this information will be used later in this lesson as well as in a later chapter.

Continue the discussion of the problem until students realize that if they want to calculate the amount of money earned by the poor soldier, they have to be able to add the numbers

\[ 2^0 + 2^1 + 2^2 + \cdots + 2^{62} + 2^{63}; \]

that is,

\[ 1 + 2 + 4 + \cdots \text{ up to the } 63 \text{rd power of } 2. \]

They will soon realize that their calculators are not able to display any of the numbers in the last part of the sequence (at least not without going to some mysterious alternative notation for numbers). So how can they possibly hope to calculate their sum?

3. After a bit of reflection, tell the students that rather than tackling this fairly difficult problem directly, they should first try some simpler examples. In other words, they should try smaller chess boards and see what the total will come to for smaller boards. Since we cannot really play chess on a smaller board anyway, we will be unconcerned if the “chess boards” we start with are not square, or even rectangular.

Figure 2.1 represents a sequence of small “chess boards” with one to ten squares. Give each student a copy of this figure - a template is available in the Appendix in Section A.3.

Ask the students to calculate the number of pennies that would be placed on each of these chess boards. The first chess board would obviously result in a single penny. Below the second chess board, the students should write \( 1 + 2 = 3 \). Make sure to ask them to write it in this form. The third chess board should get the caption \( 1 + 2 + 2^2 = 7 \). The captions should be written out in full, in order that the students will notice a pattern once they have completed the page. The remaining captions will go as follows:

\[
\begin{align*}
1 + 2 + 2^2 &= 7 \\
1 + 2 + 2^2 + 2^3 &= 15 \\
1 + 2 + 2^2 + 2^3 + 2^4 &= 31 \\
1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 &= 63 \\
1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 &= 127 \\
1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 &= 255 \\
1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 &= 511 \\
1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9 &= 1023 \\
1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9 + 2^{10} &= 2047 
\end{align*}
\]

Ask students if they notice anything about the sequence of totals created by their calculations. Do they notice any patterns? They may notice that at each step the next total is
twice the preceding total plus 1. If you have been careful to draw their attention repeatedly, in the previous section and earlier in this one, to the first 11 powers of 2, they may notice that the sum of successive powers of 2, starting at 1, is always precisely one less than the next power of 2. Thus \( 255 = 2^8 - 1 \) and \( 1023 = 2^{10} - 1 \), and in general,

\[
1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9 + 2^{10} + \cdots + 2^n = 2^{n+1} - 1.
\]

4. Tell the students that, indeed, this pattern for sums of powers of 2 is true no matter how many you add, and that you will explain later why this is so. Ask the students what this means for the amount of money placed on the real \( 8 \times 8 \) chess board in the story. If this pattern is also valid for this sequence of powers of 2, then it provides us a way to simplify the question about the total amount of money the poor soldier will receive, for then we have

\[
2^0 + 2^1 + 2^2 + \cdots 2^{62} + 2^{63} = 2^{64} - 1.
\]

In other words, “all” we have to do is to find \( 2^{64} \) and then subtract 1 from this. It may still seem like a big task, but finding one large power of 2 is simpler than calculating 64 of them and adding them all together.

5. Even so, calculating \( 2^{64} \) is no small task. How are you going to manage that since you found earlier that calculators could not compute numbers that large? You could, of course, do it by hand. But that would take a very long time. Here is an idea you should suggest: Try to do an approximate calculation rather than an exact calculation. Remind students
that $2^{10} = 1024$. You could say that $2^{10}$ is **roughly equal to 1000**. Mathematicians indicate this approximate equality by writing

$$2^{10} \approx 1000$$

6. Given that observation, ask the students what $2^{20}$ is approximately equal to. Give them time to think about it. The question tests their understanding of the meaning of $2^{10}$ and $2^{20}$. Here is the kind of answer you are looking for:

$$2^{20} = 2 \times 2 \times 2 \times \cdots \times 2 = \underbrace{2 \times 2 \times 2 \times \cdots \times 2}_{20 \text{ times}} \times \underbrace{2 \times 2 \times 2 \times \cdots \times 2}_{10 \text{ times}} \approx 1000 \times 1000 = 1,000,000.$$  

Now ask the students to find approximate values for $2^{30}, 2^{40}, 2^{50}$, and $2^{60}$ by the same method. They will find that

$$2^{60} \approx 1,000,000,000,000,000,000,000.$$  

Finally, then, since $2^{64} = 2^4 \times 2^{60} = 16 \times 2^{60}$, we conclude that the poor soldier will receive about

$$16,000,000,000,000,000,000,000,000$$ cents,

which equals $$160,000,000,000,000,000.$$  

Not bad for a poor soldier!

7. If there is time, and if you have students’ attention up to this point, you should now turn attempt an explanation of the pattern

$$1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9 + 2^{10} + \cdots + 2^n = 2^{n+1} - 1.$$  

Since this is a statement that is supposed to be true for all whole numbers $n$, and since we can never expect to check all of these infinitely many cases, the best we can hope for is discernment of an explaining pattern — some sort of pattern that ties the total $2^{n+1} - 1$ directly to the summation $1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9 + 2^{10} + \cdots + 2^n$.

One thing you notice right away is that instead of taking 1 away from the power of 2 on the right, we could add it to the sum on the left. That is, we could try to connect $2^{n+1}$ to $1 + 1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9 + 2^{10} + \cdots + 2^n$.

Again, you should have the students try this on one or two small ‘chess boards’, say one with 4 squares or one with 5 squares. Divide the class into groups and give each group enough of the pennies (or poker chips) you brought, and ask them to draw a chess board with 4 squares, where the squares are made big enough to fit the pennies or the poker chips. Then ask the students to put poker chips on them in the manner suggested in the story of the poor soldier and the king. Remind them that at this point there are $1 + 2 + 2^2 + 2^3 + 2^4$ pennies on the board. Now tell them they are going to add one more...
penny to the board by placing it on the smallest pile. Ask them if they notice anything. They should notice that the first two squares now have the same number of pennies on them (namely 2). Now tell them to take the pennies on the first pile and place them on the second. Now they should notice that the resulting pile has the same number of pennies as the next, the reason being that that third pile was made in the first place to have twice as many as the second pile. So place these two piles on top of each other, on the third square. This will then have twice as many as it had before you did this, namely 8. But twice as many is also what you put on the fourth pile in the first place. So, if as a final step you put the third pile on top of the fourth, it will double that again for a total that is precisely the double of what was on the last square in the first place.

Students should see that this process will necessarily continue in this way no matter how many squares there are on the chess board. If it seems not clear to them, you can ask them to do the same exercise with a chess board of 5 squares.

Thus, if we have an \(8 \times 8\) chess board, and if we then put pennies on them as prescribed in the story, and if then we come with one extra penny and place it on top of the penny already on the first square, we get the sum

\[
2 + 2 + 4 + 8 + \cdots 2^{62} + 2^{63}.
\]

Now combine the first two terms. Their total will be twice 2, that is 4. But now we have this sum:

\[
4 + 4 + 8 + 16 + \cdots 2^{62} + 2^{63}.
\]

If we total the two 4's at the start of this sum, their total will be \(2 \times 4 = 8\). So replace the 4's by that 8. But now there are two 8's. Their sum is \(2 \times 8 = 16\), producing two 16's at the beginning of the sum. You can see that this must continue, for every time we get two copies of a power of 2 and add them up, their total is that power of 2 multiplied by 2, which is automatically the next higher power of 2. So if we continue we eventually get to

\[
2^{63} + 2^{63} = 2 \times 2^{63} = 2^{64}.
\]

8. The same explanation can also be presented on the board in the following geometric form: Drawing squares on the board, ask students to imagine you have a row of (finitely many) stacks of boxes. The first stack has one box; the second has two boxes; the third has four; the fourth is twice as high again. Templates for the diagrams below can be found in the appendix in Section A.3. Tell the students you want to know how many boxes there are in total, without doing a lot of mental work. Tell the students you do this by borrowing one box, and placing it in front of the row of stacks. Draw an extra square on the board to represent this.
Now erase the extra square and place it on top of the first box. Observe that this makes the first stack the same height as the second stack of boxes. Place the first stack on top of the second, making it twice as high, and therefore as high as the third stack. Continuing, observe that this will end with a stack twice as high as the last of your row of stacks. In other words, after supplying one extra square we were able to combine the resulting squares into a stack twice as high as the highest stack we had originally. If the highest stack was $2^n$ squares high, the stack we get at the end will be $2 \times 2^n$ squares high, or $2^{n+1}$ squares. The first few steps in this process are illustrated below for the case $n = 4$. The fact that this needed one extra square to start indicates that $1 + (1 + 2 + 2^2 + 2^3 + \cdots + 2^n) = 2^{n+1}$. In other words,

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1.$$
Chapter 3

Division

Rain-diamonds, this winter morning,
embellish the tangle of unpruned
pear-tree twigs; each solitaire,
placed, it appears, with considered
judgement, bears the light
beneath the rifted clouds—the indivisible
shared out in endless abundance.

Denise Levertov, 1923-1997
(from Sands of the Well, 1996)

The central goal of this chapter is to get the students to understand that every number can be expressed as a product of prime numbers, and that this can be done in only one way, apart from the order of the factors. The students should come to think of the prime factors as essential building blocks that characterize a number in important ways. An analogy would be the fact that every water molecule is made up by one oxygen atom and two hydrogen atoms. Any different collection of atoms will result in a material with totally different properties.

The Chapter’s Goals

- To review and discuss tricks for checking whether numbers can be divided by 2, 3, 4, 5, and 9
- To observe that each number has a unique factorization into prime numbers
- To explore ways in which the properties of a number are characterized by its prime factors
- To learn how prime factorizations connect to greatest common factors and least common multiples
Overview of Activities

- E.A. 3.1 - Prime Number Factors
  An investigation of divisibility properties of numbers and of their prime factorizations.

- E.A. 3.2 - What Prime Factors Tell You
  A study of ways in which the prime factorization of a number determines many of its properties.

- E.A. 3.3 - Number Magic
  A magic trick that depends on prime factorizations.

- E.A. 3.4 - How Many Primes are There?
  An investigation into the infinity of primes, and the search for very large prime numbers.

- E.A. 3.5 - Greatest Common Factor
  A study of how prime factorizations can be used to identify the greatest common factor of a set of numbers.

- E.A. 3.6 - Least Common Multiple
  A study of how prime factorizations can be used to identify the least common multiple of a set of numbers.

- Problem Set 3.7 - Practice Using Prime Factorizations
  This problem set extends the ideas explored in the preceding activities and can be used to augment these activities as well as to create additional lessons.
3.1 Enrichment Activity - Prime Number Factors

In this section we introduce the idea of factors of a number, and prime factors in particular. Almost certainly, the students already know from their mathematics classes what factors and prime factors are, and how to determine the prime factors of a number. In the lessons that follow, however, the concept will be taken much further. The most important thing is to give the students an idea of the uniqueness and the importance of the factor list.

Lesson Goals

- To review the tricks for checking whether numbers divide by 2, 3, 4, 5, or 9
- To review the method for finding the prime factorization for any number
- To review the concept of prime number

Materials

- If you decide to play Multiplication Bingo at the beginning of the lesson, you will either need a computer and a data projector. In that case the game is available on the StepAhead website http://www.queensu.ca/stepahead, where it can be found under “Numbers → Games, Puzzles & Demos → Division”. If you do not have a computer and projector, you can use an overhead projection slide of the game board, together with a pair of paper clips and colored chips or a colored pen, or you can have the students play in pairs. In either case, there is a template for the game in the Appendix in Section A.4.
- The students do not need anything for this lesson other than paper and pencil

Lesson Sequence

1. Begin the class with the Multiplication Bingo game. This game allows students to practice their mental skills at factorization and multiplication. You can divide the class into two teams, or you can have students compete in pairs. If you choose the latter option you will need to print out game boards for each pair of students. The person (or team) who begins the game places the two paper clips on the digits in the bottom row. It is permitted to place both on the same number. If the computer is used to play the game, this choice is made by clicking on the appropriate boxes in the bottom row. The student (or team) then claims the square that contains the product of those two digits. The next player (or team) is then allowed to move one of the paper clips. The object of the game is to be the first to get four squares in a row either horizontally, vertically, or diagonally.

2. Divisibility Put on the board the following collection of numbers (have these ready before the class starts):
Now ask the following questions:
(a) Which of these numbers are divisible by 2?
(b) Which of them are divisible by 3?
(c) Which of them are divisible by 4?
(d) Which of them are divisible by 5?
(e) Which of them are divisible by 9?
(f) Which of them are divisible by 10?

Clearly, to do this properly, it will have to be accompanied by some discussion. Do the students know what is meant by saying that one whole number is divisible by another? Almost certainly the answer is yes, but it would not hurt to discuss it again briefly. A good way to do this is by pointing out that there are many different ways to think of divisibility. Asking the students to come up with genuinely different descriptions of the concept can be a good entry into the discussion. One way to introduce divisibility is to do it in terms of the act of long division and finding that there is no remainder at the end. Another, equivalent, characterisation of divisibility is to describe the larger number as being precisely equal to the sum of a number of copies of the smaller number. This way of thinking of divisibility comes with a geometric version: If you picture the larger number as a length on the number line, then that length can be broken up into a number of lengths each equal to the smaller number. In view of the discussion in the next chapter to explain the rules for divisibility by 3 and by 9, it is very useful (even if it seems obvious to students) to remind them that another very good way of visualising the question of dividing a number by, say, 13 is to ask whether the number can be distributed equally among 13 people. One of the goals of this chapter is to introduce yet another way of characterising divisibility, this time in terms of the sets of prime factors of the two numbers.

3. **2, 4, 5, and 10 as factors** Ask the students how they decided in each case whether the larger number was divisible by one of the smaller numbers. Students in grade eight will have been taught the tricks that allow them to answer the above divisibility questions relatively quickly. Grade seven students may not have seen them, and are in any case less likely to remember them, so here is a review of these ‘sleights of mind’: It is a good idea to teach these to the students, and try to get them to justify the tricks, except in the cases of divisibility by 3 and divisibility by 9. The explanations for divisibility by 3 and 9 are really interesting, but they will have to wait till a later chapter, where they will be presented in the context of a discussion on remainders.

However, the tricks for checking divisibility by 2, 4, 5, and 10 can be explained quite nicely at this stage.: Divisibility by 10 requires almost no explanation. The divisibility tricks are the way they are because we use a number system based on the number 10. It is therefore no surprise that the rule for divisibility by 10 is so simple. If you multiply a number by
3.1 Enrichment Activity - Prime Number Factors

10 you have to put a zero behind it, so if a number is divisible by 10 there must be a zero at the end to take away.

A number is divisible by 2 if it is in the table of 2 (that is, if it is an even number). But that can always be seen from the last digit of the number: it should be one of 0, 2, 4, 6, or 8. Here is one way to explain this: any number ending in 0 is is composed of groups of ten. For example, 12340 can be thought of as 1234 groups of 10. Thus, if we want to divide 12340 between 2 people, we simply divide each group of 10 into two groups of 5 and give one of these to each of the 2 people. If the number does not end in 0, we first set aside the quantity represented by the last digit. The quantity that remains does end in 0, so we begin by splitting that between the two people. Now we try to divide what was set aside. If the last digit was even, we can divide that number between the two people. If not, we cannot. This shows why everything hinges on the last digit.

The divisibility by 5 is treated in exactly the same way: the number must end in 0 or in 5. The explanation for this is very similar to the explanation for divisibility by 2. This time we are trying to share the quantity equally between 5 people. If the number ends in 0, we divide each group of 10 into 5 equal parts. If there is a non-zero digit at the end, we finish by dividing the quantity represented by it among the 5 people. This can be done only if that last digit is 5.

Divisibility by 4 is determined by the last two digits: The reason one digit does not suffice is that 10 is not divisible by 4. However, 100 is divisible by 4. Therefore any number ending in 00 can be divided among 4 people: It represents a certain collection of groups of 100, so all we have to do is divide each group of 100 into four groups of 25 and give one of these to each of the 4 people. If the number does not end in 00, we begin by setting aside the quantity represented by the last two digits. The number that is left will end in 00, so we can divide it among the 4 people. If the number set aside is divisible by 4, we will be successful in dividing the original total between 4 people, but not otherwise. For example, 123456 can be thought of as 123400 +56. The 123400 part can be divided by 4 (that is it can be distributed equally among 4 people) because it represents 1234 groups of 100. Since 56 can also be divided equally into 4 equal parts, we can distribute the whole quantity 123456 equally among 4 people.

4. Divisibility by 3 or 9 As you will undoubtedly remember, divisibility by 3 or 9 is remarkable: Add up the digits of the number. If the result is a big number you may do it again, until you end up with a small number, say of 1 digit. If that number is divisible by 3, so was the original. If it is divisible by 9, so was the original. For the moment, this should be demonstrated to the students by examples only.

The teacher should probably say something also about division by other numbers. In all cases, the divisibility can be checked by simply doing a long division and seeing if there is a remainder. In general there are no neat tricks, or the tricks become more complicated than the long division itself. There is a trick for divisibility by 11 that is not too bad: Add up every other digit, do the same for the remaining digits, and take the difference between the two results. If that number is divisible by 11, so was the original. It should be pointed out that numbers like 0 or -22 are considered to be divisible by 11 for this purpose. This check for divisibility by 11 will be discussed in Section 5.5. It is possible in principle to invent similar tricks for all divisors, but they are not practical.
5. **Prime numbers** It is now time to give the definition of a prime number. A *prime number* is one that is not divisible by any number (except of course by 1 and by itself). Ask the students to decide which of the following numbers is prime:

\[133, \ 29, \ 123\]

The most obvious way to check whether a number is prime is to try to divide it by the whole numbers in turn, until you find one that divides into it. In fact, it is not necessary to check all the numbers. We only need to check whether the *prime numbers* divide into them. Discuss with the students why that is enough. Another question that might be discussed is: “How far do you need to check for divisors?” Get the students to think about this. For example, if you want to check whether 133 is a prime, there is no point in checking for any divisors bigger than 12, for \(12 \times 12\) is bigger than 133, so if 12 or a larger number factors into 133, the other factor has to be smaller than 12 and will therefore have been encountered before you reached 12.

6. **The prime factors of a number** The purpose of this section is to give the students some practice at finding the prime factors of a number, and a chance to see how this information can be used. In speaking of *the* prime factors we tacitly assume that there is only one way (apart from the order of the factors) to write a number as a product of primes. This is true, but not quite obvious. In a university algebra course you would worry about this, and prove it as a theorem. At the level of this programme we will simply assume that it is so. Typically students will consider the unique factorization obvious, especially after doing several factorizations and getting consistent outcomes.

You should discuss with the students a systematic way of getting at the prime factors of a number. The students will almost certainly have been taught a particular way to approach this problem, in which case you should adopt their practice. Otherwise you can set it up as follows, which is undoubtedly equivalent to what they have been taught or may be taught later as part of their regular program:

\[
\begin{array}{c|c}
28 & 2 \\
14 & 2 \\
7 & 7 \\
1 & \\
\end{array}
\]

The numbers in the right hand column are the prime factors of 28. Another, more common way to find the prime factors is to create a “factor tree”:

```
  28
  / \n 2   14
  /   /
 2   7
```
In the end, in answer to the question "What is the prime factorization of the number 28?" they should learn to write the reply as $28 = 2 \times 2 \times 7$, preferably with the prime factor in increasing order.

7. To close the enrichment activity you can play “Factor Find” with the class. This game is found on the StepAhead web site under “Numbers → Games, Puzzles & Demos → Division”.
3.2 Enrichment Activity - What Prime Factors Tell You

We will begin with some simple exercises in finding prime factorizations. These are followed with some questions intended to demonstrate that knowing the prime factorization of a number tells us a lot about the number. There is quite a lot in this section, and some of the problems suggested at the end are very challenging, so you may wish to divide this over two successive classes.

Lesson Goals

- To review the method for finding the prime factorization for a number
- To relate division to a reduction of the list of prime factors
- To investigate properties of numbers that depend on their prime factorizations

Materials

- The students do not need anything for this lesson other than paper and pencil; it is best for this lesson that students not use their calculators.
- A list of the problems discussed in this lesson, without my comments, is found in the Appendix in Section A.5.

Lesson Sequence

1. As a warmup question ask students to determine which of the following numbers are primes.

\[
\begin{array}{ccc}
273 & 37 & 251 \\
83 & 272 & 543 \\
\end{array}
\]

2. Together with the class, write down on the board, in order, all the prime numbers under 100. Keep these on the board so that they can be used as references later.

3. Ask the students to find the prime number factorizations of the following numbers:

\[
\begin{array}{cc}
28 & 39 \\
63 & 34 \\
\end{array}
\]
3.2 Enrichment Activity - What Prime Factors Tell You

Remember that we would like the students to write the prime factors in order of increasing magnitude when they give their answers. After having the students work on this question it is probably a good idea to discuss with them the idea of the uniqueness of the list of prime factors. The analogy with molecules and constituent atoms is good if the students have been exposed to those ideas. Another good analogy is to taking x-rays. Finding the prime factorization of a number is like taking its x-ray. The resulting factorization is the skeleton of the number - it does not tell you everything about the number, but it does tell you quite a bit. As another option, the situation could be compared to essential ingredients in a food item, say a Big Mac with cheese. Try to think of a recipe that will not tolerate any substitution, or an object that changes its designation completely if its components are changed even a little. Perhaps a symbol such as the Canadian or American flag could also serve as an example. If it does not have one red maple leaf and two red stripes, it is a flag, but not the Canadian flag.

4. Ask the students to decide whether 5477 a prime number, and to note down, in order, the calculations they perform in order to come to a decision. When they are finished, or even sooner, when they have done a number of calculations, discuss with them what calculations they choose to perform. There are two important issues here:

(a) Are students trying all factors, or are they trying only prime factors. Remember that the prime numbers up to 100 are on the board for their benefit. Why is it not necessary to try all factors? Have students reflect on this.

(b) How many prime factors are the students testing? $80 \times 80$ is clearly more than 5477, so if there were a prime factor of 5477 greater than 80, the other factor that, together with it, makes the product 5477 would necessarily be smaller than 80 and will therefore have been encountered as a factor before getting to 80. In other words, roughly speaking, when you try prime factors you never need to go beyond the square root of the number. If you had a calculator you could check that $\sqrt{5477} = 74.0067$, but you obviously do not need to know the square root exactly to know when to stop.

It will turn out that, indeed, 5477 is a prime number.

5. Now pose this question to the class: Note that the prime number factors of the number 420 are given by the equation $420 = 2 \times 2 \times 3 \times 5 \times 7$.

(a) If you multiply 420 by 3 you get 1260. Can you decide in your head simply by looking at the list of prime number factors of 420 what the prime number factorization of 1260 is?

(b) If you divide 420 by 2 you get 210 of course. Can you decide in your head by looking at the list of prime number factors of 420 what the prime number factors of 210 are?

(c) If you divide 420 by 3 you get 140. Decide in your head what the prime number factors of 140 are.

(d) If you multiply 420 by 11 you get 4620. Decide in your head what the prime number factors of 4620 are.

(e) If you multiply 420 by 10 you get 4200. Decide in your head what the prime number factors of 4200 are.
The goal of this exercise is to get students to see that multiplying by a number amounts to appending the prime factorization of that number to the prime factorization of the number you started with, and that dividing by a number amounts to removing the associated prime factors.

6. A number $n$ has the following prime number factors:

$$n = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 5 \times 7 \times 11 \times 11 \times 13 \times 13 \times 19.$$ 

Without finding the value of the number $n$, answer the following questions:

(a) you want to divide $n$ by 2, then divide the answer by 2, then divide the answer by 2 again, and so on. How many times can you do this before you stop getting a whole number?

(b) Can the number $n$ be divided by 12? Write down a reason for your answer.

For part (a) the students have to realize that if you divide the number by 2, then the prime factorization of the answer will be the same as the given product, except that one of the 2’s will be missing. You could say that dividing by 2 amounts to “taking away” one of the factors 2 from the list of prime factors. Since there are four factors 2 in the original number, we can divide it by 2 four times before we run out of 2’s. Note that this is the same as dividing by 16.

For part (b) the students have to realize that $12 = 2 \times 2 \times 3$. Therefore, dividing by 12 amounts to “removing” two factors 2 and one factor 3 from the list of prime factors. Note that there are more than enough factors of each type to make this possible.

7. The preceding exercise makes use of the fact that if you know the prime factorization of a number, then other ways to factor that number are obtained by grouping the prime factors. If you have a computer and a data projector available, this point can be demonstrated very well using the demo “Factors” found on the StepAhead web site under “Numbers → Games, Puzzles & Demos → Division”.

7. This question, as well as the next three, are quite challenging. You may not be able to do more than one of them, or perhaps it will be possible to do one with the entire class, working from the board, and then challenging students to do one of the others themselves. Tell the students you have a whole number $x$ in mind. You multiply the number by 5, then you divide the result by 2, then you multiply the result by 7, then you divide the result by 5. When you have done all this you are left with a prime number. What is the value of $x$?

This is a challenging exercise that really forces students to come to terms with the idea that
multiplying a number by a prime amounts to “adding that prime number to the list of primes”, and to similar observations about dividing and squaring. For example, if $a$, $b$ and $c$ represent prime numbers then if you multiply $a \times b \times c$ by $a$ then the prime factorization of the answer is $a \times a \times b \times c$. If you divide $a \times b \times c$ by $c$ the prime factorization of the answer is $a \times a \times b \times c \times c$. To do this problem think of $x$ as if it were written as a product of its prime factors. Now multiplying it by 5 appends 5 to the list of prime factors. If next we divide by 2 we remove a factor 2 from the list of prime factors. In other words, there must have been at least one 2 among the prime factors of $x$. Next we append a factor 7 and then we remove a 5. Thinking in terms of the list of prime factors: We “added” a 5, “subtracted” a 2, “added” a 7, and then “subtracted” a 5. Since we supplied an additional factor 5 and removed it later, the net effect is one of removing a factor 2 and appending a factor 7. The question says that the end result is prime. But we appended a factor 7 to whatever was there already. If the end result is a prime, the end result can only be 7 (the 7 we brought in). This means that when we removed the factor 2, we removed the one and only prime factor that was there to begin with. In other words, $x = 2$.

8. Now tell them you have a number $a$ in mind. If you square the number and then divide the result by 45 you are left with a prime number. What is the value of $a$?

Squaring $a$ amounts to doubling the number of prime factors of each type. Since $45 = 3 \times 3 \times 5$ we next remove these factors from the factor list of $a \times a$. If we are left with a prime number, that number must be 5. That is, $a$ must have been equal to $3 \times 5 = 15$.

9. You have a number $b$ in mind. You multiply the number by 3, then you square the result, and then you divide that by 99. If the final answer is a prime number, what is the value of $b$?

After multiplying by 3 and then squaring, we have $3 \times 3 \times b \times b$. Dividing by $99 = 3 \times 3 \times 11$ removes those factors. In particular, both the 3’s we introduced earlier have been removed, together with a factor 11, which must have been one of the factors of $b$. In other words, $b \times b$ has two factors 11 in it. One of these has been removed, as we said already, so the other one must be the prime factor left. This means that there are no other prime factors in $b$. That is, $b = 11$.

10. You have a number $c$ in mind. You multiply the number by 3, then you square the result, then you divide that by 21, and finally you divide by 12. If you are left with a prime number, what was the value of $c$?
After multiplying by 3 and squaring we have $3 \times 3 \times c \times c$. Dividing by $21 = 3 \times 7$ and then by $12 = 2 \times 2 \times 3$ removes two 2’s, two 3’s and one 7 from the factor list. The 2’s and the 7 must have come out of the product $c \times c$. Since the final answer is prime, it must correspond to the remaining factor 7. That is $c = 2 \times 7 = 14$.

11. To end this lesson you could play a game with the students. The games “Matho”, “Factor Find”, and “Multiplication Bingo” are available on the StepAhead web site under “Numbers → Games, Puzzles & Demos → Division”
3.3 Enrichment Activity - Number Magic

In this section we present a number trick. It was shown to us by Dr. Bill Higginson, a colleague in the Queen’s University Faculty of Education. We are grateful to him for his permission to include his trick in this manual. The game is excellently suited for the teaching of prime factors. When it is first presented it really arouses the students’ interest because it seems magical; when it is later explained to the students it constitutes an excellent application of what they have learned about prime number factors; later still, when the students themselves demonstrate the trick to others, it forces them to practice the factoring skills they have learned.

The Section is not essential to subsequent lessons, so it can safely be skipped if you want to get to other material.

Lesson Goals

- To intrigue students with a number trick that depends on prime factorization
- To let students investigate how the trick works
- To practice finding factorizations for numbers
- To think about the role of place value in multiplication

Materials

- For this lesson students will need their calculators, and of course paper and pencil.

Lesson Sequence

1. Demonstrating the trick To do this lesson you should have a calculator handy. Begin by telling the students that you have the ability to read their minds, at least to some extent, where numbers are concerned. Ask each student to think of a three-digit number, say $abc$, and then to turn this into a six-digit number by repeating the three digits in the same order. In other words, they now have the number $abcabc$. Next ask the students to add the digits in their numbers. Single out one of the students and ask her to tell you the sum of the digits in her number. You will then pretend to concentrate hard and then give her a number which you claim is a factor of her six-digit number. She will check on her calculator, and find your claim correct.

In order to complete this part of the demonstration of you telepathic powers, you, as teacher, will need to know how to choose the factor, and for that you need to know the theory behind it. It turns out that by repeating the three digits to create the six digit number each student has automatically created a number that is divisible by 1001. We will discuss in a moment why this is so. It also turns out that 1001 has just three prime factors:
1001 = 7 \times 11 \times 13. To do the demonstration you have to have this fact memorized. It follows, of course, that no matter what number a student gives to you, any factor of 1001 will automatically be a factor of her number. Thus you could give any one of 7, 11, 13, 7 \times 11 = 77, 7 \times 13 = 91, 11 \times 13 = 143, or even 1001 itself as a factor. So why did you ask for the sum of the digits? Well, doing that expands the range of possibilities: If the sum of the digits is divisible by 3, the list of possible factors of the number is augmented by 3 and by three times each of the factors listed already: for example, 3 \times 77 = 231 will then be a factor. Similarly, if the sum of the digits is divisible by 9 then the list of initial factors can be extended by including the factors obtained by multiplying them by 9. You can see why you should have a calculator handy, for unless you have a much better memory than we have, you will not be able to keep all the possible factors for the different cases in mind. The idea is to vary your answers as you ask each student to give you the sum of the digits, so that the students will not see a pattern.

Here is an example to illustrate the game. Suppose one of the students has picked the number 258. The student turns this into 258258 and tells you that the sum of the digits is 30. You notice that this is divisible by 3 (but not by 9). You decide that you will respond by using this factor 3, together with the factors 7 and 13 of 1001. That is, you claim that the number 3 \times 7 \times 13 = 273 is a factor. And, sure enough, the student finds that 273 does divide into 258258.

2. Eventually, of course, you should reveal to the students how the trick works. This is how you should do this: Ask each student to write down a three-digit number \(abc\) and then to “double” it in the way discussed above to form the number \(abcabc\). Now ask each student to find the prime factorization of this six-digit number, using their calculators if necessary. Collect their results on the board. Thus, one of the results could be

\[
133133 = 7 \times 7 \times 11 \times 13 \times 19
\]

After all the factorizations have been listed on the board in this way, ask students whether they notice anything. They will undoubtedly remark on the fact that every one of the lists of prime factors contains (at least) one 7, one 11, and one 13. Ask the students what that tells them about all of the numbers on the list. Try to nudge them toward the observation that this means that each of them is divisible by the product of these three primes, and that this product happens to be 1001.

3. Ask them why it is that each of the numbers is divisible by 1001. What is it about the way the numbers were constructed that made this so? Once they have had lots of time to struggle with it, you can help them by giving one or both of the following explanations:

\[
abcabc = abc000 + abc
\]

\[
= 1000 \times abc + abc = 1001 \times abc
\]

. This shows that \(abcabc\) is always divisible by 1001.

A second way of showing why \(abcabc\) is divisible by 1001 is to carry out the multiplication
1001 \times abc \text{ by hand. It would go as follows:}

\[
\begin{array}{ccc}
  & a & b & c \\
1 & 0 & 0 & 1 & \times \\
\multicolumn{1}{l}{a} & b & c \\
\multicolumn{3}{c}{+} \\
\multicolumn{1}{l}{a} & b & c & a & b & c
\end{array}
\]

4. Once the students are persuaded that 1001 is always a factor of the six-digit number, ask them to tell you what other numbers must therefore be factors of each of the six-digit numbers. Try to get them to come up with the reasoning. Eventually ask them why you asked for the sum of the digits, and how that added information was used.

5. To reinforce students’ understand the mathematics behind the trick, you could ask them whether a similar trick could be created with two digit numbers. For example, you could ask students to pick a two digit number ab and ask them to repeat it to form the number abab or to form ababab, and then ask them whether the result is divisible by 5. Which of these two alternatives would make a better trick?

6. As always, you can end the session with one of the games suggested earlier.
3.4 Enrichment Activity - How Many Primes are There?

This section is optional. It should be used only if the class is thought to be ready for it. It is perhaps not necessary for the students to understand all of the following, but the teacher should feel secure that they will grasp enough of it for it to hold their interest and not cause frustration. My experience indicates that this is possible with students at this level. The principal difficulty in the section is the use of a form of reasoning known as indirect proof or proof by contradiction.

Lesson Goals

- To intrigue students with the fact that there are infinitely many prime numbers
- To provide students with an example of indirect proof
- To interest students in the search for very large prime numbers
- To give students an occasion to estimate the size of a very large number

Materials

- If you have one available, you should bring a globe to class. If not, a large ball will do.

Lesson Sequence

1. Begin by raising this question: “Does the list of prime numbers go on forever?” The answer is yes, and the students will probably feel quite sure about it. The first challenge is to persuade the students that they really do not have good reason to be so sure. Ask them why they think it goes on. They may respond that it seems to, that it must because the numbers go on forever, or that a book or a teacher said that they do. You could ask them whether they do not think that it is conceivable that once you get to really large numbers you have passed so many prime numbers that after that all further numbers are products of the factors that you have come past.

   The proof that the set of prime numbers is infinite is not long, but it is logically sophisticated, being an indirect proof. Now indirect proof (also called proof by contradiction) is often used in daily life, so it might be useful to give the students an example that is not abstract, and point out to them that it is an indirect proof. For example, suppose there are 400 students in the school. Then there must be some students who have the same birthday. For if all students had different birthdays, there would have to be at least 400 days in the year. We know that there are only 365, so the supposition in italics must be false. A proof by contradiction always has the same basic form. You begin it by supposing the opposite of what you want to prove, and you end by showing that this assumption (the italicized clause) leads to a consequence that is patently false, or even contradicts the supposition itself.
Here is the example we have used to great effect. It is a true story. When one of us (Leo Jonker) was about 14 years old he had never heard of the international date line. One Saturday afternoon, as he was playing with a friend, building something in his front yard (the friend lived in the country), the following thought occurred to him: There must be a line somewhere on the globe, where it is one day on one side and the next day on the other. *because if there were no such line*, it would be possible to do the following: Get on a plane in Toronto at noon on a Monday and start flying West at a speed that is exactly the same as the rate at which the earth turns about its axis. That is, the plane would be traveling in such a manner that it would be noon throughout the trip. From time to time you could radio down to airports on the ground, and to ships when you are over water, and ask them what time and what day it is. Imagine doing this about every five minutes or so, or perhaps even more frequently. Of course the answer would always come back “It is 12 o’clock noon (or at least some time between 11am and 1pm”, and because you are supposing that there is no international date line, they would always add “and it is Monday”. Eventually you would arrive back in Toronto, and it should still be Monday, for that is what people kept telling you when you radioed them. But what has happened in Toronto while you were gone? There it eventually became dark and then light again, the next day, Tuesday. Here we have arrived at a contradiction. The supposition that there is no date line gave rise to this contradiction, and must therefore be wrong. Somewhere there has to have been a point on your trip around the world where at one moment they told you it was noon Monday, and just a few minutes and a few kilometers later they told you it was noon on Tuesday.

In fact, to minimize the inconvenience, the international date line runs through the emptiest part of the Pacific Ocean. The beauty of this story is that it is an interesting illustration of the power of logical (and mathematical) thinking. Without any reference to a geography teacher or an atlas, it is possible to prove that there is an international date line. Of course, this is no guarantee that others will accept the argument. In fact, the discovery led to a frustrating argument with the friend, who refused to accept that something as absurd as an international date line could exist.

Here is the proof that there are infinitely many prime numbers: *Suppose that there were only finitely many prime numbers*. Then we would be able to multiply all of them together. Let us give names to all the prime numbers. We could call the first one $p_1$, the second one $p_2$, and so on. Let us say that the total number of primes is $N$, so that the last prime number should be called $p_N$. Let $A$ be the product of all the prime numbers. (Remember we are working with the supposition that there are only finitely many - otherwise you could not expect to multiply all of them).

$$A = p_1 \times p_2 \times \cdots \times p_N.$$  

Now consider the number $A + 1$. What is the remainder if we try to divide $A + 1$ by $p_1$? Since $A$ is exactly divisible by $p_1$, the remainder will have to be 1. The same conclusion holds for each one of the other prime numbers $p_i$. Thus $A + 1$ is not divisible by any of the prime numbers $p_1,p_2,\cdots,p_N$. Can it be divisible by some other prime? Well, no! For there are no other prime numbers, we supposed. Thus $A + 1$ must itself be a prime number. Can it be one of the ones in our list? Well, no again! For it is bigger than a multiple of each of the numbers in the list. We have reached an impasse: $A + 1$ is prime
and yet it is not on our (supposedly) complete list of prime numbers. The only way out of this is to conclude that the assumption that there are only finitely many prime numbers is false. This is the end of the proof.

2. Here is another question which, though difficult, may be accessible to really good students at this stage: “Are big numbers more likely to be prime numbers, or less likely?” Put differently: When you go along the infinite list of prime numbers do they get farther apart, or closer together? To approach this question you should think of it in terms of the following picture:

Imagine all the whole numbers in order in front of you. The top line in Figure 3.1 represents the numbers 1 to 144 as little rectangles. Below that, for each of the first few prime numbers, from 2 to 41, there is a row of rectangles representing that prime (a white rectangle) and its multiples (grey rectangles). Obviously, a multiple of a prime number is not prime, so in the top row, the copies of all the grey rectangles in the lower rows are also colored grey. This means that the primes between 1 and 144 are indicated by the white rectangles in the first row. Not all of those primes are represented by rows of multiples. By doing this for the primes up to 41 and coloring in all the squares above the multiples of those primes in the top row we guarantee that the white rectangles between 1 and 144 really do represent primes. In fact we did not need to go as far as 41 to achieve that. After constructing the rows up to and including the one for 11, the remaining rows did not produce any new grey rectangles in the top row. Can you guess why?

Each sequence of rectangles in each of the rows following the first is uniformly distributed. So if you pick any number anywhere on the number line, you have a one in three chance of
hitting a multiple of 3, and a one in five chance of hitting a multiple of 5, etc. However, if you pick a large number, say over 10,000, you have to worry about all the prime numbers up to 10,000 (actually only about those up to the square root of 10,000) to see if it is a multiple of any of them, while if you pick a number under 20, you only have to worry about all the prime numbers under 20. Figure 3.1 shows a large white “triangular” area in the lower left hand corner. These blank spaces occur in the rows that correspond to primes larger than the number attached to the columns to which the spaces belong. Thus, large numbers are more likely to be a multiple of something, and therefore less likely to be prime numbers. In other words, the large prime numbers are farther apart (on average) than the small prime numbers. You can see in the top row that the little white rectangles are somewhat farther apart (on average) near the right side of the row.

3. History of large primes Although we showed that the list of prime numbers never stops, it is in fact very very difficult to identify a large prime number and establish beyond doubt that it is one. The search for large prime numbers has a long and interesting history. In recent years, this search has added commercial motivation to its basis in curiosity, for the modern science of cryptography, so central to secure transmission of sensitive information on the internet and on other public networks, depends on the identification of large prime numbers.

In the 1600’s Mersenne became interested in numbers of the form $2^p - 1$, where $p$ is a prime, as a way to find very large examples of primes. These numbers are now known as Mersenne numbers. It looks initially as if all of them are going to be prime numbers. For example, $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 31$, $2^7 - 1 = 127$, are all prime. However, $2^{11} - 1 = 2047 = 89 \times 23$. Very quickly these numbers become very large. Even so, in 1588 the mathematician Pietro Cataldi was able to check that $2^{17} - 1 = 131071$ and $2^{19} - 1 = 524287$ are also prime numbers. Imagine doing this by hand! In 1640 the famous mathematician Fermat showed that $2^{23} - 1$ is not a prime number. In other words, this pattern, which continued successfully for 8 successive prime numbers and encouraged many people to think that it would continue forever, broke down at the 9th prime number. Nevertheless, the hunt for large prime numbers continued. In 1772, The mathematician Euler was able to show that $2^{31} - 1 = 2147483647$ is a prime number. In 1867 it was shown by someone named Landry that $3203431780337$ is a prime number. It is not a Mersenne number, but was discovered as a factor of $2^{59} - 1$. At least two further large prime numbers were discovered before the advent of computers: In 1876 it was shown that

$$2^{127} - 1 = 170141183460469231731687303715884105727$$

is a prime number of 39 digits, and in 1951 it was shown that

$$20988936657440586486151264256610222593863921$$

is a prime number of 44 digits. Although a mechanical calculator was used for the second of these numbers, imagine the cleverness and determination required to check these numbers without the help of a computer! When computers became available, even larger prime numbers were found. Already in 1951 79 digit prime number was identified. In 1952, it was shown that $2^{521} - 1, 2^{607} - 1, 2^{1279} - 1, 2^{2203} - 1$, and $2^{2281} - 1$ are prime numbers. Since then, many other record large numbers have been discovered at a steady rate: $2^{3217} -$
1, $2^{4253} - 1$, $2^{4423} - 1$, $2^{9689} - 1$, $2^{9941} - 1$, $2^{11213} - 1$, $2^{19937} - 1$. Every year or so a larger one would be announced. The last one in this list was obtained in 1971. The search continues.

4. The Great Internet Mersenne Prime Search One of the most interesting projects in the quest for large primes is the Great Internet Mersenne Prime Search launched in 1996. Information about this project, and about prime numbers generally, can be found at the website http://www.utm.edu/largest.html. Through this website, anyone interested can obtain software that allows the use of a home computer for the search of ever larger prime numbers. In 2001, Michael Cameron, a student from Owen Sound, Ontario, discovered the prime number $2^{13,466,917} - 1$. He had to run his computer for 42 days to check that it really is a prime number. More recently, on November 17, 2003, Michael Shafer discovered that $2^{20,996,011} - 1$ is a prime number. This number keeps going up as bigger primes are discovered, so you should consult the website to check on the most recent record holder.

5. As an interesting exercise, you could ask the class to estimate how many digits there are in these numbers. To answer this question, make use of the fact that $2^{10} - 1024 \approx 10^3$. To make sure students understand the logic, you should first do it for one of the smaller primes, say the one discovered by Euler: $2^{31} - 1$. In that case you should not tell students (till later) what the actual value of this number is. First of all, if we are looking for an approximate size of the number, we should not worry about the $-1$. So we are looking for the magnitude of $2^{31}$. We know the approximate value of $2^{10}$, so the first question is “how does $2^{31}$ compare to $2^{10}$?” This involves thinking about the meaning of these expressions. Namely,

$$2^{31} = 2^{10} \times 1024 \times 1024 \times 1024.$$ 

That is,

$$2^{31} = 2 \times 1024 \times 1024 \times 1024.$$ 

We can change this to the approximation

$$2^{31} \approx 2 \times 1000 \times 1000 \times 1000 = 2,000,000,000.$$ 

We saw in an earlier paragraph that the precise value is 2,147,483,647, so our estimate is not too bad!

If the students understand this argument, you can then try it for one of the larger primes. For example if we did it for the prime discovered by Michael Cameron, we would note that $13,466,917 \approx 1,346,692 \times 10$. Therefore

$$2^{13,466,917} - 1 \approx 2^{10} \times \cdots \times 2^{10} (1,346,692 \text{ times})$$

$$\approx 10^3 \times \cdots \times 10^3 (1,346,692 \text{ times})$$

$$= (10 \times 10 \times 10) \times (10 \times 10 \times 10) \times \cdots \times (10 \times 10 \times 10) (1,346,692 \text{ times})$$

$$= 10^{3 \times 1,346,692} = 10^{4,040,076},$$

which is a number of over 4 million digits! Similarly, $20,996,011 \approx 2,099,601 \times 10$. Thus

$$2^{20,996,011} - 1 \approx 2^{10} \times \cdots \times 2^{10} (2,099,601 \text{ times})$$
3.4 Enrichment Activity - How Many Primes are There?

\[ \approx 10^3 \times \cdots \times 10^3 \text{ (2,099,601 times)} = 10^{\frac{3}{1} \times 2,099,601} = 10^{6,298,803}, \]

a number of more than 6 million digits. In fact the website for the Great Internet Mersenne Prime Search indicates that the number has exactly 6,320,430 digits. \(^1\)

\[^1\text{There are many good resources for those who want to look more closely at the history of the search for large primes. Paulo Ribenboim’s “Book of Prime Number Records”, Springer Verlag, 1988, gives an excellent account of that history.}\]
3.5 Enrichment Activity - Greatest Common Factor

In this lesson we continue to pursue the goals set at the beginning of the Chapter. We try to get the students to think of numbers as made up in a unique way of its prime factors. We will focus, among other things, on showing that this understanding is helpful, and even central, when we want to find the greatest common factor of a set of numbers. If you wish, you can supplement this session by selecting problems from Section A.7 in the Appendix.

Lesson Goals

- To reinforce the idea that the prime factorization of a number tells a lot about its properties
- To use prime factorizations to find greatest common factors.

Materials

- You will not need anything special for this lesson, but you may find it convenient to supplement the lesson with one or two of the problems found in the Appendix in Section A.7
- One of the difficulties that may emerge during the lesson is that the students will have an almost irresistible urge to reach for their calculators even for the most basic calculations. Once again, however, calculators are definitely a hindrance to the project at hand. The students should be told to leave them off their desks and to do the calculations by hand or in their heads.
- If you have access to a computer and a data projector, you could plan to play the game “Greatest Common Factors”, or to use the demo “Factors” with your students at some point in the lesson. These are found on the StepAhead web site under “Numbers → Games, Puzzles & Demos → Division”.

Lesson Sequence

1. To begin the session, discuss the following types of sentences: “28 is divisible by 14”, “14 is a factor of 28”, and “14 is a divisor of 28”. Ask the students what we mean when we say that 28 is not divisible by 13. In all cases, throughout this chapter, we are interested in division without remainder. Another way to say this is that we are interested only in cases when numbers divide exactly into other numbers. On a calculator, this would be indicated by getting a whole number when we do a division. In the next two chapters we will be interested in divisions that leave remainders, but not in this chapter.

2. Ask the students to find the prime number factorizations of each of the following numbers: 60, 462, 117, 51. Collect the solutions on the board or on an overhead transparency.
3. Now ask them to use these factorizations to find the prime number factors of the number
\[60 \times 462 \times 117 \times 51.\] Tell them not to multiply these four numbers together to answer this question. There is an easy way. The idea is, of course, to remind them that the prime factorization of a product is obtained by “adding” the prime factorizations of the numbers together to form the list

\[60 \times 462 \times 117 \times 51 = 2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 3 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17.\]

4. Now write the following products on the board or on an overhead transparency,

\[
\begin{align*}
2 \times 3 & \times 5 \times 7 & & 5 \times 5 \times 13 \\
2 \times 2 \times 2 \times 17 & & & 11 \times 19 \\
16 \times 9 & & & 3 \times 3 \times 3 \times 3 \times 3 \\
21 \times 77 & & & (2 \times 3 \times 5 \times 5 \times 7) \div (2 \times 5)
\end{align*}
\]

and ask students to decide in their heads which of them can be divided into the number
\[60 \times 462 \times 117 \times 51.\] Note that this is the number whose prime number factorization you calculated in the preceding question.

5. Now, to begin the discussion of greatest common factor, ask students to solve the following set of questions:

(a) Find all the factors of 28 (not just the prime factors this time, but all the factors).
(b) Find all the factors of 231.
(c) Find all the common factors of 28 and 231.
(d) What is the greatest common factor of 28 and 231?

This discussion can be supplemented, or replaced, by the demo “Factors” available on the StepAhead web site under “Numbers → Games, Puzzles & Demos → Division”.

The idea in question 5 is that all the factors of a number can be obtained by multiplying some of the prime factors of the number. Here is a good exercise to discuss at the board at this point: Without writing them all down, how many different factors are there for the number \(2 \times 3 \times 5\)? The idea is that because the prime factors are all distinct, the number of factors is the same as the number of ways of choosing to include or exclude the three factors. For example, if you include 2 and 3, but not 5, you get the factor 6. Now the number of ways of including and excluding can be counted by the following method (the instructor will have to decide whether the students are ready for this): There are two choices for the factor 2, “include” or “exclude”. For each of these there are two choices for 3 and two choices for 5. Thus the total number of possible combinations is \(2 \times 2 \times 2 = 8\). The choice that leaves out all three of the factors should be thought of as giving the factor 1, and the choice of including all three of the factors gives the factor 30.

At this point, as the students get ready to start question 4, it would be appropriate to talk at
the board about the concept of greatest common factor, to make sure they all understand clearly
what it means. At the same time it should be pointed out that “greatest common divisor” is another expression with the same meaning.

6. Now challenge the class with this set of problems: Here are the prime number factors of two large numbers, which we will call \( m \) and \( n \):

\[
\begin{align*}
n &= 2 \times 2 \times 3 \times 3 \times 5 \times 7 \times 7 \times 7 \times 11, \\
m &= 2 \times 2 \times 2 \times 3 \times 5 \times 5 \times 7 \times 7 \times 13.
\end{align*}
\]

In each of the following cases decide whether the number is a common factor of the two numbers \( m \) and \( n \). do not multiply the given factors of \( m \) and \( n \) to find the values of \( m \) and \( n \). There is a much easier way.

(a) \( 2 \times 2 \times 5 \times 5 \),
(b) \( 2 \times 3 \times 7 \times 11 \),
(c) \( 2 \times 2 \times 3 \times 7 \times 7 \),
(d) \( 2 \times 2 \times 3 \times 5 \times 7 \times 7 \times 7 \),
(e) \( 2 \times 2 \times 2 \times 5 \times 7 \times 7 \times 7 \),
(f) 100.

One of the numbers in the preceding list is the greatest common factor of \( m \) and \( n \). Which one is it?

7. Now ask the students to do this question in their heads: If \( 140 = 2 \times 2 \times 5 \times 7 \) and \( 36 = 2 \times 2 \times 3 \times 3 \), what is the greatest common factor of 140 and 36?

1. As a final activity you could play the game “Greatest Common Factors” available on the StepAhead web site.
3.6 Enrichment Activity - Least Common Multiple

Lesson Goals

- To reinforce the idea that the prime factorization of a number tells a lot about its properties
- To use prime factorizations to find least common multiples.

Materials

- You will not need anything special for this lesson, but you may find it convenient to select some problems from Section A.7 in the Appendix to supplement (or replace) some of the problems below.
- You may also want to make use of the games “Lowest Common Multiples” and “Matho”, or the demo “Factors” as part of the lesson.
- One of the difficulties that may emerge during the lesson is that the students will have an almost irresistible urge to reach for their calculators even for the most basic calculations. Once again, however, calculators are definitely a hindrance to the project at hand. The students should be told to leave them off their desks and to do the calculations by hand or in their heads.

Lesson Sequence

1. To (re-) introduce students to the concepts multiple and least common multiple, write on the board the equations

   \[ 140 = 2 \times 2 \times 5 \times 7 , \]
   \[ 36 = 2 \times 2 \times 3 \times 3 . \]

   Ask the students which of the following are multiples of 140:

   \[ 2 \times 3 \times 3 \times 5 \times 7 , \]
   \[ 2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 7 \times 11 , \]
   \[ 2 \times 2 \times 3 \times 3 \times 17 . \]

   Also ask them which are multiples of 36. Finally ask them which are common multiples of 140 and 36, and use that to initiate a discussion of the concept of least common multiple. Point out to the students that g.c.d. is often used as an abbreviation of greatest common divisor, and that l.c.m. is often used as an abbreviation of least common multiple.

2. Ask them what is wrong with the following reasoning:

   \[ 50 = 2 \times 25 , \]
and 

\[ 15 = 3 \times 5 . \]

Therefore the least common multiple of 50 and 15 is equal to 

\[ 2 \times 3 \times 5 \times 25 . \]

3. Invite students to note that 

\[ 48 = 2 \times 2 \times 2 \times 2 \times 3 . \]

Then ask: if \( 48 \times n \) is divisible by 9, what is the smallest \( n \) can be?

It is important here and in the questions that follow that the students begin to think of numbers in terms of their prime factors. For example, in this question, 9 “contains” two factors 3, while 48 “contains” only one. Thus the number \( n \) should “contain” a factor 3 as well. Since there are no other requirements put on the number \( n \), the smallest it can be is 3. Similar comments apply to the next few questions.

4. Now present some or all of the following problems:

(a) If \( 65 \times n \) is divisible by 26, what is the smallest \( n \) can be?

(b) If \( 84 \times n \) is divisible by 90, what is the smallest \( n \) can be?

(c) If \( m \times 27 = n \times 12 \), what is the smallest \( n \) could be?

What matters in the third question is that \( 27 = 3 \times 3 \times 3 \), while 12 has only one prime factor 3 in it. This means that \( n \) must be \( 3 \times 3 = 9 \).

5. Challenge the students with this question: If \( 28 \times m = 8 \times n \), what is the smallest \( n \) can be?

(6. Ask students to find a common multiple of \( 2 \times 2 \times 2 \times 3 \times 5 \times 11 \) and \( 2 \times 2 \times 3 \times 3 \times 5 \times 7 \). Tell them they are not required to find the actual value of the common multiple; it is enough to find the prime factors of the common multiple. Then ask them what is the prime factorization of the least common multiple of these two numbers?)
Notice that by asking the students *not* to evaluate the number, you help them focus on the prime factors as the constituents that matter.

7. Ask the students to find the least common multiple of 48 and 9 by first finding the prime factorization of these numbers.
3.7 Problem Set - Practice Using Prime Factorizations

Goals

The exercises in this problem set can be used in two ways: (1) All at once to fill an entire enrichment session, and (2) to supplement the preceding sections of this chapter. The problems constitute more than merely a set of exercises to practice what was learned in those sections. Some of them are indeed mainly that, but others are deep problems that deserve a lot of time and attention. Some of them are sufficiently deep that they could make up the core of a session all on their own. The goals of the problems can be summarized as follows:

- To practice finding prime factorizations
- To practice using prime factorizations to find other factors, and especially greatest common factors
- To practice using prime factorizations to find least common multiples
- To explore other ways in which prime factorizations tell you things about the way numbers behave

Materials

- A copy of the problem set in the Appendix, Section A.7 for each student.

Lesson Sequence

Tell the students to do the following exercises. The first nine are relatively straight-forward. Problems 10 to 16, however are very challenging, and can almost certainly not be done in one hour. You can select specific ones, or you can assign one of two specific ones to the students. Remind them to attack each question by focusing on the prime factorizations of the involved.

1. Find the greatest common divisor of 230 and 180.
2. Find the greatest common factor of 128 and 72.
3. Find two numbers whose greatest common factor is 12.
4. Find two numbers whose greatest common factor is 1234.
5. Find the l.c.m. of 65 and 26 by first finding their prime factorizations.
6. Find the l.c.m. and the g.c.d. of 84 and 35.
7. Find the l.c.m. of each of the following pairs of numbers:
   (a) 22 and 33  (b) 24 and 36  (c) 25 and 35

8. Find the l.c.m. and g.c.d. of 72, 180 and 300.

Conceivably, this is the first time the students have been asked to consider a least common multiple or greatest common divisor of a set consisting of more than two numbers. It is clear how the ideas extend to include any finite set of numbers, but you may have to point out to them what it means. In this case, we have

\[72 = 2 \times 2 \times 2 \times 3 \times 3,\]
\[180 = 2 \times 2 \times 3 \times 3 \times 5,\]
\[300 = 2 \times 2 \times 3 \times 5 \times 5.\]

Thus the greatest common divisor is
\[2 \times 2 \times 3,\]
and the least common multiple is
\[2 \times 2 \times 3 \times 3 \times 5 \times 5.\]

For the greatest common divisor, all you have to do is find the longest list of prime factors that is a subset of each of the three lists found for the given numbers. To get the least common multiple, you look for the shortest list that has all three of the lists as subsets. It is really important that you get the students into the habit of worrying less about the final answer than about the method for getting it. One way this is achieved is by not allowing the use of calculators. The other is by leaving the answer in the form of a product of primes, and refusing steadfastly to do the multiplication, excusing this by saying that “that is the easy part, I already know that you can multiply”.

9. Which of the following numbers is equal to \(n \times n\) for some whole number \(n\)?
   (a) \(2 \times 2 \times 3 \times 3 \times 3 \times 3\)
   (b) \(2 \times 5 \times 5 \times 11 \times 11\)
   (c) \(13 \times 13 \times 17 \times 17\)

The answer is, of course, that only the first and the third are of the form \(n \times n\), because in those cases the lists of prime factors can be split into two equal parts. Another way to put this is that only in those cases does each prime factor occur an even number of times.
10. Is
\[ 2 \times 2 \times 2 \times 3 \times 7 \times 7 \times 11 \times 19 \times 23 \times 29 \times 31 = 116887742667 \]?

Explain your answer.

The correct answer is “no”. The simplest way to see that is by noticing that the given list of prime factor contains a 2 (several even) and that their product must therefore necessarily be even, while 116887742667 is not. Alternatively, you could note that one number is divisible by 9 (add up the digits) while the other is not.

11. How many zeros are there at the end of the number
\[ 2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 5 \times 5 \times 5 \times 7 \times 7 \times 11 \times 13 \times 13 \times 17 \times 17 \]?

Give a clear explanation of the reason for your answer.

Each zero at the end of a number indicates a factor 10. Since the prime factorization of 10 is 2 \times 5, therefore we should count the number of two's and the number of 5's in the list of prime factors of the product. To count the 2's we note that there are 3 of them at the beginning, but that the other factors are odd. Counting the number of 5's, we see that there are 4 of them (none of the other factors are divisible by 5). Since for each 10 we need one 2 and one 5, it follows that the product ends in three zeros.

12. If 123123123122000 is multiplied by 456456456455000, how many zeros will there be at the end of the answer, and why?

Clearly there will be at least six zeros, since there are three factors of 10 in each of the two given numbers. Neither of the two numbers includes additional factors of 10. However, further factors of 10 may be created in the product when one number supplies 2’s and the other supplies 5’s. So is this the case for 123123123122 and 456456456455? You can see that the second of these numbers is divisible by 5 but not by 25. (numbers divisible by 25 end in 00, 25, 50, or 75). At the same time the first number is clearly divisible by 2. Thus when the two numbers are multiplied, there is one additional factor 10, created by combining the factors 2 and 5. Thus the product will end in 7 zeros.
13. How many zeros are there at the end of the greatest common factor of

\[ 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 7 \times 11 \times 13 \times 13 \times 13 \times 19 \]

and

\[ 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 5 \times 5 \times 11 \times 11 \times 17 \times 17 \times 17 \times 19 \]?

Explain how you got your answer.

The greatest common divisor will have three 2’s and two 5’s in its list of prime factors, so there will be two zeros at the end of the greatest common factor.

14. When I multiplied

\[ 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 7 \times 11 \times 11 \times 17 \times 19 \times 19 \]

I wrote down 36782351000 as the answer. My friend pointed out that I wrote the first digit down incorrectly. Can you decide what the first digit should be?

There are two 3’s in the list of prime factors, so the number is divisible by 9. However, when you add the digits you get 35, which is not divisible by 9. If you alter the first digit in 36782351000, you can change the digit total to anything as low as 33 (by changing the first digit to 1) or as high as 41 (by changing the first digit to 9). In the set of numbers 33, 34, 35, 36, 37, 38, 39, 40, 41 only 36 is divisible by 9, and this total is obtained by choosing 4 as the first digit.

15. How many zeros are there at the end of the number

\[ 1 \times 2 \times 3 \times 4 \times \cdots \times 19 \times 20 ? \]

Remember from a similar question in the preceding section that to answer this question we have to count the number of prime factors 2 and the number of prime factors 5, and then take the smaller of these numbers, for that will then be the number of factors 10 in the product. You can see that there are 17 factors 2 in the product, but only 4 factors 5. Thus there are just four zeros at the end of the number.

16. Suppose we have three cog wheels, one with 42 cogs, one with 34 cogs, and one with 39 cogs. The wheels are touching each other as shown.
Each wheel has an arrow painted on it that points up. How many times does the wheel with 34 cogs have to go around before all arrows point up again?

Suppose that after the wheel with 34 cogs has gone around $n$ times all the arrows point up again. Now, when one wheel moves by one cog, so do the other two. That means that each of the wheels moves by $34 \times n$ cogs. Since all three arrows point up at the end, this means that $34 \times n$ is a multiple of 42 and a multiple of 39. Therefore, we want the value of $n$ that will produce the least common multiple of these three numbers. To find this least common multiple we first find the prime factors:

\[
42 = 2 \times 3 \times 7 ,
\]
\[
34 = 2 \times 17 ,
\]
\[
39 = 3 \times 13.
\]

Therefore the least common multiple is $2 \times 3 \times 7 \times 13 \times 17$. Thus $n = 3 \times 7 \times 13$. 

Chapter 4

Fractions

This lunar beauty
Has no history,
Is complete and early;
If beauty later
Bear any feature,
It had a lover
And is another.

W. H. Auden, 1907-1973
(from Collected Poetry of W. H. Auden, 1945)

The purpose of this short chapter is to discuss and explore with the students the relationship between the representation of numbers as fractions and their representation in decimal form. We will then try to understand why it is that the decimal expression obtained from a fraction always either terminates or repeats.

One of the key outcomes of this discussion is the relationship between the “discrete” point of view, which sees numbers as concepts used for counting and for dividing into equal parts, and the “continuous” point of view from which numbers are first of all points on a line, and only afterwards receive numerical designation. From the geometric standpoint the numerical designations are secondary and derivative. The numbers serve as instructions for locating the points. In particular, a single point can (and always does) have many distinct numerical designations.

The paradoxes arising from the juxtaposition of these two points of view date back to the time of Pythagoras. It was thought at the time that all geometric measurements were commensurate. In other words, it was thought that every point on the number line (that is, every “real” number) corresponds to a fraction (that is, a “rational number”). The discovery by the followers of Pythagoras that this is not so caused great consternation. In modern terms, it amounted to the discovery that the square root of 2 cannot be written as a fraction. Of course, like all paradoxes,
this fact leads to an enormous amount of new and beautiful mathematics, some of which will be explored in this chapter.

Just as it was initially assumed by the Pythagoreans that all numbers are expressible as fractions, so also students will be accustomed to think of all numbers that way. It is of great importance, therefore, to persuade the students to take a new view of numbers in this chapter. They should be thought of geometrically, as points, rather than arithmetically, as fractions, or as other outcomes of calculations.

The Chapter’s Goals

- To learn how to put a fraction in decimal form, and how to turn a terminating decimal expression into a fraction.
- To learn how to turn a repeating decimal expression into a fraction.
- To discover that there are numbers that cannot be expressed as fractions.
- To understand decimal expressions as specifiers of a location on the number line.

Overview of Activities

- E.A. 4.1 - Fractions and Decimals
  An exploration of the ways to convert fractions to decimal expressions and to convert a terminating decimal to a fraction.

- E.A. 4.2 - Terminating or Repeating?
  An exploration of the conversion of repeating decimal expressions to fractional form, and a discussion of the relationship between the length of the repeating pattern in a repeating decimal and the fraction equivalent to it.

- Problem Set 4.3 - Practice with Fractions and Decimals
  A set of problems to reinforce the ideas presented in the first two activities.

- E.A. 4.4 - Rational and Irrational Numbers
  An exploration of the number line with an emphasis on numbers that are not rational. Proof that the number $\sqrt{2}$ cannot be written as a fraction.

- E.A. 4.5 - The Difference Between 0.9 and 1
  A further exploration of the concept of number and its expression in decimal form.
4.1 Enrichment Activity - Fractions and Decimals

Lesson Goals

- To see (real) numbers as primarily points on the number line - “measurements” if you like - and to see (terminating) decimal expressions as indicating locations on the number line.
- To learn (or review) how to convert a fraction to a decimal
- To learn how to convert a terminating decimal into a fraction
- To review place value in the process

Materials

- Use of calculators should be avoided in this lesson.
- If you have a computer and a data projector available, you should be prepared to use the demo “Magic Line” in class. This demo is found on the StepAhead web site under “Numbers → Games, Puzzles & Demos → Fractions”.
- If you do not have a computer, you should prepare one or several copies of the template provided in Section A.9 for each student, and make an overhead transparency of that template.

Lesson Sequence

1. Points on the number line Before the class begins, draw a number line on the board on a very large scale, say from a little before the number 0 up to a little past 2. Put large tick marks at the whole numbers. Then divide each unit into ten equal pieces and put a smaller tick mark at each of the resulting points. If you drew the original line on a sufficiently large scale, you should be able to divide these into tenths again, and indicate the new points by even smaller tick marks.

When the class begins, tell the students that they should think of your number line as a “magic” number line that not only goes on forever and ever in both directions, but also is subdivided into tenths at every scale. In other words, between any two tick marks there are more tick marks, ad infinitum. Agree on a way to designate the tick marks of various sizes. For example, the original two tick marks could be referred to as “size 1” tick marks. The next level as “size 1/10” and so on.

Of course, the smaller tick marks are going to be so close together that you cannot distinguish them. To get around this, we can use the template found in Section A.9 to zoom in to a part of the number line to see more clearly what it looks like, and then to do this again, and so on. Figure 4.1 illustrates this process of zooming in.
The idea is now to get students to locate numbers on that magic number line. Hand out to the students copies of the template found in Section A.9, and put the corresponding transparency on the overhead projector. Begin with decimal expressions. You can write some decimal expressions (for numbers between 0 and 2) on the board. 0.73, 1.38, ... are examples. Have students work in groups to locate these numbers on their templates. Have them come to the front to indicate where these numbers are located on your number line. If you have access to a computer and a data projector, you should complement the discussion by using the demo “Magic Line” available on the StepAhead web site. To help the students understand the process you could ask them the following question: You wrote down a decimal expression. It starts with the digits 1.53, but the digits after the 3 have been smudged so that you cannot read them. What can you say about the location of the number from the information given by the three digits you are able to read?

2. Now do a similar exercise with fractions whose denominators divide into powers of 10. That is, these denominators should have only 2 and 5 as prime factors. For example, you could begin with

\[
\frac{3}{2}, \quad \frac{11}{10}, \quad \frac{1}{20}, \quad \frac{3}{20}, \quad \frac{3}{40}, \quad \frac{7}{8}
\]

and ask students to locate these. Reflect on the methods students use.

3. Once these simple fractions have been located, turn to some that are a little more difficult to decide, such as

\[
\frac{21}{125}, \quad \frac{31}{250}, \quad \frac{25}{40}, \quad \frac{97}{80}
\]
Notice that these continue to be fractions whose denominators divide into powers of 10. This is deliberate; other fractions will be considered a little later.

In fact, one of the ways to decide where the number should be located is to first change the fraction into an equivalent fraction whose denominator is a power of ten. Another way is to perform the long divisions suggested by these fractions. Eventually the students should begin to notice that the location for the fraction $\frac{127}{100}$ is found by looking for the size-one tick mark corresponding to the number 1, then for the second size-$1/10$ tick mark to the right of that, and then the seventh size-$1/100$ tick mark to its right. Similarly, if long division is used, the number 0.35 is found by starting at the number 0, then moving three size-$1/10$ tick marks to the right, followed by five size-$1/100$ tick marks. Both procedures are instances of our understanding of place-value. Try to prevent students’ using calculators to find their answers. The lesson is more valuable if the calculations are done by hand, or mentally. In fact, in this discussion you are reviewing the idea of place value, especially as it relates to places to the right of the decimal point.

Of course, it is just as likely that student will find the decimal expression for, say, $\frac{23}{25}$ by hand. You may find that not all students will have been taught the long division algorithm. In some cases they may have learned an other division algorithm, and in some cases they may not have learned a particular method at all. Find out by asking how they would do the division of $\frac{23}{25}$ by hand. You may get some interesting suggestions. If the standard long division algorithm does not emerge, you may have to adapt some of the discussion in the next section to the division algorithm used by the class. It is important for what follows that students be familiar with a division algorithm that allows them to do divisions by hand.

4. Ask the students whether they feel confident that every fraction will land on one of the tick marks. At some point in the discussion, ask them where they would locate the fraction $\frac{1}{3}$. This time students will find they do not have the option of finding an equivalent fraction whose denominator is a power of 10. Eventually they will have to resort to long division or some equivalent process, only to discover that the decimal expression for $\frac{1}{3}$ does not terminate, indicating that the search for the right tick mark never ends. Suggest to the students that this means the number $\frac{1}{3}$ does not exist. They will not agree, but it should lead to an interesting discussion about the fact that even though the tick marks seem to be everywhere, there are many points on the number line that do not correspond to any tick mark. As a result of the discussion, the students should come to feel that even for non-terminating decimal the sequence of decimal digits indicate where the number is located on the number line. In other words, the decimal expression can be thought of as a road map to the number. You should be tempted to think of it as the address of the number, but we prefer not to use that analogy, for an address is unique, and we shall see later that numbers can be designated by more than one decimal expression.

5. **Conversion between fractions and decimals** Since the decimal expansion of a number indicates where it is located, and since fractions correspond to points on the number line, it should be possible to write fractions in decimal form and vice-versa. In fact, expressing fractions as decimals is essentially what we were doing all along, and you may want to skip some of the following suggestions if it is clear that the students have already grasped what is happening. For grade 8 students the conversion of fractions to decimals and back will be
Fractions

a review of material seen before, and for grade 7 students it may or may not be, depending on the time of the year the lesson is taught. In any case, if more practice is needed, ask some of the questions in this revised form: Ask the students whether they know how to write 1/2, 3/4 and 7/5 as decimal expressions. For simple fractions, the students should be able to work out the answers in their heads. However, when the numbers involved become at all large, they should do the calculation by long division. Especially here it is essential that the students put their calculators aside, for calculators will hide what is really going on. The examples above were picked so that the resulting decimal expressions will terminate. If it seems necessary to do another example, then you should pick another one that will give an answer that terminates. For example, you could do 47/625 or 312/125.

6. Now ask the students if they know how to go the opposite way: given decimal expressions such as 3.7 or 0.35, can they write these as fractions? This, too, is something that should have been seen before, at least by students in grade 8. Of course, the secret (for terminating decimal expressions such as these two) is to multiply them by a power of 10. For example, $0.35 \times 100 = 35$, and so

$$0.35 = \frac{35}{100} = \frac{7}{20}.$$  

Make sure that the significance of the last step is discussed with the students: The fractions should be reduced to simplest form. That is, any common denominators between the numerator and the denominator should be factored out. Not only does this tie this discussion to the preceding, but the matter of reducing a fraction to simplest form is important for the discussion that follows next.

7. Non-terminating decimals At this point, remind the students that we saw earlier in this lesson that the fraction 1/3 does not result in a terminating decimal. Ask students if they can think of others that don’t. Examples are 1/9 and 1/11. When we perform the long divisions indicated, we see that these examples repeat rather quickly. When you do the long divisions on the board, ask the students how they can be sure, from the pattern of the calculation, that the repeating pattern will go on for ever and ever. At first the students will probably say that as soon as you notice repetition, then you can be sure that it will continue. When they say that, remind the students of the fact that when we studied number patterns earlier on in the programme it was important to be suspicious of such claims. We saw examples where patterns continued for a while, but failed later on. How can we be sure that the same might not be the case here? If necessary, ask students to find the decimal expression for the fraction 109092/90000. It will seem to repeat in one way, but then settle down to a different pattern.

The answer can be found by looking closely at the long division process. What matters here is that after a certain number of steps all the digits in the dividend (the numerator) have been used up, and that when we continue the division process beyond this stage we bring down zero digits which we supply ourselves. At the end of each step in the long division process we subtract, resulting in a number smaller than the denominator. If we get a number that is larger than or equal to the denominator, we should re-do the step for then we have made an error. If the number we get after subtraction is 0, then the long division is finished, and we have a terminating decimal. In all other cases we will get a non-zero number strictly smaller than the number we are dividing by. Since there are
only finitely many numbers that can occur here, therefore we must eventually come back to a number we have had before. When this happens, conditions are exactly the same as they were the first time this number came up: we have the same number as a result of the subtraction, and we are bringing down the same zero digit. Therefore, the process has no option but to evolve exactly as it did the first time around, causing the repetition.

Rather than telling this story to the students, though, you should challenge them with the question, give them some calculations to do and ask them to look out for the answer to the question. Here are some numbers that form good exercises:

\[
\begin{align*}
\frac{1}{7} &= 0.142857, \\
\frac{1}{13} &= 0.076923, \\
\frac{3}{14} &= 0.2142857, \\
\frac{44}{65} &= 0.6769230, \\
\frac{1}{31} &= 0.032258064516129.
\end{align*}
\]

The third, fourth and fifth examples show that you cannot conclude that you have discovered the repeating pattern as soon as you see a digit occurring for the second time. In fact, if the students think about this for a while they will realize that it is not reasonable to expect that, for there are only 10 digits, so if the repeated list of digits is long, some digits will have to occur many times within that list.

Note that there are several other patterns in this list of decimal expressions. Both \(\frac{1}{7}\) and \(\frac{3}{14}\) have the same set of repeated digits, and so do \(\frac{1}{13}\) and \(\frac{44}{65}\). This has to do with the fact that 7 divides 14, and that 13 divides 65. If you and the students are interested, you could find decimal expressions for each of the fractions \(\frac{1}{13}, \frac{2}{13}, \frac{3}{13}, \ldots, \frac{12}{13}\) and see what happens. However a thorough discussion would take us too deep.
4.2 Enrichment Activity - Terminating or Repeating?

Lesson Goals

- To learn that the denominator of a fraction determines whether the corresponding decimal expression terminates
- To learn that if a fraction has a non-terminating decimal form, then the length of the repeated pattern is always less than the magnitude of the denominator
- To learn how to express non-terminating repeating decimals as fractions

Materials

- The students do not need anything for this lesson other than paper and pencil. Use of calculators should be avoided.

Lesson Sequence

1. At the beginning of this lesson you should review the discussion of the previous week. Ask the students how they would go about finding the decimal expression for a fraction (long division) and ask them what sort of decimal expression they would get (terminating or repeating).

2. Once they have had adequate time for this, challenge the students with the following question: Is there any way to tell from the fraction you start with whether the decimal will terminate or whether it will repeat? This is a good problem to do with the class at the board. Make two spaces on the board. In one space collect fractions that resulted in terminating decimals, and in the other collect those that result in repeating decimals. Ask the students if they notice any difference between the two groups. To keep the discussion going for a while, you can add fractions to the two categories yourself, using the criteria explained next. It turns out that the denominator is the key. To convince the students of this, look at an example with the class. Note that the fraction $9/8$, when written as a decimal, gives the answer 1.125. On the other hand, if we were to try to write 1.125 as a fraction we would first convert it to $1125/1000$. In other words, the fractions $9/8$ and $1125/1000$ must be equal. But that means that 1000 is a multiple of 8! In fact, whenever a fraction is equal to a terminating decimal, then the fraction must be equal to a fraction with 10, 100, 1000, or another power of 10 in the denominator. That is, the denominator must be a factor of a power of 10. But what kinds of numbers are factors of powers of 10? The secret, we saw in the preceding chapter, lies in the prime factors of the number. The prime factors of 10 are 2 and 5. Therefore, all the prime factors of any power of 10 are 2’s and 5’s. So if the denominator is to be a factor of a power of 10, its prime factors, also, must be 2’s or 5’s. So, for example, $12/25$ will give a terminating decimal, while $25/12$ will not. Indeed, the discussion provides us with another way of finding the decimal expression
if it is going to terminate: Multiply the denominator and the numerator by the number required to turn the former into a power of 10. For example,

\[
\frac{12}{25} = \frac{48}{100} = .48
\]

3. Ask students if they remember from the previous section how to find decimal expressions for numbers whose denominators do not divide into 10 or 100, or into any power of 10. In other words, what do we do with fractions like \(\frac{1}{3}\) and \(\frac{3}{7}\)? Make sure that they remember how the division algorithm did this for you. Review with them how the way the algorithm works ensures that these fractions (all fractions) produce a repeating decimal pattern, allowing of course for the possibility that at some point you get nothing but 0's. This would normally be described as a terminating decimal.

4. Once you feel sure that the students understand what it is that causes the repetition in a repeating decimal expression, you should ask them whether there is any way to predict, when looking at a fraction, how long the repeated string of digits will be. The answer is implicit in the discussion that showed that the decimal expression must terminate or repeat: Since the repetition is occasioned by the recurrence of a particular remainder in the long division process, the length of the repeated string of digits cannot be longer than the set of possible remainders. But the only possible remainders are 1, 2, 3, ... up to the number that is one less than the denominator. Thus the length of the repeated set of digits is always smaller than the denominator, though of course it may be much smaller.

5. After completing this thorough discussion of the mathematics of finding the decimal expression for a fraction, you should now return to the question of finding the fraction when the decimal expression is given. We did this already for terminating decimal expressions, so you should begin by reviewing the procedure. Ask the students what they would do to find the fractions for the decimal expressions 0.532 and 7.805.

Once it is clear that the students are comfortable with the procedure, you are ready to teach them how to convert a repeating decimal to a fraction. Here is the technique: Suppose we wanted to write the number \(x = 13.\overline{23}\) in decimal form. You would do the following: Since the block of digits that gets repeated is two digits long, multiply \(x\) by 100 and subtract \(x\) from the product. In other words,

\[
100 \times x = 1323.232323\ldots
\]

\[
x = 13.232323\ldots
\]

\[
99 \times x = 1310.
\]

Thus

\[
x = \frac{1310}{99}
\]

Follow this with one or two other examples on the board. When presenting these it may be preferable not to write 13.2323... symbolically as \(x\) but instead to write the number out in full (i.e. as in \(100 \times 13.2323\ldots = 1323.232323\ldots\)).

The last step in the process may be difficult for younger students to follow. In other words, why should \(99 \times x = 1310\) imply that \(x = \frac{1310}{99}\)? It is important that the students not see
this as a mechanical operation unless they have first really understood it as a statement about the meaning of fractions. It would be good to draw a number line of length $\frac{1310}{10}$ on the board, or act one out by stretching out your arms, and then indicate a small piece of it that represents the number $x$. To say that $99$ of these little pieces fit exactly into $1310$ is precisely what we mean when we say that $x = \frac{1310}{99}$. 
4.3 Problem Set - Practice with Fractions and Decimals

Goals
The exercises in this problem set can be used in two ways: (1) All at once to fill an entire enrichment session, and (2) to supplement the preceding sections of this chapter. The Goals of the problems are

- To practice turning fractions into decimals
- To practice turning decimal expressions (repeating or terminating) into fractions
- To practice discerning which fractions produce terminating decimals

Materials
- A copy of the problem set in the Appendix, Section A.8, for each student.

Lesson Sequence
1. Write the following numbers as decimals:
   \[
   \frac{1}{5}, \quad \frac{31}{125}, \quad \frac{1102}{250}.
   \]

2. Express the following numbers in decimal form:
   \[
   \frac{33}{14}, \quad \frac{1}{11}.
   \]

3. Each of the following numbers is written in decimal form. Find the fraction to which the number is equal. Your answer must be a fraction in simplest form.
   \[
   0.25, \quad 1.024, \quad 0.525.
   \]

4. First write the following number in decimal form, and then express it as a fraction.
   \[
   2 + \frac{1}{10} + \frac{7}{100} + \frac{6}{1000}.
   \]

5. Without actually finding the decimal expressions for the following fractions, decide which of them will produce a terminating decimal and which will produce a repeating decimal:
   \[
   \frac{377}{160}, \quad \frac{87}{620}, \quad \frac{251}{13251}, \quad \frac{3291}{625}.
   \]
6. Convert the following decimal expressions to fractions:

\[3.781, 0.123, 3.238, 7.2123\]

The last of these will require some ingenuity on the part of the students. Challenge them to think of a way to generalize the examples done in class. The first digit 2 past the decimal point is in the way. However, it will be moved to the left of the decimal place if we multiply by 10. So instead of multiplying by 1000 and by 1 and subtracting, we should multiply by 10000 and 10 and subtract:

\[
\begin{align*}
10000 \times x &= 72123.123123123\cdots \\
10 \times x &= 72.123123123\cdots \\
9990 \times x &= 72051.
\end{align*}
\]

Thus \(x = \frac{72051}{9990} = \frac{24017}{3330}\). This fraction is in fact in simplest form, though it would take a bit of work to verify that.
4.4 Enrichment Activity - Rational and Irrational Numbers

Lesson Goals

- To discuss the existence of decimal expressions (and therefore points on the number line) that do not terminate or repeat
- To observe that this means that such numbers cannot be expressed as fractions.
- To introduce the terms ‘rational number’ and ‘irrational number’
- To prove that $\sqrt{2}$ is irrational
- To show that (probabilistically) there are more irrational numbers than there are rational numbers

Materials

- It will be helpful at one point in this lesson if students have a calculator handy.

Lesson Sequence

1. To start the lesson, ask the students whether all fractions produce decimal expressions that either terminate or repeat. Of course, the outcome of the last two lessons is that they do. However, it is quite possible that when you ask the question this way some students will say that this is not the case, because they have been taught that the decimal expression of the number $\pi$ goes on forever without repeating. If $\pi$ comes up this way, repeat the outline of last week’s finding, now with the above question in mind: We found that a fraction always produces a decimal expression that either repeats or terminates. Ask the students how we can resolve this apparent contradiction. The answer is, of course, that the discussion of the previous lesson applies only to numbers that can be written as a fraction in the first place. In other words, the only reasonable conclusion is that it is not possible to write $\pi$ as a fraction. Once when one of us asked a class (rhetorically) what sort of number this is, that cannot be written as a fraction, a student suggested that it should be thought of as a “made-up number”. Not a bad name, and not far from the way the Greek mathematicians looked at irrational numbers when they first discovered them! Numbers whose decimal expressions neither repeat nor terminate are called irrational, while numbers that can be written as fractions are said to be rational.

2. To make the concept of non-repeating decimal expression very clear, ask the students whether the number 0.12112111211112... is equal to a fraction. The answer is no, for even though this number has a pattern to it, it is not a pattern that is repeated, but rather one that grows. So 0.12112111211112... is an irrational number. To be a rational number, the pattern should repeat exactly.
3. At this point you might hazard a discussion about the relative quantities of rational and irrational numbers. That is, ask the students whether there are more rational numbers than irrational numbers or vice versa. Students will almost certainly decide that there are more rational numbers. After all most of the numbers they encounter are rational. Among many other reasons for this is the fact that calculators cannot display irrational numbers. Even the number $\pi$ is approximated, on the calculator, by a rational number!

However, try putting the question this way: Suppose we wanted to make a number by the following method. We have a ten-sided die with one of the digits on each side. It is not difficult to make one that rolls - make it in the form of a decagonal prism. We put down a decimal point, and then let the die decide on the digits that we will write down behind it. The plan is that we should go on adding digits forever and ever. It is possible (and not unreasonable) that the students will object that the result is not a number. If this happens, remind them that we saw in an earlier lesson that whenever we have a decimal expression it can be thought of as a road map to a point on the number line and should therefore be called a number. Ask them to accept this for today at least, and tell them that we will discuss this more in the next class.

Ask the students whether it is more likely that after some stage the digits will repeat or become all zeros (this is really also a kind of repeating!), or whether the sequence of digits is more likely to be random. The answer is obvious when you look at it in this way, and it is indeed the case that there are far more irrational numbers than rational numbers in some precise sense suggested faithfully by this discussion.

4. If indeed there are more irrational numbers than there are rational numbers, it would be nice to know of at least one good example, so if there is time, and if you think the students can handle it, you should finish the class with a discussion of the fact that the square root of 2 is not a rational number. Ask the students to calculate the square root of 2 on their calculators. Write their answer on the board. It will depend on the number of digits displayed by their calculators, of course. Suppose you get the answer $\sqrt{2} = 1.414213562$. Point out to the students that all you can really know from the answer given by the calculator is that the true value is at least 1.414213562, and that it is less than 1.414213563. That still leaves a range of possibilities of width 1/100000000 on the number line. It does not tell us whether the list of given digits repeats immediately, or with a longer list of digits, or whether it repeats at all. Tell the students that if it were possible to get a calculator to produce all the digits, they would never terminate or repeat. That is, the square root of 2 is an irrational number as well. Remind the students that to say this is to say that $\sqrt{2}$ is not equal to a fraction. You will prove this now.

5. As was the case for the proof in Section 3.4 showing that there are infinitely many primes, this is also a proof by contradiction: In other words, we begin by trying to imagine that $\sqrt{2}$ can be written as a fraction and see that it is not possible.

In order to do the proof, the students have to be very clear on the way fractions are multiplied. That is, they must know that when you multiply two proper fractions, you have to multiply their denominators as well as their numerators. If necessary, you should review this a moment.

There are two slightly different forms for the proof. You should pick one of them for your class.
6. **The first version of the proof.** Suppose

\[ \sqrt{2} = \frac{m}{n} . \]

Then, by squaring both sides of this equation, we have

\[ 2 = \frac{m \times m}{n \times n} . \]

Multiplying both sides of this equation by \( n \times n \) we get \( 2 \times n \times n = m \times m \). However, from the preceding chapter we know that the number of prime factors of a perfect square is always even. This means that in the equation \( 2 \times n \times n = m \times m \) the number of prime factors on one side is odd, while on the other it is even. Clearly this is an impossibility. We can only conclude that our supposition that \( \sqrt{2} \) is rational was not correct.

7. **The second version of the proof.** To start the second proof, remind the students that every fraction may be written so that the numerator and the denominator have no common factor. So suppose \( \sqrt{2} \) were written as a fraction whose numerator and denominator have no common factor. Then, squaring both sides as in the first proof, we have

\[ 2 = \frac{m \times m}{n \times n} . \]

This means that \( n \times n \) divides exactly into \( m \times m \). However, we supposed that there were no common factors between \( m \) and \( n \), other than 1 of course. This can only mean that \( n = 1 \). But then \( 2 = m \times m \). Since 2 is a prime number, this is not possible.

8. Note that it is not difficult to generalize these proofs. Certainly, both proofs can be used with only very minor adjustments to show that the square root of any prime number is irrational. However, the second proof can easily be adjusted to apply to an even larger set of square roots: If a number is not a perfect square, then its square root is irrational. In other words, unless \( \sqrt{m} \) is a whole number, it is irrational.
4.5 Enrichment Activity - Are 0.\bar{9} and 1 Different?

The purpose of today’s lesson is to reinforce the point made at the beginning of the chapter, that real numbers should be thought of as points on the number line rather than as expressions composed of digits that can be added and subtracted, and which can be made to appear on a calculator by pressing buttons.

Not all numbers should be regarded in the first place as points on the line. In some sense there is an important conceptual and ontological gap between rational numbers and real numbers. When we are doing arithmetic using the whole numbers, we are doing mathematics that has a discrete character. The natural numbers are devices for counting first of all. By the same token the rational numbers are, as the name suggests, numbers used to express ratios.

Without getting too philosophical about it, it can be said that by contrast the real numbers should be regarded geometrically, as locations on a continuous line, before they are given their symbolic representations with the help of digits. This is not to deny that there are very clever ways of expressing real numbers entirely in terms of the set of rational numbers. However, when this is done, the conceptual leap hidden in the attempt exacts its cost by involving us in the considerable abstraction of infinite sets and sequences.

In any case, the habit of interpreting numbers primarily or exclusively as discrete objects involved in calculations is destructive of all geometric thinking. Even at university we have come across honors mathematics students well into their programmes who had difficulty with their work because they were in the habit of thinking of numbers primarily in terms of their digital representations.

Lesson Goals

- To reinforce the idea of numbers as points on lines
- To interpret even an infinite decimal expression as a road map to a location on the number line
- To reflect on the paradoxical consequence that, for example, 0.\bar{9} = 1

Materials

- You should make an overhead transparency of the template provided in Section A.9 in the Appendix, or provide copies of this sheet for each student.

Lesson Sequence

1. To start the day’s lesson draw a number line on the board, as we did in the first session, and mark zero and the positive integers 1, 2, 3, 4 on it. Ask the students where they
would locate some numbers given by decimal expressions. For example, to begin with the familiar you could ask them to locate 2.3, 1.35, but then you should quickly move up to more challenging questions such as locating 0.1211211121112121112... It is best to use as examples only numbers that are positive (negative numbers behave in the opposite way: -1.35 is located to the left of 1 rather than to the right) and which lie between 0 and 10.

The purpose of the discussion is to review what the other digits of the number indicate about the location of the number on the line.

2. Now draw the portion of the number line from 0 to 1 and divide it into ten equal parts. You should use the template provided in Section A.9 in the form of an overhead transparency for this, or copies of that template provided for each of the students. Notice that the divisions of the line from 0 to 1 are numbered 1 to 9 on this template. These numbers are the decimal digits. Ask the students where the number 0.357 is located, using the top line on the template. The answer is of course that the 3 indicates that it is located between the third and fourth of the divisions. Now think of the second line on the template as the portion of the number line between 0.3 and 0.4, enlarged to a reasonable size, and ask the students to locate the number 0.357 on this segment. It is useful to describe the procedure as a succession of “zoomings in”, each time enlarging a portion of the previous diagram by a factor of 10. On their sheets and on your overhead transparency this can be demonstrated by putting in appropriate curved lines, as in Figure 4. You could point out to the students that when you get to the digit 7, you can draw the conclusion that the number is located exactly at one of the division points.

3. Now repeat this discussion for the number 0.34343434... . In this case the process never finishes! It is important to try to get the students to imagine this sequence of enlargements. You could ask them to imagine a movie that enlarges the number line gradually, keeping the number 0.34343434... at the center of the screen.

4. In order to help them think of numbers in this way, you could have them do an exercise at their desks. To begin the exercise, discuss the concept of cube root with them. The cube root of a number is a number such that if you take its third power (its cube) you get the original number back. Ask the students to tell you what the cube root of 8 is (it is 2) and what the cube root of 27 and 125 are (they are 3 and 5 respectively). Teach them the notation. For example, \( \sqrt[3]{8} = 2 \) now ask them what the cube root of 7 is. They will not be able to give a satisfactory answer. Furthermore, the reason we suggest a cube root rather than a square root for the exercise, is that then the students will not be able to use their calculators either (at least not in a straightforward manner they are likely to know about). Ask the students if they can say anything about the cube root of 7? Is it less than 2 (yes, for the third power of 2 is 8, which is more than 7)? Is it more than 1 (yes, for the third power of 1 is 1, which is a lot less than 7)? In other words, the cube root of 7 is somewhere between 1 and 2. It must be of the form 1.\ldots. Tell the students you are going to have them find the next three digits of the number. To record the result, they should have another copy of the template provided in Section A.9, or use some of the remaining line segments.

The idea is to have the students indicate, after a trial-and-error experiment, where the cube root of 7 is located. Since we have already discovered that the cube root of 7 is located between 1 and 2, the students should be invited to use their pencils to thicken
the interval between 1 and 2 in the top scale as shown below. Once this interval has been highlighted, you should explain that the second scale is to be interpreted as an enlargement of the interval between 1 and 2 on the top scale. To indicate this, have the students draw two (curved) lines like the ones shown in Figure 4.

5. The next step in the exercise is to get the students to decide what the next digit is. To discover this they should do calculations such as $1.2 \times 1.2 \times 1.2$ and $1.3 \times 1.3 \times 1.3$ until they discover the pair between which the correct value of $\sqrt[3]{7}$ is located. They should then thicken that interval on the second scale, and draw two curves from it to the third scale, to indicate that the latter is to be interpreted as an enlargement of that interval. At some point, as the students are doing their calculations, or when you discuss their findings, you might ask the students how long a piece of the number line is represented by the third scale. The answer is that the second scale represents a unit length (enlarged 10 times) while the third represents a length of one-tenth (enlarged 100 times). Have the students continue the exercise until the third decimal place has been found. Incidentally, $\sqrt[3]{7} = 1.91293118 \cdots$

It is important to point out to the students that the decimal representation of the number should be thought of as a set of instructions about where to locate the number on the line. It is essential that the students try to think of the number itself as the point on the line, and that they do not think of the decimal expression as the number. A little later we will see that just as there may be several sets of instructions that lead to the same house, so also there are numbers for which there are more than one decimal expression.
6. It is appropriate at this point to mention that when you look at it in this way, every decimal representation, even one that goes on forever, and whose digits may be generated randomly, corresponds to a number. The decimal expression provides you with a set of intervals, one for each digit, of lengths $1$, $1/10$, $1/100$, and so on, each contained in the preceding. In fact, when a digit terminates, you might say that it really ends in an infinite sequence of $0$’s, and that this means that after a certain stage, you will always be told to pick the first of the intervals that result when the latest interval is subdivided into ten pieces.

7. One of the consequences of this point of view is that sometimes a number can have more than one decimal expression. After all, if the decimal digits tell you with successively greater accuracy where to look for the number, it is conceivable that two sets of instructions lead to the same destination, much as there are usually more than one way to give directions to your house. To discuss the possibility of numbers that have more than one decimal expression, ask the students which is larger, $1.0$ or $0.99999...$? They will say that the former is larger. If so, ask them by how much? This should lead to an interesting discussion of the fact that numbers that can be written as terminating decimals always have a second expression ending in an infinite string of $9$’s. The students find this surprisingly difficult to accept.

There are several ways to justify thinking this way, and you should probably try them all. In the first place, remind the students that decimal expressions are supposed to be instructions how to get to the number. $1.0$ means that once you have chosen the interval from $1$ to $2$ (because of the digit $1$ before the decimal point), after that you will encounter nothing but zero digits, telling you to keep picking the first interval. On the other hand, $0.999...$ tells you to pick the interval between $0$ and $1$ and then to keep picking the last interval. You can see that the number $1$ is located in both of the resulting sequences of nested intervals.

Another way to see that $1.0 = 0.999...$ is to ask the students to use the method learned a few lessons ago to convert $0.999...$ to a fraction. You will see that the result is $1$.

A closely related third method is to remind students that $1/3 = 0.333...$ Students never quibble with that. Now ask them what they get if they multiply this by $3$.

A fourth method is to get the students to agree that if the number $0.999...$ is less than $1$, it must be between $0.999999$ and $1$. But the difference these two numbers is one one-millionth. So the difference between $1$ and $0.999...$ will have to be less than that. By adding more nines to the first of the two numbers we can make that difference as small as we like. Now if the numbers are not equal (that is, if they represent two distinct points on the line) they must be a certain distance apart. Clearly that is not possible, so $0.999... = 1$.

8. Paradoxically, the rational numbers are densely distributed on the number line. This can be seen by drawing the points corresponding to $k/10$ for integers $k$. Note that they are apart by $1/10$. Now do the same for the numbers $k/100$ and then for $k/1000$, and so on. After a while it begins to look as if the whole number line is covered with dots! This is a paradox because we observed earlier that there are more irrational numbers (holes that are left) than there are rational numbers! Part of the problem is, of course that when we draw dots on the line, they are not infinitely thin.
Chapter 5

Remainders

Above the boat,
bellies
of wild geese.

Kikaku, 1661-1701
(translated from the Japanese by Lucien Stryk and Takashi Ikemoto)

The purpose of this short chapter is to do what mathematicians call modular arithmetic. This is a kind of arithmetic produced when we ignore all properties of a number except its remainder upon division by a fixed number. For example, if we want to do modular arithmetic using the number 3, then we regard the numbers 5 and 11 as the same for our purposes, because if you divide 3 into them you get 2 as remainder in each case. Doing modular arithmetic is fun for the students, it helps them to see that there are other mathematical structures that share features with the system of integers. As an added bonus, once we get into the subject, it will allow us to show why it is that divisibility by 3 and 9 can be tested by adding digits.

The Chapter’s Goals

- To learn some simple modular arithmetic
- To learn why the tricks for division by 3 and division by 9 work
- To devise a similar trick for division by 11

Overview of Activities

- E.A. 5.1 - Some Strange Division Problems
  An introduction to modular arithmetic.
• E.A. 5.2 - Jelly Beans
  An explanation for the tricks for division by 3 or 9.

• E.A. 5.3 - Multiplication
  A fuller discussion of modular arithmetic including both addition and multiplication.

• Problem Set 5.4 - Apples and Oranges
  Problems applying and extending the ideas of this chapter.

• E.A. 5.5 - Leftovers
  A session that starts with number problems involving modular arithmetic and ends with a trick for divisibility by 11.
5.1 Enrichment Activity - Some Strange Division Problems

Lesson Goals

- To introduce modular arithmetic for division by 2 and division by 3

Materials

- You should have overheads or boards ready with the two story problems used to introduce the subject

Lesson Sequence

1. Give the class the following story problem, and tell them that they have to solve it in their heads. No calculators are allowed, nor even pencil and paper!

   **The king’s two sons**
   A king has two sons. He is getting old, and wants his sons to help him rule the kingdom by putting each of them in charge of a number of cities. He asks the minister of internal affairs to list the provinces with the number of cities in each of them. Here is the minister’s list:

   
<table>
<thead>
<tr>
<th>Province</th>
<th>Cities</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>123</td>
</tr>
<tr>
<td>B</td>
<td>58</td>
</tr>
<tr>
<td>C</td>
<td>211</td>
</tr>
<tr>
<td>D</td>
<td>318</td>
</tr>
<tr>
<td>E</td>
<td>87</td>
</tr>
<tr>
<td>F</td>
<td>153</td>
</tr>
<tr>
<td>G</td>
<td>273</td>
</tr>
</tbody>
</table>

   Can the king divide the cities evenly?

   There are several ways to think of this problem without writing anything down. For example, a student might suggest the following: “When the king tries to divide the cities in province A there will be one left over, because 123 is an odd number. The same is the case for 211, 87, 153, and 273. Therefore after he has tried to divide all the cities like that there will be five cities left to divide. Unfortunately, 5 is odd, so he cannot divide these equally. Therefore we must conclude that the king cannot do it.”

2. If it has not already come up as an option, suggest the following alternative to the students: It sounds more sophisticated, but really expresses the same idea: “We know that an even number plus an even number produces an even number, and that an odd number plus and odd number produces an even answer as well. In fact, we have an addition table
that goes like this:

<table>
<thead>
<tr>
<th></th>
<th>odd</th>
<th>even</th>
</tr>
</thead>
<tbody>
<tr>
<td>odd</td>
<td>even</td>
<td>odd</td>
</tr>
<tr>
<td>even</td>
<td>odd</td>
<td>even</td>
</tr>
</tbody>
</table>

Therefore we have

\[
\text{odd + even + odd + even + odd + odd + odd} = \text{odd},
\]

so if we add the numbers of cities we get an odd number of cities. This tells you that the king cannot divide the cities evenly between his sons."

If students do not themselves come up with this addition table, try to bring it into the discussion, since it relates to the arithmetic that constitutes the heart of the chapter. Make sure as well that you draw students’ attention to the strange ‘arithmetic’ that allows you to write ‘odd + even + odd + even + odd + odd + odd = odd’

3. Now present the following story problem to the students:

**The king’s three daughters**

Suppose the king also has three daughters, and wants them to take over management of the vineyards. Once again the minister of internal affairs is enlisted. This time he produces the following list of numbers of royal vineyards in each province:

<table>
<thead>
<tr>
<th>Province</th>
<th>Vineyards</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>29</td>
</tr>
<tr>
<td>B</td>
<td>57</td>
</tr>
<tr>
<td>C</td>
<td>85</td>
</tr>
<tr>
<td>D</td>
<td>93</td>
</tr>
<tr>
<td>E</td>
<td>83</td>
</tr>
<tr>
<td>F</td>
<td>67</td>
</tr>
<tr>
<td>G</td>
<td>138</td>
</tr>
</tbody>
</table>

Can the king divide his vineyards equally among his three daughters?

Once again the students are not to use a calculator. In particular they should not try to add up the numbers of vineyards. The point of the exercise is to get them to think about the problem differently. This time, they should be allowed to use pencil and paper to record partial results if they wish to.

Chances are the students will find this problem more difficult. The problem sounds similar to the preceding, but they cannot use the “odd” and “even” number idea. You could ask the students if they can think of something like “odd” and “even” that is suited to division by three. Suggest that they try to divide the 29 vineyards first. What will happen when they try? How many vineyards will be left to be distributed later? Ask students to think through each number of vineyards in that way. What are the remainders that occur? Why are these the only possible remainders? Suggest to the students that looking for remainders
upon division by three is completely analogous to the role of “odd” and “even” when we are dividing by two.

Encourage students to summarize the findings as follows:

- When you try to divide 29 by 3 you have 2 left over;
- When you try to divide 57 by 3 you have 0 left over;
- When you try to divide 85 by 3 you have 1 left over;
- When you try to divide 93 by 3 you have 0 left over;
- When you try to divide 83 by 3 you have 2 left over;
- When you try to divide 67 by 3 you have 1 left over;
- When you try to divide 138 by 3 you have 0 left over;

The problem can then be completed by adding up these remaining vineyards: $2 + 0 + 1 + 0 + 2 + 1 + 0 = 6$, and these six remaining vineyards can be divided equally among the three sisters. So, yes, the king is able to distribute his vineyards evenly.

4. Tell the students that Mathematicians have a special way to express the thoughts in this list of sentences. Instead of saying that

When you try to divide 29 by 3 you have 2 left over

They write

$$29 \equiv 2 \pmod{3}.$$ 

We pronounce this as “29 is congruent to 2 modulo 3”. In fact the sentence $29 \equiv 2 \pmod{3}$ can be also be understood in a second slightly more general way:

When you try to divide by 3, both 29 and 2 leave the same remainder.

Of course, it seems silly to think of dividing 2 by 3, but we can, for $2 = 0 \times 3 + 2$, and, as you can see, the remainder is 2 when you do it. Thinking of the sentence $29 \equiv 2 \pmod{3}$ this way also allows us to write things such as

$$29 \equiv 5 \pmod{3},$$

for these two numbers also both leave exactly the same remainder (namely 2) when you divide by 3. You could say that “for the purposes of remainders left when dividing by 3, the numbers 29 and 5 are equivalent”. Interpreting congruence modulo 3 this way allows you to think of $\equiv$ as a special kind of equal sign.

The list of sentences produced by our attempt to divide the vineyards now takes the following form:

$$29 \equiv 2 \pmod{3},$$
$$57 \equiv 0 \pmod{3},$$
$$85 \equiv 1 \pmod{3},$$
$$93 \equiv 0 \pmod{3},$$
83 \equiv 2 \pmod{3}
67 \equiv 1 \pmod{3}
138 \equiv 0 \pmod{3}

5. At this point of the discussion you could ask students if they can think of a way to create an addition table for division by 3 analogous to the “odd - even” table created earlier. What should the entries be for such a table? What replaces “odd” and “even” in this situation? Well, actually we already answered this question when we decided to concentrate on the remainders after division by 3. The entries should be chosen from the set of possible remainders: 0, 1, and 2. Here is the table you should get:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

For example, 2 + 2 \equiv 1 \pmod{3} because if you have two quantities, both of which are congruent to 2 (mod 3), then if you add them together, the answer will be congruent to 1 (mod 3). In particular, if you add 2 and 2 you get 4, which is congruent to 1 (mod 3).

6. Once this table is understood you should summarize the solution to the vineyards problem by means of a “calculation \pmod{3}”:  

2 + 0 + 1 + 0 + 2 + 1 + 0 \equiv 6 \equiv 0 \pmod{3}.

Note that as far as division by 3 is concerned, there is no difference between 0 and 6, since both leave the same remainder.
5.2 Enrichment Activity - Jelly Beans

Lesson Goals

• To explain why the tricks for division by 3 and division by 9 work the way they do.

Materials

• The students do not need anything for this lesson other than paper and pencil.
• The lesson will work particularly well if you can bring to class several boxes of various sizes, as well as some loose jelly beans (or equivalent candy) and a pad of sticky notes.

Lesson Sequence

1. Begin by posing this question:

A treasure trove of jelly beans
Suppose you have 3425 jelly beans, packaged in boxes of 1000, boxes of 100, and boxes of 10, plus some loose jelly beans. How many boxes of each type will there be?

There is more than one answer to this question of course, but one answer is the best because it is the tidiest. Which is that? Notice the connection between tidiness and place value. Decimal place value is what you get if your containers have sizes 10, 100, 1000, etc. (and you are inclined to keep you jelly beans well-organized). Of course, the numbers of boxes of each size are precisely the digits of the number 3425: There are 3 boxes of size 1000, 4 boxes of size 100, 2 boxes of 10, and 5 loose jelly beans. Note as well (and this is going to be key to the explanation) that if you simply count the boxes (regardless of size) and add in the number of loose jelly beans for good measure you get precisely the sum of the digits in the original number.

2. Now ask students this further question:

How exactly would you go about dividing these 3425, mostly boxed, jelly beans among three people?

There are many ways in which to divide you cache of jelly beans. for example, you could begin by giving each person one of the boxes of 1000. But then you will not be able to do the same with the boxes of 100 since there are four of those.

3. In order to explain the division trick you have to ask students to imagine a very particular way to distribute the jelly beans. Tell them that you have decided to open each of the boxes and distribute as many as the jelly beans in each box as you can. So, ask students to consider dividing up the first box of 1000 jelly beans. This is where it will help if you
have box with you, with the number 1000 written on it in large letters. You are desperate to know how many will be left after you have tried to divide that one box, for there is a chance you will get to keep that one for yourself. Ask them to imagine the process: The box has been opened, the contents have been distributed to the extent that this was possible, and where there was a box of 1000 there is now a single jelly bean in its place. So take a sticky note and stick it on top of the number 1000 written on the box, and write a large ‘1’ on the sticky note. There are 1000 jelly beans in the box, but as far as your personal (possible) benefit is concerned, there might as well be one - the other 999 are going to be shared out.

How many jelly beans will we have once we have done the same to each box?

Notice that each box is replaced by a single jelly bean. As you discuss it, you can act it out using the boxes you brought, replacing the original number on each box by a sticky note with the number 1. So the total number of jelly beans, at the end of the process of breaking open all of the boxes, is precisely the same as the sum of the digits of the original number: $3 + 4 + 2 + 5 = 14$. If you distributed the jelly beans exactly as described, then at this (intermediate) stage of the process you will be left with as many jelly beans as the sum of the digits of the original number.

If we can divide these remaining jelly beans by 3, we will have succeeded in dividing 3425 by 3. In other words, if the number obtained by adding the digits can be divided by 3, then so can the original number. In fact we can go a step further: even if the original cannot be divided by 3, the remainder that is left will be exactly the same as the remainder we get if we try to divide the sum of its digits by 3. In the case of 3425, when we try to divide 14 by 3 we find we get a remainder of 2. Therefore, dividing 3425 by 3 will leave a remainder of 2. Notice that we have shown why divisibility by 3 works the way it does:

**The trick for checking divisibility by 3:**
When you try to divide a number by 3, the remainder is the same as the remainder you get if you divide the sum of its digits by 3.

4. **Division by 9** At this point the students should be ready to discover and explain the trick for checking divisibility by 9. Challenge them to think about it. You could ask them to find an easy way to determine whether the number 56274 is divisible by 9; and if not, what its remainder will be. The trick is in fact completely identical to the procedure that works for 3:

**The trick for checking divisibility by 9:**
When you try to divide a number by 9, the remainder is the same as the remainder you get if you divide the sum of its digits by 9.
5.3 Enrichment Activity - Multiplication

This section is optional. In it the students explore the possibility of multiplication tables in modular arithmetic. The results will not be used in later chapters.

Lesson Goals

• To develop modular arithmetic a little further

Materials

• The students will not need anything other than paper and pencil.

Lesson Sequence

1. Begin the session by reminding the students that if you multiply an even number by an even number you get an even number, and that the product of an even number and an odd number is always even. In fact we could, and often do, display this information by making a multiplication table to complement the addition table we produced earlier. Ask the students to help you decide what should go into the table.

<table>
<thead>
<tr>
<th>×</th>
<th>even</th>
<th>odd</th>
</tr>
</thead>
<tbody>
<tr>
<td>even</td>
<td>even</td>
<td>even</td>
</tr>
<tr>
<td>odd</td>
<td>even</td>
<td>odd</td>
</tr>
</tbody>
</table>

2. Note that this table, together with the addition table produced earlier make it look as if we have made an arithmetic involving just two numbers, one called “even” and one called “odd”, for between them they tell us exactly how to add and multiply these “numbers”. Ask the students to give the answer to some calculations in this arithmetic, such as

\[(\text{even} \times \text{even}) + (\text{odd} \times \text{even}) = ?\]

or

\[\text{even} \times (\text{odd} + \text{odd}) + \text{odd} = ?\]

3. Remind students that, in keeping with the focus on remainders in this chapter, since an odd number is a number that produces the remainder 1, and an even number the remainder 0 when we divide by two, we decided in an earlier lesson to use the symbols 0 and 1 in place of the words “even” and “odd”. In other words, the multiplication and addition tables could be expressed as follows:
4. Spend some time with the students reflecting on the features of these tables. At first there seems to be nothing strange about them. We already know that $0 \times 1 = 0$ and that $0 + 1 = 1$. But, wait a minute, the last entry in the second table seems to say that $1 + 1 = 0!$ Can this be correct? Well, it is of course not correct if we interpret 0 and 1 in the usual way. However, in this “even - odd” arithmetic the symbol 0 represents all even numbers, 1 represents all odd numbers, and the sentence $1 + 1 = 0$ means “when you add two odd numbers the sum is even”. In fact, of course, we are doing arithmetic modulo 2 here. Rather than writing $1 + 1 = 0$, which is not correct unless it is interpreted in a non-standard way, we should write $1 + 1 \equiv 0 \pmod{2}$. Similarly, $1 \times 1 \equiv 1 \pmod{2}$ means that if you multiply two numbers both of which leave a remainder of 1 when divided by 2 (i.e. both are odd) then their product will also leave that remainder.

5. Remind the students of the addition table modulo 3 constructed earlier. Here it is again:

$$
\begin{array}{c|ccc}
+ & 0 & 1 & 2 \\
\hline
0 & 0 & 1 & 2 \\
1 & 1 & 2 & 0 \\
2 & 2 & 0 & 1 \\
\end{array}
$$

6. Challenge the students to construct a multiplication table for arithmetic modulo 3. It will look like this.

$$
\begin{array}{c|ccc}
\times & 0 & 1 & 2 \\
\hline
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 2 \\
2 & 0 & 2 & 1 \\
\end{array}
$$

7. Once the tables are finished, or perhaps while you are still working on them, you should also challenge the students by asking them what the sentence $2 \times 2 \equiv 1 \pmod{3}$ really means. Of course the “English” translation of it is “Suppose you have two numbers whose remainders will be 2 if you divide them by 3, and suppose you multiply these number together; then the product, if divided by 3, will give a remainder equal to 1”. When
we produced the multiplication table we probably only tested this on a particular pair of
numbers whose remainder is 2, namely 2 and 2. What if we took two others? Get the
students to try multiplying some other pairs of numbers both of whose remainders are 2.
Here are some examples: 20, 35, 122, 101. For example, 20 = 6 × 3 + 2 and 35 = 11
× 3 + 2. Look what happens if we multiply 20 and 35: 700 = 233 × 3 + 1! After the
students have tried a number of examples, and are inclined to believe that it will always
work, ask them whether it might not be a coincidence that all the examples tried seem to
work. Could it be the case that the rule works for most cases, but that there are some
exceptions? In other words, can we find an explanation why it will always turn out that
2 × 2 ≡ 1 (mod 3)?

8. In order to produce an argument that should convince the students, you should first discuss
with them the following very concrete way of doing arithmetic (mod 3). Suppose you have
three friends, Alan, Barb and Cindy. Let us call them A, B, and C for short. Suppose
you have 20 bags of oranges which you would like to divide among them. If there are 35
oranges in each bag, how many will be left over after you have given each of your three
friends as many as you can?

One way to see the structure of the question, and thus the reason why these calculations
work the way they do, is as follows. If you are going to distribute the oranges, there is no
need to open all of the bags. The first 18 bags can be dealt with by giving 6 to each of
your friends. This leaves 2, because 20 ≡ 2 (mod 3), so our multiplication problem has
the same answer (that is, the same remainder) as the multiplication 2 × 35 (mod 3). Here
is a sequence of diagrams illustrating the principle: The number of rows is ≡ 2 (mod 3)
in the illustration that number is 20). Each row represents a bag containing a number of
oranges ≡ 2 (mod 3). The first diagram illustrates the first step in which all but 2 of the
bags are distributed to Alan, Barb and Cindy.

In the next step, the two remaining bags are opened, and all but 2 of the oranges in each
bag is distributed to A, B, and C.
Notice that the number of oranges that remains is precisely the remainder of one of the numbers we are multiplying, times the remainder of the other.

9. **A card game for three players?** Here is another question you can ask to support this unit: Imagine a card game involving three players. You could ask the students to invent a name for it. The game has become so popular that the schools in a district are planning a tournament. Typically a school team will not be exactly divisible by 3, so each school brings some “extras”. To test addition you could suppose that there are four schools in the tournament, one bringing 1 extra, the second bringing 2 extra players, the third also bringing 2 extras, and the fourth bringing no extra players. At the tournament, how many players at a time will be sitting out a game? To test multiplication you could imagine 25 schools each bringing 2 extras.
5.4 Problem Set - Apples and Oranges

This section can safely be skipped. Future lessons do not depend on it.

Lesson Goals

• To continue the exploration of modular arithmetic

Materials

• The students will need copies of the problems in Section A.10 in the Appendix.

Lesson Sequence

The idea of this problem set is to get them to develop an arithmetic (mod 5) on their own. There is no reason why we go from 2 and 3 in the preceding lesson to 5 in this one, other than that modular arithmetic is more interesting if a prime number is used. It may help to have the addition and multiplication tables (mod 3), which were studied in the preceding lesson, on the board when the class starts, and to review them briefly. At the same time you could do some examples on the board of calculations (mod 3) that involve products of larger numbers. For example, you could discuss the calculation

\[ 2 \times 46 + 7 \times 28 + 1 \times 68 \equiv ? \pmod{3}. \]

The point is that it is not necessary to do the multiplications, but that the large numbers may first be replaced by their remainders (mod 3). This is like saying (in the case of the first term in the summation) that if you have two bags of 46 oranges each, then if you were to distribute each bag to three friends, each bag would produce a leftover of 1 orange. That is,

\[ 2 \times 46 \equiv 2 \times 1 \equiv 2 \pmod{3}, \]

or if we do this to the whole calculation,

\[ 2 \times 46 + 7 \times 28 + 1 \times 68 \equiv 2 \times 1 + 7 \times 1 + 1 \times 2 \equiv 11 \equiv 2 \pmod{3}. \]

In fact we could even replace the factor 7 by its remainder 1. However, this is probably best illustrated by another example: consider the calculation

\[ 83 \times 32 + 40 \times 62 \equiv ? \pmod{3}. \]

Here you have to imagine 83 bags containing 32 oranges each, and 41 bags containing 62. If you were to distribute these to three friends, you would probably not bother to open most of the bags. In fact, you would begin by distributing 81 of the 83 bags to your three friends by giving
each 27 of these bags. This leaves you just two of the 83 bags. Then you would do the same
with 39 of the 41 bags. In terms of a calculation, what you have done is this:

\[ 83 \times 32 + 40 \times 62 \equiv 2 \times 32 + 1 \times 62 \pmod{3}. \]

At this point you would open the remaining bags to distribute the contents. That is, you would
continue the calculation as follows:

\[ \equiv 2 \times 2 + 1 \times 2 \equiv 6 \equiv 0 \pmod{3}. \]

Here is the handout for the day, together with comments. A copy without the comments can
be found in the Appendix as Section A.10.

\[ \circ \circ \circ \]

1. Finish the following calculations in arithmetic \( \pmod{3} \):

\[
1 + 2 \times 1 + 2 \equiv \? \pmod{3}
\]

\[
2 \times (1 + 0 + 2) + 2 \equiv \? \pmod{3}
\]

\[
245 \equiv \? \pmod{3}
\]

\[
12 \times 13 + 36 \times 38 \equiv \? \pmod{3}
\]

2. Make addition and multiplication tables for arithmetic \( \pmod{5} \); that is, the arithmetic
of remainders you get when you divide numbers by 5.

This is what the answer will look like:

\[
\begin{array}{c|cccc}
+ & 0 & 1 & 2 & 3 & 4 \\
0 & 0 & 1 & 2 & 3 & 4 \\
1 & 1 & 2 & 3 & 4 & 0 \\
2 & 2 & 3 & 4 & 0 & 1 \\
3 & 3 & 4 & 0 & 1 & 2 \\
4 & 4 & 0 & 1 & 2 & 3 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\times & 0 & 1 & 2 & 3 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 2 & 3 & 4 \\
2 & 0 & 2 & 4 & 1 & 3 \\
3 & 0 & 3 & 1 & 4 & 2 \\
4 & 0 & 4 & 3 & 2 & 1 \\
\end{array}
\]
3. When you try to divide a bag of 237 apples equally among five friends, how many apples will be left over? How would you write this fact as an equation?

Of course the remainder is 2. There are actually two ways to write this: You could express the fact by the equation $237 = 47 \times 5 + 2$, or in the spirit of this chapter, you could express it by writing $237 \equiv 2 \pmod{5}$. Actually, the latter of these equations is the one that should be emphasized. The first one actually says more than the question asks, for it tells not only how many apples will be left, but also how many apples each of your friends will receive (47). So it could be argued that the first equation is not an accurate translation of the answer to the question.

4. When you have a bag of 13 oranges which you want to distribute equally among 5 friends, how many oranges will be left over? Express this as an equation.

The answer in this case is of course $13 \equiv 3 \pmod{5}$.

5. When you have 24 bags of 13 oranges each, and you try to divide them equally among five friends, describe how you would distribute the oranges if you want to leave as many bags undisturbed as possible, and avoid unnecessary calculations. How many oranges will be left over at the end? Express this sentence by means of an equation.

You would probably do the following: You give 4 bags to each friend, and have 4 bags left. This part could be expressed by saying that $24 \times 13 \equiv 4 \times 13 \pmod{5}$. Next you might distribute one of these bags by giving each friend 2 oranges, so that you have 2 left out of that bag. This says that $13 \equiv 3 \pmod{5}$. In other words, if you do this to each bag, $4 \times 13 \equiv 4 \times 3 \pmod{5}$. Next you would take these 12 remaining oranges, give two of them to each of your friends, and be left with 2. Thus the whole operation could be expressed by

$$24 \times 13 \equiv 4 \times 13 \equiv 4 \times 3 \equiv 2 \pmod{5}.$$
6. Suppose you have a lot of bags of oranges, each with 14 oranges in them. How many bags would you have to use, as a minimum, if you want to distribute the contents of the bags to six friends and have nothing left over?

The answer is that because $14 \equiv 2 \pmod{6}$, therefore each bag will produce a remainder of 2 oranges. In order to be able to distribute these exactly among your six friends you will need 3 bags, for $3 \times 2 \equiv 0 \pmod{6}$. After the students have had sufficient time to work on these problems, you should discuss them at the board.
5.5 Enrichment Activity - Leftovers

In this session we complete our unit on modular arithmetic. We begin with some interesting problems for the students to tackle at their desks on their own or in groups. We will end with a discussion about divisibility by the number 11. You can safely skip this section, as it will not be referred to in succeeding lessons.

Lesson Goals

- To practice some modular arithmetic
- To develop a trick for divisibility by 11.

Materials

- You should make copies of the problems provided in the Appendix in Section A.11

Lesson Sequence

Have students work on the following problems. A copy of this problem set is included in the Appendix as Section A.11.

1. How many numbers between 1 and 100 are congruent to 1 (mod 3)? How many are there between 1 and 1000?

The idea is to force students to notice that to be congruent to 1 (mod 3) the number must be one greater than a multiple of 3, and that they are therefore spaced three units apart. Now the multiples of 3 between 1 and 100 are 3, 6, 9, ... , 99. That is, there are 33 of them. The corresponding numbers congruent to 1 (mod 3) are 4, 7, 10, ... , 100. Finally we should check whether there might be an additional one at the beginning or at the end of our list. Well, there cannot be anything more at the end, for our list included the number 100. However, at the beginning of the list we should include the number 1. Thus in total there are 34 numbers congruent to 1 (mod 3)

2. Find a number which is congruent to 1 (mod 3) and congruent to 2 (mod 4).

If we write down the list of positive integers that are congruent to 1 (mod 3) we get 1, 4, 7, 10,
13, 16, 19, 22, 25, ... and if we write down the numbers congruent to 2 (mod 4) we get 2, 6, 10, 14, 18, 22, 26, .... This immediately shows a couple of numbers of the type asked for, namely 10 and 22. You could ask the students, when you discuss this question at the board, “how far apart are these numbers going to be?” The answer is that they will continue to be spaced apart by 12 units. The reason is that the numbers in the first list are 3 units apart, while the numbers in the second list are 4 units apart. Therefore, the distance between numbers that are common to both lists should be a common multiple of 3 and of 4. The least common multiple is 12. In any case, this kind of consideration is needed to answer the next question.

3. How many numbers are there between 1 and 100 satisfying the description of question 2?

4. If you add all the odd numbers from 1 to 99, what is the sum congruent to (mod 3)?

The best way to do this question is to replace the list of odd numbers from 1 to 99 by their remainders after division by 3. If we do this we get

\[
1, 0, 2, 1, 0, 2, 1, 0, 2, \cdots, 1, 0
\]

The zeros correspond to the multiples of 3, the 1’s correspond to numbers that are one larger than a multiple of 3, etc. Thus there are 17 0’s, corresponding to 3, 9, 15, \cdots, 99. There are also 16 2’s corresponding to 5, 11, 17, \cdots, 95, and 17 1’s corresponding to 1, 7, 13, \cdots, 97. Thus the remainder, if we add them all, is the same as the remainder we get if we add all the 1’s and all the 2’s:

\[
17 \times 1 + 16 \times 2 \equiv 17 + 32 \equiv 2 \equiv 1 \quad (\text{mod } 3).
\]

5. Suppose \( m \equiv 2 \) (mod 3). Can \( m \) be a square number?

The answer is NO, for one of the following must be the case: Either \( m \equiv 0 \), or \( m \equiv 1 \), or \( m \equiv 2 \) (mod 3). But \( 0 \times 0 \equiv 0 \) (mod 3), \( 1 \times 1 \equiv 1 \) (mod 3), and \( 2 \times 2 \equiv 1 \) (mod 3). In none of the three cases do we get a remainder of 2.

**Divisibility by 11.** To prepare for the last topic in this chapter, the construction of a method for checking divisibility by 11, you should start by reviewing the proof of the trick for divisibility
by either 3 or 9. Ask the students to try to explain the proof to you, perhaps with you playing the role of the skeptic.

For divisibility by 9 the explanation can be given as follows: We have the congruences

\[
\begin{align*}
1 & \equiv 1 \pmod{9} \\
10 & \equiv 1 \pmod{9} \\
100 & \equiv 1 \pmod{9} \\
1000 & \equiv 1 \pmod{9} \\
& \vdots
\end{align*}
\]

Thus we have, for example,

\[
3572 \equiv 3 \times 1000 + 5 \times 100 + 7 \times 10 + 2 \times 1 \equiv 3 + 5 + 7 + 2 \pmod{9}.
\]

Now for the discussion of divisibility by 11. To do this it is necessary first to allow negative numbers into the discussion. Earlier in this session you discussed with the students the relationship between the ordered list of natural numbers and their remainders mod three. We could simply extend this list to include the negatives, as follows:

\[
\begin{array}{cccccccccccc}
\cdots & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
\cdots & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & \cdots
\end{array}
\]

Where the remainders \( \pmod{3} \) are entered directly below the numbers. It seems a little difficult at first to attach a concrete meaning to something like

\[-4 \equiv 2 \pmod{3}.
\]

However, it can be done as follows: Suppose that instead of having oranges to distribute to your three friends, you are in desperate need of 4 oranges. In other words, the - sign indicates a need instead of a supply. You do not mind asking your three friends to help out, but you like to treat them equally, so you will ask them each to give you the same number of oranges. Clearly they would have to give you two each. Now you have the four oranges you needed, but you also have 2 left over. Thus the remainder associated with \(-4\) is 2.

In order to develop a trick for divisibility by 11, we should now obtain congruences \( \pmod{11} \) for the powers of 10, just as we did in the case of divisibility by 3 and by 9. From the preceding paragraph we see that we have \(-1 \equiv 10 \pmod{11}\). Of course, we could turn this around, and write \(10 \equiv -1 \pmod{11}\). Let us see what we can say about other powers of 10: \(100 = 99 + 1\), and 99 is divisible by 11, so we have \(100 \equiv 1 \pmod{11}\). What about 1000? Unfortunately, 999 is not divisible by 11. However, 990 is. Thus \(1000 \equiv 10 \equiv -1 \pmod{11}\). If we continue to
other powers of 10 we get this list:

\[
\begin{align*}
1 & \equiv 1 \pmod{11} \\
10 & \equiv -1 \pmod{11} \\
100 & \equiv 1 \pmod{11} \\
1000 & \equiv -1 \pmod{11} \\
10000 & \equiv 1 \pmod{11},
\end{align*}
\]

If we apply this to a large number, just as we did for the number three, we get this sort of thing:

\[
6243975 \equiv 6 \times 1000000 + 2 \times 100000 + 4 \times 10000 + 3 \times 1000 + 9 \times 100 + 7 \times 10 + 5 	imes 1 \\
\equiv 6 \times 1 + 2 \times (-1) + 4 \times 1 + 3 \times (-1) + 9 \times 1 + 7 \times (-1) + 5 \times 1 \\
\equiv 6 - 2 + 4 - 3 + 9 - 7 + 5.
\]

What we see here is that to determine divisibility by 11 we should look at the sum of every other digit, and subtract from that the sum of the remaining digits. In this example we get \(24 - 12 \equiv 1 \pmod{11}\). Thus the number is not divisible by 11, and in fact if we were to divide by 11 we would get a remainder of 1.
Chapter 6

Counting

Brown bird, irresolute as a dry leaf, swerved in flight just as my thought changed course, as if I heard a new motif enter a music I’d not till then attended to.

Denise Levertov, 1923-1997
(from Sands of the Well, 1996)

When you tell students that you are going to do a unit on counting, they will think that you are joking, and may even feel insulted. The fun in this short chapter is to demonstrate to students how really difficult counting can be. The point is that if you have to count the number of objects in a very large collection, it is important that you can find an arrangement of the collection that makes the counting process possible. A simple illustration may help you to see this: Counting the students in a large class is quite difficult if they are crowded together in the hallway waiting for someone to open the door to let them into the classroom. The counting problem becomes much easier if you first let the students into the room and into their seats. Once they are settled, all you have to do is multiply the number of rows by the number of seats in each row, and subtract the number of empty seats. What makes the counting easy once the students are seated is the fact that the students are then arranged in a pattern. Usually the ability to count a large set depends on our finding an arrangement of the set that makes the count possible. Often, finding this arrangement is akin to taking a particular point of view on the set.

This chapter is very much a preparation for the following chapter on probability, for probability is always essentially a count of equally likely possible events.
The Chapter’s Goals

- To learn that to count a large collection you have to find a way to organize it
- To explore a variety of ways of organizing large collections
- To develop skill in interpreting a counting situation and finding the appropriate mathematics to model it

Overview of Activities

- E.A. 6.1 - Finding a System
  An introduction to counting problems with emphasis on the need to find appropriate organization of a collection before it can be counted.

- Problem Set 6.2 - Counting Practice
  A set of counting problems.

- Problem Set 6.3 - Coins and Dice
  A set of counting problems involving situations often used in the teaching of probability.
6.1 Enrichment Activity - Finding a System

In this section, you should stress again and again the importance of finding a pattern or system when counting large and complicated sets.

Lesson Goals

- To emphasize, through examples, that organization is the essence of counting
- To practice careful reading of the description of a set before modeling it mathematically

Materials

- Prepare an overhead transparency of the icosahedron, the first template provided in the Appendix in Section A.12.
- It is possible to conduct the last part of this lesson as a set of problems done by the students in groups or individually. In that case you should prepare a copy of the problem set provided in the Appendix in Section A.12 for each student.
- If you decide to conduct the entire class in discussion mode, you should also make an overhead transparency of the dodecahedron pictured on the second page of Section A.12 in the Appendix.

Lesson Sequence

1. Ask the students the following question:

   How many two-letter words are there, if nonsense words are allowed?

   To get across to the students the idea that an good arrangement of the set is essential, you could begin at the board by writing down some random two letter words and suggesting that you will keep doing this until you have them all. Soon the students will probably suggest that this is a hopeless task, that you should be systematic about it, writing them down in alphabetic order - first the ones that begin with A, then the ones that begin with B, and so on. let the students suggest the way to organize the collection. An extended discussion about this issue is more important than the count at the end.

   Students will soon notice that for each initial letter there are 26 words, so that there will be 26 columns, each containing 26 words, as in Table 6.1, and that therefore the total number of words is $26 \times 26 = 676$. You could point out to the class that this is a lot like asking the class to take their seats before counting it.

2. Now put up your overhead transparency of the icosahedron, and ask students this problem, which beautifully illustrates the importance of taking a particular point of view in order to make counting possible:
Table 6.1: A possible arrangement of all two-letter words

Suppose you want to know how many edges there are in an icosahedron, given that it is a regular solid with 20 triangular faces.

Even with a picture in front of you, it is not easy to be sure of the correct number of edges. Let students try ideas. One very nice way to do it is as follows: We know that each of the twenty triangles has three edges. Therefore, as a first attempt we might say that there are $3 \times 30 = 60$ edges. However, each edge belongs to two triangles, so we will have counted the edges twice by doing this multiplication. That is, there are $60 \div 2 = 30$ edges in total.

This is an instance of counting problems in which the best procedure is to first count the set with duplications and then account for these duplications at the end.

3. Here is another question of the same type: Ask the students how many vertices the icosahedron has. Again, give the students time to see themselves if they can create a count analogous to the one in the preceding discussion. Here is one solution: Every triangle has three vertices around it, so that suggests there might be $20 \times 3 = 60$ vertices. However, you can see from the picture that every vertex is shared by five triangles, so we have counted every vertex five times. Thus the correct answer is $20 \times 3 \div 5 = 12$.

At this point you can continue your discussion at the board, using the problems that follow, or you can give students sheets with these problems to work on them in groups or individually. A copy of this list of problems is provided in the Appendix in Section A.12.
It is important to get the students in the habit of indicating the calculation that leads to the answer, rather than the numerical answer itself. Also make sure that the solutions are reviewed at the board, even if the students did them on their own, for it is crucially important that it is clearly understood by all that in each case the calculation of the total is made possible by an arrangement of the set. The numbering of the problems in the Appendix is necessarily different from the numbering presented here.

4. How many different two-letter words are there if the first letter has to be a consonant and the second one a vowel? Nonsense words are allowed, and it is agreed that the letter Y is to be used only as a vowel.

The answer is found by counting the number of vowels (6 if you include Y), and the number of consonants (26 - 6 if you do not include Y here), and then multiplying them: $6 \times 20 = 120$.

2. How is the total changed if you allow Y to be used both as a vowel and as a consonant?

This time there are 6 vowels, and 21 consonants, since Y is included in each set. Thus the total is $6 \times 21 = 126$, as can be seen by imagining the words arranged in 21 rows, each 6 words long. That is, there is one row for the initial letter B, this row containing the words BA, BE, BI, BO, BU, BY; then there is a row starting with the letter C and ending in the same vowels, and so on.

3. How many different three-letter words are there if the first letter must be a vowel? Once again, nonsense words are allowed, and Y may be used as a vowel only.

This time, to imagine an arrangement of the set that will allow you to count it, you could think of a book that has one page for each of the six vowels. On the page assigned to the letter A you have written down all the permissible three letter words beginning with the letter A. They are arranged in 26 rows of 26 words, each row determined by the second letter in the word. Thus on the page assigned to the letter E, the fourth row of words begins with the words EDA, EDB, EDC, ... . Once the students agree that this is a good way to organize the words, one for which it is clear that none have been missed and that none have been repeated, you can start the count. There are 6 pages, each with $26 \times 26 = 676$ words. Thus there are 4056 words in total.

4. How many edges does a dodecahedron have, given that a dodecahedron is a regular surface made up of 12 pentagons? Also, how many vertices does it have?
Once again, to get the number of edges you multiply the number of faces by the number of edges around each face, and then divide by two because each edge will have been counted twice. Thus the answer is $5 \times 12 \div 2 = 30$. The number of edges is obtained by multiplying 12 by 5 and then dividing by 3 (the last step because, as can be seen from the picture, each vertex gets counted three times).

5. How many different pairs of whole numbers add up to 40? (The whole numbers are defined to be the numbers 1, 2, 3, 4, ...).

The way to count these is to pay attention to the smaller of the two numbers only. Thus we have $1 + 39, 2 + 38, \cdots, 20 + 20$. If we continue we get $21 + 19$ which is the same pair as $19 + 21$, so we have included this pair already. The same holds for $22 + 18, 23 + 17$, etcetera. Thus the total is 20.

6. If there are seven puppies in a pet store, and you have to choose one for yourself and one for your friend, in how many ways can you do this?

The way to assess this is to focus on the act of choosing. If you choose one for yourself first, there are 7 choices. For each of these choices there are 6 possible choices for your friend. Thus the total is $7 \times 6 = 42$. 
7. If there are seven puppies in a pet store, and you have to choose two for yourself, in how many ways can it be done?

At first this may seem to ask the same question, but in fact it is not. For now it does not matter in which order you choose the two puppies. That is, if you count the number of ways you can choose two puppies by choosing one first and then the other you will get a total of 42 again, but in this case each outcome is counted twice. That is, the answer is 21.

8. If there are seven puppies in a pet store, and you have to decide which one is the cutest and also which one is the friendliest, in how many ways can you do it?

This question is different again, for once you have selected the cutest in one of seven possible ways, you still have seven choices left for the friendliest, for it is possible that the cutest puppy is also the friendliest. Thus, the answer in this case is $7 \times 7 = 49$. 
6.2 Problem Set - Counting Practice

Counting problems are a lot of fun. Doing them well depends more on seeing a variety of situations and interpreting subtle changes in descriptions of sets than on a single deep concept that needs to be developed carefully. For that reason, this chapter contains a lot of problems.

Lesson Goals

• To give students more good counting problems to work on
• To learn how to infer, in the context of a word problem, what collection must be counted

Materials

• You should make copies for the students of the problems found in Section A.13 in the Appendix.

Lesson Sequence

You can decide how you want to structure this problem session. It is probably a good idea to do one or two of the questions in the form of whole class discussions at the beginning. This will give you a chance to make sure that the students remember from the preceding lesson that a collection needs to be organized before it can be counted. It will also give you an opportunity to help students see that interpreting a question correctly requires close attention and careful thought.

The list of problems is quite long, and some of them are not easy. You should feel free to select from this list. It is much more important that the students be given lots of time and encouragement to reflect on the solutions. Often in a counting problem the solution seems easy. For example, in the first question there are 20 students in the class and the answer to the question is $20 \times 20$. What could be easier? There will be an enormous temptation for students to jump to the conclusion that all you have to do is look for some numbers to multiply. Spend lots of time discussing with them the reasons why a certain calculation is appropriate for the solution. They need to understand the reasoning behind each solution, so have them do a few questions carefully rather than a lot of questions quickly.

1. A class has 20 students in it. Both MacDonald’s and Burger King want to give a free burger to the student whose essay they think is best. In how many different ways can the two burgers be awarded?

It is very important in a counting question to start with a judgment as to which outcomes are
allowed, as well as a judgment as to which outcomes are really one and the same. In other words, once the students have spent a bit of time thinking about this question, it is a good idea to have a discussion with the class to settle these issues. Here is how such a discussion might go: In this question it is important to note that because the judgments of the Burger King representative and the MacDonald’s representative as to what constitutes a good essay may differ, it is important to notice that there is no reason to suppose that there is one and only one "best" essay, and that both prizes will go to the same person. On the other hand, if you try to make a rule that both prizes not go to the same student, you have a big problem on your hands, for in that case you have to decide whose burger will count as first prize and whose as second. When the burgers are provided by two very competitive fast food outlets, it is wise to avoid making a pronouncement on that issue. Thus, we will have to allow both burgers to go to the same person if that is how the two company representatives judge the essays. This settles the collection of outcomes, except for one matter: Does it matter to the winners whether they get a Burger King burger or a MacDonald’s burger? If the burgers were identical in the minds of the students, then it would not matter whether Sally won the Burger King prize and Roger the one provided by MacDonald’s, or whether it should be the other way around. In that case we would have to divide by two at the end of the counting process. However, you will find that students have strong opinions on the relative qualities of the two companies, so who wins what prize is important to them.

Given all these considerations, and assuming that the students really agree that the rules of procedure should be as suggested here, the number of possible outcomes is 20 × 20, 20 possible choices for the Burger King representative, and for each of these outcomes, 20 independently made choices for the MacDonald’s representative.

2. A class has 16 students in it, 7 boys and 9 girls. The teacher wants to pick one boy and one girl to play the lead roles in a performance of Romeo and Juliet. In how many different ways can the teacher choose the pair?

Here the class will undoubtedly agree that Romeo should be played by a boy and Juliet by a girl. Thus there are 7 choices for Romeo, and 9 for Juliet. You can imagine listing all the possible choices on a page by listing pairs of names, the boy who plays Romeo on the left and the girl who plays Juliet on the right. These pairs of names have to be arranged in a way that ensures that all possibilities are included and that none of them are included twice. The most natural way to ensure this is by doing the boys in alphabetical order creating for each of them a column in which the girls are listed in the same order. This will create 7 columns each consisting of 9 boy-girl pairs. That is, the total number of choices is 7 × 9. It is of course not necessary, or even advisable, to write out all these name pairs for some imagined class, but it is essential that all the students have a good understanding of the justification for the claim that the correct answer is 7 × 9. The same principle applies to all of the questions in this section, though it is not mentioned every time.
3. Suppose instead there were 8 boys and 8 girls in the class. Now how many ways are there to choose Romeo and Juliet?

This is rather simple. The answer is $8 \times 8$.

4. Once again, suppose we have a class with 8 boys and 8 girls. The teacher wants to pick two students to clean the blackboards. In how many ways can the teacher choose?

The teacher begins by choosing one of the sixteen students. It does not matter whether she picks a boy or a girl. Obviously, since the teacher needs two students, she is not allowed to pick the same student again when she makes her second pick. That is, she has only 15 students to pick from once the first one has been chosen. This indicates that there are $16 \times 15$ choices. However, once she has picked the two students it does not matter which of them has been picked first. In other words, in the calculation $16 \times 15$ we have counted each outcome twice. Thus the number of truly distinct possible outcomes is $16 \times 15 \div 2$.

5. Suppose instead that the teacher wants to award one prize for the best poem written by a boy and one for the best poem written by a girl. This time, how many different ways are there to award these two prizes?

In this case, there are 8 ways to choose the boy, and 8 ways to choose the girl. Thus the total is $8 \times 8$.

6. Four married couples get together for an evening. When they arrive they shake hands. Of course married people do not shake each other’s hands. How many handshakes will there be?

Each person shakes hands with 6 other people. This makes the count $8 \times 6$. However, we have then counted each handshake twice. So the correct answer is $8 \times 6 \div 2 = 24$. You can also obtain this answer by first counting how many handshakes there would be if the married couples did shake each others’ hands. This gives $(8 \times 7) \div 2$. Then subtract the handshakes of the four couples, giving $(8 \times 7 \div 2) - 4 = 24$.

7. Three girls in one family are going to get married to three boys in another family. In how many ways can this be done?
This is different from the preceding questions. At first students may think along the following lines: Each boy can choose one of three girls, so the answer is $3 \times 3$. However, this is not correct, for once the first boy has chosen his wife (or the first girl has chosen her husband), the second one has only 2 choices left; and after that, the third, only one. Therefore the correct answer is $3 \times 2 \times 1$. Here is a more detailed justification for this solution, in terms of a listing of all the possibilities: If you name the girls 1, 2, and 3, and the boys A, B, and C, then a good way to represent one of the possibilities is to list the three boys in some order, indicating that the first of these marries girl number 1, the second, girl number 2 and the third, girl number 3. Thus the problem is really one of counting the different ways the letters A, B, and C can be ordered. In terms of writing down the possibilities, the most natural arrangement is the following:

ABC  BAC  CAB
ACB  BCA  CBA

8. You have two candy bars, and you would like to give them to your friends. If you have four friends, in how many ways can you do this, assuming that you do not want to give them both to the same friend?

The answer is relatively straightforward: You have 4 choices for the first bar, and 3 for the next. However, since the bars are identical, it does not matter which of your friends you choose first and which you choose next, so the answer is $4 \times 3 \div 2$.

9. Suppose you decide that you will allow yourself to give both candy bars to the same friend. Now in how many ways can it be done?

The best way to do this question is to use the answer to the preceding question. You are now allowed four new possibilities in addition to the outcomes allowed in that question. Thus the answer is $4 \times 3 \div 2 + 4$.

10. Five people are competing in a chess tournament. Each player has to play every other player exactly once. How many games will be played?

The answer is that each player has to play four games. If we multiply this by the number of players we will have counted each game twice. Thus the answer is $5 \times 4 \div 2$. 

11. Twenty people are competing in a badminton competition. After every game, the loser is eliminated from the tournament, so he is not allowed to play any more. The tournament continues until only one competitor is left. How many games have to be played?

This is an excellent example of a problem where point of view is everything. Students may be tempted to set up possible schedules and then to see how many games will be involved. The problem with that approach is that it is then not clear whether the number of games is dependent on the particular schedule. On the other hand, if you notice that each game produces a loser, and that the rules are such that each person, other than the ultimate champion, will lose exactly once, it becomes clear that there will have to be 19 games no matter how the schedule is planned.
6.3 Problem Set - Coins and Dice

The chapter on counting is intended, at least in part, as a preparation for the next chapter, in which we will discuss probability. A lot of probability questions center on cards, coins and dice, and all of them boil down to counting a collection of outcomes.

Lesson Goals

- To give students more good counting problems involving coins and dice
- To learn how to infer, in the context of a word problem, what collection must be counted

Materials

- You should make copies for the students of the problems found in Section A.14 in the Appendix.

Lesson Sequence

The lesson is based on the following counting questions involving coins and dice. They are also found in the Appendix as Section A.14. You should decide which of them you want to use in your lesson.

1. Suppose we throw two dice, one coloured red and the other coloured black. How many different outcomes are there?

Once again, a lot hinges on the determination, best done in discussion with the class, which possible outcomes are considered equivalent and which are not. For example, one of the possible outcomes is that the red die comes up 5 and the black one 4, while another possible outcome is that the red die comes up 4 and the black one 5. These two outcomes will be counted as being different. Thus the answer is $6 \times 6$.

2. Suppose we throw two dice which look identical. How many different outcomes are there?

In this case, we cannot distinguish between a 4 and a 5 on the one hand and a 5 and a 4 on the other. Thus it would seem that we should take the solution to the previous question and simply divide it by 2, because the 4,5 combination will have been counted twice. However, it is more complicated than that, for while it is true that every combination of two different numbers will
have been counted twice, a pair of identical outcomes will only have been counted once. Thus we should count unequal combinations and equal combination separately: \(6 \times 5 \div 2 + 6\).

3. Suppose we throw a single die three times and write down the numbers we get.

(a) How many different outcomes are there if we pay attention to the order in which the numbers come up?

The answer is \(6 \times 6 \times 6\).

(b) How many of these outcomes have the same number repeated three times?

The answer is 6, by writing down these outcomes: 111, 222, \(
\cdots\) 666.

(c) How may of the outcomes have a total of 12?

This takes a bit of work. First think about the different ways you can get a total of 12 (without worrying about the order of the numbers), and then decide how many times these come up. Here are the relevant summations: \(1 + 5 + 6 = 12\), \(2 + 4 + 6 = 12\), \(2 + 5 + 5 = 12\), \(3 + 3 + 6 = 12\), \(3 + 4 + 5 = 12\), \(4 + 4 + 4 = 12\). You will notice that in the count, the numbers of outcomes represented by these sums are 6, 6, 3, 3, 6, 1 respectively. Thus the answer is \(6 + 6 + 3 + 3 + 6 + 1\).

4. Suppose we flip a coin 3 times, and write down the total numbers of heads and tails at the end. How many different outcomes are there?

The outcomes are best counted by keeping track of the number of heads. There can be 0, 1, 2, or 3 heads. That is there are 4 possible outcomes.

5. What if we flip the coin 4 times?

By the same reasoning, the answer is 5.
6. Suppose we flip the coin 5 times and write down the H’s and T’s in the order in which they occur.

   (a) How many possible outcomes are there?

   This time outcomes are counted as distinct when the orders of H’s and T’s are not the same, even if their total are equal. Thus in this case there are two choices for the first throw, two for the second, etc. That is, the total is $2 \times 2 \times 2 \times 2 \times 2$.

   (b) How many of these outcomes will have three heads and two tails?

   To list the outcomes we only need to indicate where the two tails are located in the list of five letters. This can be analysed by imagining the decision where to locate the two T’s. The first T can be placed in one of five locations. Once its location has been selected, we have four choices left for the second T. However, once these two choices have been made, we notice that the same outcome results if the first and second choices are reversed. Thus the correct total number of outcomes is $5 \times 4 \div 2$.

   (c) How many of the outcomes will have four heads and one tail?

   In this instance the outcomes are distinguished by the location of the single T. Thus there are five possible outcomes.

   (d) How many of them will result in an even number of heads?

   An even number of heads means 0 heads, 2 heads, or 4 heads. There are two ways to come up with the answer this time. One way is the more pedestrian. It involves counting separately the outcomes with 0 heads (there is one), 2 heads (this means two heads and three tails, and so should have the same total as that found in the case of 3 heads and 2 tails; that is, 10), and 4 heads (which, being the same as one tail, comes to 5 outcomes). Thus the total count is $1 + 10 + 5 = 16$.

   The more clever solution goes as follows. There is a symmetry between heads and tails in this problem. That is, the number of outcomes with an even number of heads is the same as the number of outcomes with an even number of tails. However, the total number of throws is odd, so when there are an even number of H’s there are automatically an odd number of T’s, and
vice-versa. This means that precisely half of the total collection of outcomes involve an even number of heads. In other words, the total is $2^5 \div 2 = 16$. 
Chapter 7

Probability

From the bough
Floating down river,
Insect song.

Issa, 1763-1827
(translated from the Japanese by Lucien Stryk and Takashi Ikemoto)

The Chapter’s Goals

• To learn that a probability problem is always a problem about counting a set of equally like outcomes

• To discern when it is reasonable to judge outcomes equally likely

• To learn how to calculate probability

• To learn about the concept of risk, a concept combining probability and cost

Overview of Activities

• E.A. 7.1 - Equally Likely Outcomes
  An introduction to probability.

• Problem Set 7.2 - Practicing Probability
  A chance to get lots of practice.

• E.A. 7.3 - Should We Be Surprised?
  An activity in which probability is connected to surprise.
• Problem Set 7.4 - More Practice
  As the title suggests.

• E.A. 7.5 - What is the Risk?
  An introduction to the concept of risk.
7.1 Enrichment Activity - Equally Likely Outcomes

Lesson Goals

- To discuss the importance of probability in daily life
- To help students develop a sense that things that are “more probable” will happen more often in the long run, though not with certainty
- To define probability in terms of a counting of equally likely outcomes

Materials

- You should prepare “score cards” for the students. Templates for these cards can be found in the Appendix in Section
- If you think there will be time for it, you may want to select some problems from the next section to include at the end of this lesson.

Lesson Sequence

1. A discussion on probability could begin with a discussion of some of the ways in which probability calculations are important to us. Ask the students where probability is used. It is likely that they will suggest that it is important in the area of betting and gambling. They may find it harder to come up with more significant examples. To help them out, you could ask them what their parents will say if, upon reaching age 16, they ask to learn how to drive and to use the family car, or even to have a car of their own. One of the objections will surely be that the cost of insurance will rise with an under-age driver in the family. You could then ask the students the same question with respect to life insurance: Who should expect to pay more for $100,000 life insurance, a 16 year old or a 60 year old? Explore with the students why the insurance company feels it has to raise the premiums, and how it decides the amount by which they should go up. You could mention that the mathematicians who are specialists in probability, and who set the premiums for insurance companies are called actuaries. The actuary wants to know how much you use the car and how likely it is, on any given trip, that you will have an accident that will cost the insurance company several thousand dollars. in the case of life insurance the actuary will want to know whether you have any history of illness. The actuary’s purpose is to make sure that the premiums paid by all the various customers are sufficient to cover the cost of the accidents or deaths suffered by the few.

A broad area where probability is very important is the area of risk analysis. For example, when an oil company builds an oil rig for operation off the coast of Newfoundland, the cost of the platform becomes a significant part of the cost of extracting the oil. It is important for the company to have a good estimate of the probability of a collision with an iceberg and the likelihood of wind force at a level that could cause damage. As well, the company
will want to estimate the general rate of wear and tear. A risk analyst will use weather information as well as other scientific data to determine the probable lifetime of the oil platform, so that the oil company can set the oil prices at a level that will allow for its eventual replacement.

2. **Shooting Gallery** A good way to prepare for the definition of probability is to play “Shooting Gallery”. Tell the students that you are going to roll a pair of dice repeatedly, and that they are going to record the total each time. The idea is to guess at the beginning of the game how many times each total will appear. Distribute “score cards” to the students in which they can mark their scores by crossing out a square. These score cards are available in Section A.15 in the last chapter.

```
X  X  X  X
X  X  X  X  X  X
1  3  2  5  5  3  5  1  3  4  3  1
1  2  3  4  5  6  7  8  9  10 11 12
```

Figure 7.1: Shooting Gallery

The students are supposed to enter numbers into the second row adding up to 36. As the game progresses they will enter X’s in the squares above these. The winner is the first one to have at least as many X’s in each column as the number guessed for that column. The diagram shows what one of the score cards might look like after the dice have been cast 12 times.

The purpose of the game is to introduce students to the idea that some totals are more likely to come up than others. Another benefit of the game is that the X’s form a histogram that provides a visual representation of the probability distribution of the various totals.

Students will soon discover that they should not put any number in the first column at all. They will also begin to notice that the numbers near the centre of the scale appear much more frequently than the ones at the ends. You should discuss this with them, and ask them for explanations why that is so.

Eventually students will be analyzing situations such as these in terms of numbers of
equally likely outcomes. When you throw the dice, one pair of numbers is as likely to appear as any other, if we may assume that the dice are not biased. However, some totals are produced in more ways than some others. For example, 2 can appear as a total only when both dice turn up the number 1. The same goes for the total 12. However, the total 6 is produced by 1+5, 2+4, 3+3, 4+2, and 5+1. In other words, this total can be expected five times as often as either 2 or 12.

After playing the game a number of times, and increasingly throughout discussions in this chapter, students should become aware that while probabilistic calculations can help us guess the outcomes intelligently, they cannot predict the outcomes exactly. The outcomes remain highly uncertain in any game of chance. In particular, if the Shooting Gallery game is played repeatedly, a given outcome is very unlikely to be repeated.

3. Equally likely outcomes

You are now ready to become precise about probability calculations.

Ask the students to imagine flipping a coin, and ask them for the probability of getting heads (as opposed to tails). Usually someone will say that the probability is one half, or fifty percent. It may be appropriate to digress here and remind the students that one half and fifty percent mean the same thing. Then ask the students what they really mean when they say that the probability is one half. The understanding we are aiming for, and which is almost certainly what the students mean by their answer, is that there are two possible outcomes, “tails” and “heads”; that these two outcomes are equally likely to occur.

In putting it this way we are thinking about probability theoretically and a priori. That is, the reason we say that the two outcomes are equally likely is that we are confident that there is nothing structural that favours one outcome over the other. Our flipping is sufficiently variable that we cannot make the coin spin one way rather than another; we believe the coin to be physically symmetric, so there is nothing in the shape of the coin that would cause it to land one way as opposed to the other.

A consequence of this a priori understanding is that we expect that in a large number of experiments about half of the outcomes will be “heads” and half will be “tails”. Of course, while we expect this kind of outcome, we also realize that it need not come out this way exactly, for if after a certain number of coin tosses the number of “heads” and the number of “tails” are equal, there is a small possibility that the next two tosses will both give us “heads” and thus destroy the equilibrium. This understanding of the independence of a toss on the tosses that took place before is a necessary part of this understanding of probability. Give the class some time to discuss these ideas.

If we agree that in this problem we may refer to “heads” as the “favourable” outcome, we can describe the probability calculation by saying that the probability is equal to the number of favourable outcomes divided by the total number of outcomes. Try to get the students to come up with most of this themselves. Stimulate the process by asking leading question, and help them to organize their thinking by suggesting the terms “outcomes”, “equally likely outcomes”, and “favourable outcomes”.

4. To continue the study of this mathematical model, you could now ask the students to consider an experiment in which two coins are flipped. Ask them which is more likely, to get two different results or to get two results that are the same. The answer is found by
listing all the possible outcomes (HH, HT, TH, TT) and noting that, again, these outcomes are equally likely, that precisely half of them show two different results, and that half have two identical results. Thus, both probabilities are equal to 2/4, or 1/2. In other words, the answer to this question is that the two possibilities are equally likely.

5. Before considering the next problem, tell the students that “chance” is another, less formal, word for “probability”. Thus, to say that the probability is 1/3 has the same meaning as saying that the chance is 1/3. Furthermore, especially when the word chance is used, it is also possible to express the same thing by saying that the chance is “one in three”. There is a third way of expressing probability. In popular parlance, especially in the prediction of the outcomes of horse races and elections, it is customary to express probabilities by speaking of “odds”, as in “the odds that our candidate will win the election are 2:1.” It is probably best not to introduce this language unless the students suggest it themselves. The reason is that the meaning is slightly different. To say that the odds of an event are 2 to 1 is to say the the probability that the event will occur is twice as big as the probability that it will not. In other words, (since these two probabilities must add up to one) they are 2/3 that the event will occur and 1/3 that it will not. Thus to say that the odds for an event are 2 to 1 is to say that the probability of its occurrence is 2/3.

6. Now ask the students to consider an experiment involving three coins. This time ask them what is the chance that all three coins will come up the same? This time the total number of possible outcomes is $2^3 = 8$ and the ones in which all three coins come up the same is 2. Thus the probability in this question is $2/8 = 1/4$.

7. At this point you could initiate a simple discussion can be used to illustrate the importance of counting events that are equally likely when calculating probabilities. For example, you could ask the students what is wrong with the suggestion that there are really three possible outcomes when you flip a coin: heads, tails, and on its edge, and that therefore the chance of getting heads is really only one third.

Here is another possible suggestion that could be used to make the same point: When you throw a pop can into the air it can land in one of three ways: right side up, upside down, or on its side. Therefore the chance that it will land right side up is one third. What is wrong with this line of reasoning? In both of these examples we make the mistake of considering possibilities that are not equally likely, and treating them as if they were.

8. The following slightly more subtle example can also be used to impress on the students the importance of being very careful to make sure that the events they count are equally likely. Tell the students you want to give them one more simple example of a probability calculation. Describe an experiment in which you shuffle a deck and then turn up one card. Suggest to them that one possible outcome is that it will be a face card, while the other possible outcome is that it will be a number card, and therefore that the probability of getting a face card is one half.

There is a good chance that the students will know that there is something wrong with your solution. The resulting discussion will reveal that while in a probability question there are often different ways of designating “outcomes”, the calculation is not valid unless it is based on outcomes that are equally likely. Although the question was framed to suggest that there are just two outcomes, a face card or a number card, students will probably
decide that we should think of the problem in terms of 52 possible outcomes, since there are 52 cards. It is important to let the students realize that it is up to us to decide what constitutes an outcome before we try to count them, and that the key to the decision is the condition that the outcomes we count should be equally likely.

Since there are far more number cards than there are face cards, there is no reason to believe that the two outcomes “turning up a face card” and “turning up a number card” are equally likely. The correct analysis notes that each of the 52 cards is equally likely to turn up, and that these constitute the 52 possible outcomes on which we should base our analysis. Of these, 12 produce the desired outcome, so the probability of getting a face card is $12 \div 52 = 3/13$. 
7.2 Problem Set - Practicing Probability

Lesson Goals

The exercises in this problem set can be used in two ways: (1) All at once to fill an entire enrichment session, or (2) in parts to supplement several of the enrichment sessions in this chapter. The goals of the practice problems are:

- To give students practice at discerning sets of equally likely outcomes
- To give students practice calculating probabilities

Materials

- You should make copies for the students of the problems found in Section A.16 in the Appendix.

Lesson Sequence

You can simply give students copies of the problems to work at their desks in groups or alone, or you could pick one or several for whole-class discussion. Here are the problems:

1. There are four coloured balls in a bag, a red one, a white one, a green one and a black one. If you reach into the bag without looking, what is the chance you will choose the red ball?

Of course, there are four equally likely outcomes, and we are interested in just one of them, so the answer is 1/4.

2. Suppose we have the same bag with four coloured balls, and without looking we take out two. What is the probability that we will choose the black and the white ball?

This time, counting the possible outcomes means counting the different pairs of balls that could have been picked. Upon reflection, the answer is 4 × 3 ÷ 2 = 6. Since we are interested in just one of these outcomes, the probability is 1/6.
3. Now suppose we have the same bag again, with the same four balls, but we do the experiment a little differently: We reach in and take one ball, without looking, and then put that ball back into the bag, shake the bag around and then reach in again to take a ball. In this case, what is the chance of picking the black ball and the white ball?

This is a very different question. While in the preceding question you automatically ended up with two different balls because you picked them at once, in this case you could pick the same ball twice. In other words, the list of possible outcomes is larger. In effect there are $4 \times 4$ possible and equally likely outcomes, if we pay attention to the order in which we pick the balls. Of these, two outcomes are the ones we are interested in ("black, then white", and "white, then black"). So the answer is $2 \div 16 = 1/8$.

4. Suppose you roll a die. What is the chance of getting more than 3?

The answer is of course $1/2$.

5. Suppose you flip two coins. What is the chance of getting two heads?

The possible outcomes are HH, HT, TH and TT. It is pretty clear that these are equally likely. Thus the probability of getting HH is $1/4$.

6. Again suppose you flip two coins. What is the chance of getting at least one head?

This time, of the four possible outcomes listed in the preceding question, three include at least one head, so the probability is $3/4$.

7. Suppose you flip three coins.
   
   (a) What is the chance of getting three tails?
   (b) What is the chance of getting all three coins to come up the same?
   (c) What is the chance of getting one head and two tails?
By the kind of counting procedure we learned in the preceding chapter, the number of possible, equally likely, outcomes is $2 \times 2 \times 2 = 8$. Just one of these outcomes includes three tails (namely TTT), two of the outcomes fit the description that all three coins come up the same (namely HHH and TTT), and one head and two tails can be achieved in three ways (namely HTT, HTH, and THH). Thus the probabilities are $1/8$, $2/8$ (=1/4) and $3/8$ respectively.

8. Suppose you roll two dice.
   
   (a) What is the chance of getting a double six?
   (b) What is the chance of getting a double number of any sort?
   (c) What is the chance of getting a total of seven?

When you roll two dice, to count the possible outcomes, you imagine them written down in column of pairs of numbers, the first number (between 1 and 6) indicating the number on the first die, and the second number indicating the result on the second die. These outcomes are equally likely, and they are naturally arranged in six rows of six entries, for a total of 36 possible outcomes. Of these, only one corresponds to a double six, six of them correspond to a double number, and six of them give a total of seven (namely 1&6, 2&5, 3&4, 4&3, 5&2, and 6&1). Thus the probabilities are, respectively, $1/36$, $6/36$ and $6/36$.

9. Suppose you have a brother and a sister. At Christmas time you draw names to buy each other a gift. Of course, the name drawing is set up so that if anyone draws his or her own name, the name drawing is done over.

   (a) You hope that your sister will get your name. What is the probability that she will?
   (b) What is the probability that the first time you draw names no-one gets his or her own name?

It is more difficult this time to think of a suitable list of equally likely outcomes. The best way to do it is to think of the three of you listed in order, say first yoursister, then your brother and finally yourself. Now imagine writing down the three of you in any order, and interpreting the resulting list to mean that the first person you wrote down has drawn your sister’s name, the next one your brother’s, and the third one has drawn yours. For example, if we use the letters with the obvious meanings, when we write down ISB, this means that you drew your sister’s name, your sister drew your brother’s, and your brother drew your name. In this manner there is a possible outcome for each of the possible arrangements of the three letters. However, some of these arrangements are not allowed, for as soon as someone draws his or her own name the draw has to be repeated. So how do we organize and count the outcomes that are allowed? In
this case the numbers are small enough to simply go at it. For example, you can begin by writing down all the possibilities, say in a table, including the ones that are not allowed:

\[
\begin{array}{ccc}
SBI & BIS & ISB \\
SIB & SBI & IBS \\
\end{array}
\]

Now delete those possible outcomes that would result in a new draw:

\[
\begin{array}{cc}
BIS & ISB \\
\end{array}
\]

Thus there are just two possible outcomes that are also allowed, and just one of these has your sister drawing your name. Thus the probability for part (a) is 1/2. For the second part of the question note that if we include the outcomes that are not acceptable, there are 6, of which just two are acceptable. Thus the probability that you will not have to do the draw over again is 2/6.

Now some of the students, or you the teacher, may have noticed that there seems to be a simpler way to do part (a). What is wrong, you may ask, with the following line of reasoning: “Since I am not allowed to end up with my own name, I will end up with either my sister’s or my brother’s. These two possibilities are equally likely, so the probability is one half.” In fact there is nothing wrong with this argument. You are making use of the symmetry of the question. In effect, you are saying there is no difference between brother and your sister where the draw is concerned, so the set of outcomes that involve your brother getting your name, and the set involving your sister getting your name are equally large.

In fact this shorter solution lends itself to the same question involving a larger group of people, for which the more detailed solution listing all the possible arrangements becomes prohibitively difficult.
7.3 Enrichment Activity - Should We be Surprised?

Lesson Goals

- To continue the discussion of probability
- To connect probability calculations to psychological reactions, such as surprise and fear

Materials

- The students will not need anything other than paper and pencil

Lesson Sequence

1. Ask the students to consider an experiment in which they roll 4 dice. Ask the students whether they would be surprised if the dice all came up showing the same number. Undoubtedly the students will agree that that would be a surprising event. Now ask them whether they would be surprised if three of the dice showed the same number. Continue the discussion, decreasing the number of identical outcomes by one each time. The purpose of the discussion is to raise doubt in the minds of the students about the justification for their feeling more or less surprised. You may have to improvise a little. For example, you could ask them at some point whether one should be surprised at finding the four dice all giving different outcomes.

2. At this point it is a good idea to discuss with the students whether there is some way in which we can use probability to determine whether we should be surprised. Hopefully the students will realize, after some thought, that if an event has a probability less than 1/2 and yet occurs, then we should be somewhat surprised at it, with the degree of surprise increasing as the probability goes down. So if there is an event whose probability has been calculated at one in a million (i.e. one millionth), we will be very surprised indeed!

3. Once the students realize that our dice question is difficult, or at least worth asking, ask them how many possible outcomes there are in total when you roll four dice. This is, of course, the kind of counting problem we have studied before. After some discussion they should come to the conclusion that the first die can turn up six different numbers, that for each of these outcomes, the second die also has six possible outcomes, and then also for the third, etc. In other words, the total number of outcomes for the whole experiment is $6 \times 6 \times 6 \times 6$, or $6^4$. Make sure the students understand that each of these outcomes is equally likely to occur. That the outcomes are indeed equally likely becomes evident when you consider that when the first die is rolled, each outcome is equally likely to occur. Then when the second die is rolled, this is again the case. In particular, the second die is not influenced by the outcome of the first. Continuing this way, it is clear that each of the $6^4$ outcomes is equally likely.
The next step is to determine how many of the equally likely outcomes are “favorable”; that is, how many of them correspond to the event described in the question. When we are asking the dice to come up equal, the favourable outcomes are 1111, 2222, 3333, 4444, 5555, and 6666. Thus there are 6 favourable outcomes out of a total of $6^4 = 7776$. In other words, the probability is

$$\frac{6}{7776} = \frac{6}{6^4} = \frac{1}{6^3} = \frac{1}{216}.$$ 

4. Now ask students to consider the case in which we are looking for three identical outcomes. Now the problem has become more difficult. The total number of outcomes remains the same, but the number of favourable outcomes is much larger and is more difficult to count. We also have to make an interpretive decision: Does the question ask whether we should be looking for at least three identical outcomes or should we be looking for exactly three identical outcomes? Before settling this interpretation, it would be a good idea to ask the students which of these has the greater probability. It may be that some students will reason that getting more dice to turn up the same is hard, so getting “at least three the same” is less probable. However, this is not correct. Either way, the denominator in the probability calculation is the same, representing the total number of possible outcomes. The question, therefore, revolves around the size of the numerator. By allowing more than three equal numbers we increase the collection of favourable outcomes, and so increase the numerator and the probability. Let us agree that we really meant to ask about the probability of getting exactly three equal numbers.

To count the number of favourable outcomes, we now proceed as follows: Suppose we begin by counting the number of outcomes that include exactly three 1’s. We can write them down: They are 111x, 11x1, 1x11, x111, where x represents one of 2, 3, 4, 5, and 6. This adds up to $4 \times 5 = 20$ different outcomes. Similarly three 2’s, three 3’s, etc. each correspond to 20 different outcomes. This makes for a total of $6 \times 20 = 120$ favorable outcomes. That is, the probability of getting exactly three equal numbers when rolling three dice is

$$\frac{120}{7776} = \frac{6 \times 2 \times 2 \times 5}{6^4} = \frac{5}{6 \times 3 \times 3} = \frac{5}{54}.$$ 

Even in this case, it is reasonable to be surprised.

5. Now what about getting exactly two dice to show equal numbers? Would that be reason for surprise? Again, precise interpretation of the question is important. In particular, we have to decide whether the question includes the possibility of getting, say, 2255 among its favourable outcomes. We would judge that the question is asked in such a way as to exclude that outcome. Then the count could go as follows: Suppose we get two 1’s. This can happen in any of the following ways: 11xy, 1x1y, 1xy1, x11y, x1y1, xy11, where x and y represent a pair of distinct numbers chosen from the list \{2, 3, 4, 5, 6\}. How many ways are there to choose this pair? By the counting methods discussed in the preceding chapter, we see that the answer is $5 \times 4$. Thus the total number of ways to get two 1’s and no other repeated numbers is $6 \times 5 \times 4$. The same goes for a pair of 2’s, etc. In other words, the number of favorable outcomes is $6 \times 6 \times 5 \times 4$. Therefore, the probability is

$$\frac{6 \times 6 \times 5 \times 4}{6^4} = \frac{20}{36} = \frac{5}{9}.$$
So this time we should not be surprised.
Lesson Goals

The exercises in this problem set can be used in two ways: (1) All at once to fill an entire enrichment session, or (2) in parts to supplement several of the enrichment sessions in this chapter. The goals of the practice problems are:

- To practice more probability calculations
- To associate the outcomes to feelings of surprise

Materials

- You should make copies for each student of the problem set listed at the end of this lesson, and available in the Appendix as Section A.17

Lesson Sequence

1. Suppose you ask two friends to pick a number between 1 and 10 (1 and 10 are allowed). Should you be surprised if they pick the same number? What is the probability that they will?

   There are 100 equally likely outcomes, of which 10 are favourable. This results in a probability of 1/10.

2. Suppose you ask the two friends to pick a number between 1 and 5. What then?

   This time the probability is 5/25 = 1/5.

3. Suppose your drawer contains three pairs of socks, a red pair, a white pair, and a blue pair. Unfortunately, it is dark in the room, and the socks are all in a jumble.

   (a) Suppose you pick out two socks. Should you be surprised if they match?
This problem is more difficult than it appears. In order to identify the equally likely possible outcomes we have to distinguish between socks of the same colour: We could call the two blue socks B1 and B2. Similarly for the white and red socks. Thus the total collection of socks is

B1 B2 W1 W2 R1 R2

When we make our choices we should pay attention to the order in which we pick the socks. Thus, it is just as likely that our choice would be [B1,W2] (meaning first B1 and then W2) as it is to choose [W2,R2] or any other pair. These possible pairs constitute equally likely outcomes. Students may be tempted to suggest that we do not need to distinguish between socks, or order in which they are picked, and instead consider BB, BW, BR, WW, WR, and RR as equally likely, leading to a suggestion that the probability is 3/6 = 1/2. But it is not at all clear that we can consider these outcomes as equally likely. In fact, if we agree that distinguishing socks and order of picking as leading to equally likely outcomes, then the outcome BB is much less likely than BW, for the former consists of [B1,B2] and [B2,B1] while the latter includes the outcomes [B1,W1], [B1,W2], [B2,W1], [B2,W2] plus four more in which these socks are picked in reverse order. That is, BW is four times as likely as BB. In any case, here is the correct calculation: There are 6 ways to pick our first sock, and then there are 5 to pick the second. This makes for a total of 6 × 5 = 30 equally likely outcomes. Six of these produce two socks of the same colour, so the probability is 6/30 = 1/5.

(b) Suppose you pick out three socks, What is the probability of getting a matching pair?

This time there are 6 × 5 × 4 = 120 ways to pick the three socks, counting the order in which they are picked. This time it seems easier to count the “unfavourable” outcomes: If we want to ensure an unfavourable outcome it does not matter what sock we pick first, so we have 6 ways to do that; but then we should not pick the other sock of that colour, leaving only 4 choices for the second sock; the third sock must then be picked from the two remaining socks that have a colour not yet picked. So, the number of unfavourable choices is 6 × 4 × 2 = 48. This means the number of favourable outcomes is 120 − 48 = 72 and the probability of getting a matching pair is 72/120 = 3/5.

(c) Suppose you pick out four socks. What about the probability now?

This is a trick question. If you pick three socks, when there are only three colours, two of them are necessarily going to be of the same colour.
4. You have two friends over at your house. You mention that you were born on a Sunday. What is the chance that at least one of your friends was also born on a Sunday?

There are $7 \times 7 = 49$ equally likely outcomes. Of these the ones that are favorable can be counted as follows: If your first friend is born on a Sunday there are 7 possible days for the second friend. Conversely, if the second is born on a Sunday there are 7 possible days for the first. This makes for a total of 14. However, now we have counted twice the possibility that both of them are born on Sunday, so we should subtract one, leaving a total of 13. Thus the probability is $13/49$.

5. Suppose you have three friends over. What is the chance that at least two of you are born on the same day of the week?

There are $7^4 = 2401$ equally likely outcomes. Of these the ones that have at least a pair of equal days is best counted by first counting how many outcomes have all four people born on a different weekday. This is counted as $7 \times 6 \times 5 \times 4 = 840$. But then the number of outcomes that have at least two people born on the same weekday is $2401 - 840 = 1561$. This means that the probability is $1561/2401 = 223/343$. 
7.5 Enrichment Activity - What is the Risk?

Lesson Goals

In the introduction to the chapter we touched on the role of probability in the analysis of risk. Simply put, when something goes wrong there is a cost involved. A probability calculation is needed to help us prepare for that cost. To make the connection between probability and risk we need to begin by connecting our hitherto rather abstract account of probability to one that allows for application. The goal of this lesson is:

- To introduce the concept of risk and to practice risk calculations

Materials

- you should bring enough coins for all the students
- You should make copies of the problem set associated with this activity. It can be found in the Appendix in Section A.18

Lesson Sequence

1. Begin with a discussion of the way probability comes up in practice. Until now, we treated probability as an attribute of a collection of possible outcomes to an experiment. When we flipped a coin we said that the probabilities of heads and tails were each one half, mainly because we could think of no reason why one or the other should be more likely to occur. You might say that our decision was based on a kind of symmetry between heads and tails which persists until there is information that in fact there is no symmetry. The whole idea of equally likely outcomes is based on that kind of symmetry. In effect you say to yourself that this outcome is as likely as that one, and therefore, as far as a probability calculation is concerned, there is nothing to choose between them - you can safely exchange one (equally likely) outcome for another during your analysis.

To explore this connection between an abstract consideration of probability and actual outcomes of an experiment, you could ask the students what it means practically when we say that in a coin flipping experiment the probability of getting heads is one half. This time we want a different answer than the one used in the preceding sections. If none is forthcoming, rephrase the question as follows: “We say that the probability of getting heads is one half. What does that tell us about an experiment in which we flip a coin 2 times (or 10 or 100 or 1000)?” In other words, how many heads should we expect in this experiment? Possibly some student will answer that we should get heads onetime out of the two. But do we? In fact, the students will be quick to agree that we do not necessarily get one head out of the two flips.

You could digress and ask the students to calculate the chance that (when flipping a coin twice) you will get exactly one head. We have done this calculation before, and found that
the chance is 1/2. In other words, there is no certainty at all that we will get one head, for certainty would be indicated by a probability of 1 rather than 1/2.

Students may suggest that as we increase the number of coin tosses it becomes increasingly likely that the number of heads will be exactly one half. This is a good next attempt, but it is still not correct. In fact, when we toss a coin four times, instead of two, the chance of getting exactly two heads is easily calculated: There are $2^4 = 16$ equally likely outcomes, and of these the ones that are favorable are HHTT, HTHT, HTTH, THHT, THTH, and TTHH. Thus the probability of getting exactly two heads in four coin tosses is $6/16 = 3/8$. And, indeed, it is the case that the more coins you toss, the less likely it is that precisely half of them will come up heads.

At some point, to supplement the discussion, you should ask the students to conduct an experiment. Give each of them a coin, and ask them to flip it four times, carefully recording the outcomes. Record the results on the board. You will probably see that in most cases they do not get exactly two heads.

After the experiment, ask the students once again if there is any way to make a connection between our assertion that the probability is one half and the outcome of coin tossing experiments. One suggestion that may be offered is that as you make a longer and longer series of coin tosses, the number of heads is going to get closer and closer to one half of the total number of coin tosses. This, too, is incorrect. You can suggest why it must be incorrect, for if in four coin tosses there is likely to be a certain distance between our expected two heads and the actual outcome, then in eight coin tosses you might expect that distance to be roughly twice as great. However, a discerning student will see a flaw in this latest suggestion, for there is a chance that during the first four coin tosses there were more than two heads, while during the next four there were fewer than two. In other words, the possibility of cancelation will cause the difference between the actual number of heads and the expected four heads in eight coin tosses to be less than twice the corresponding difference for four tosses. Put differently, if you calculate at the end of an experiment the number of heads and divide that by the number of coin tosses, this quotient should get closer and closer to one half. Our experience has been that, if given the time, the students themselves will eventually come up with this suggestion for the connection between our abstract calculations of probability and the outcomes of actual experiments.

To reinforce this understanding, you could ask the students to repeat their coin tossing experiment with longer and longer runs, and to see if indeed this is their finding.

2. Another point that could be discussed with the students is that the thought process could be turned around: If you were to find that in longer and longer experiments the proportion of heads did not get closer and closer to one half, then you would begin to suspect that the coin is biased.

3. It is now time to introduce the idea of risk. To do this you might ask the class to imagine the following situation. You have a small store that sells fireworks. Especially popular are the rockets, for which the supplier asks you 50 cents apiece. To pay for the rent of your store and to make a living for yourself, you feel you must earn 40 cents on each rocket sold. However, you have noticed that on average, one in ten rockets is defective. Since you value your reputation you agree to replace at no cost any rocket that is returned because it was defective. What should you be charging for each rocket you sell?
Point out to the students that to say that “on average, one in ten rockets is defective” is just another way of saying that the probability that a particular rocket will turn out to be defective is 1/10.

So how should the probability of a rocket’s failure be factored into its price? Try to get the students to reason their way to a solution. For example, you could ask what they would expect to happen if you sold 100 rockets (or some other large number). The students will agree that you would expect to have to replace 10 of them because they turned out to be defective. That is, you will be out of pocket by the amount of $10 \times 0.50 = 5.00. In other words, you should add just enough to the cost of each rocket to cover this expense. That is, the $5.00 should be spread over the 100 rockets you sold, increasing the cost of each rocket by 5 cents, from 90 cents to 95.

If you think the students are ready for it you could use a variable name for the total number of rockets sold. That is, you sell \( n \) rockets, of which \( n/10 \) become defective. Notice that effectively you have multiplied the number of rockets sold by the probability of failure to get the (expected) total number of rockets that fail. The cost to replace these failed rockets is \((n/10) \times 0.50\). To spread this cost over the rockets sold you should divide by \( n \) to obtain the final answer \((1/10) \times 0.50\). In effect, we have developed a formula here. By selling a rocket that may turn out to be defective and may therefore have to be replaced, we are assuming a financial risk equal to the probability of defect multiplied by the cost of replacement.

This latest observation can be made in a slightly different way. Suppose every time you sell a rocket you add a small amount that is meant to cover the cost of replacement if the rocket malfunctions. If the probability of failure is 1/10, then you expect to receive this amount 10 times before you have to spend the accumulated amount on a free rocket. That is, 10 times the extra amount should be just enough to cover that cost.

4. Here is another problem that involves the analysis of a risk. Suppose your friend suggests a game, which goes as follows. If you pay your friend 30 cents, he will roll a die. If the die comes up 6, he pays you $1.00, but if it does not then you have lost your 30 cents. Should you agree to play this game with him? Are you going to be making money or are you going to be losing money in this game? At what rate, per game, will you lose money or gain it?

Once again, you should try to get the students to find the solution, by asking them leading questions when necessary. If a student comes up with the answer rather quickly, you should challenge that student to explain to everyone how the solution was obtained and the reasoning behind it. You can get things started, if they don’t on their own, by asking the students how many times the die has to be rolled, on average, before a 6 is likely to turn up. Of course, the answer you are looking for is that if you roll the die six times you expect, on average, the die to come up once with a six. What you really mean of course is that if you roll the die many times then you can expect the number of sixes to be, roughly, one sixth of the number of rolls. So ask the students how much money you will be making or losing (on average) in six rolls. The answer is that once you will receive $1.00 from your friend, but to win that prize you had to play the game six times, for a total cost of \(6 \times 0.30 = 1.80\). Thus you lose $1.80 − $1.00 per six rolls. That averages to
\[ \frac{80}{6} = \frac{40}{3} = 10.33 \text{ cents per roll}. \]
Clearly you should not play this game, for you would be losing money at a rate of approximately 10.33 cents per roll.

5. As a last example for general discussion with the whole class, you could have a discussion about lotteries. The law requires that every lottery indicates the probability of winning. Imagine a lottery for $100,000. To participate you have to pay $5.00 and pick a five digit number. If the five digit number is drawn, then each person who picked it wins $100,000. Is it to your advantage to participate or not?

When you choose the five digit number you have to choose a number between 10,000 and 99,999. In other words, there are 90,000 possible choices. Just one of these is the number that will turn up at the draw. Each time you play the game your chance of winning the $100,000 is \( \frac{1}{90,000} \). You might say that, on average, each time you play the game you earn \( \left( \frac{1}{90,000} \right) \times $100,000 = \frac{10}{9} = $1.11 \). Of course you do not get any money until you actually win, but when you multiply the prize by the probability of winning it, you get the amount that you can expect \textit{on average}. The conclusion is that by playing the game you pay $5.00 for the expectation to win $1.11. That is, your loss is (on average) $3.89 each time you participate.

Here is a slightly different way to put it: In order to get the winning number once you expect to have to play the game 90,000 times, at a cost of $5 \times 90,000 = $450,000. In fact, of course, in this instance you can make sure that you will win exactly once, by entering the same draw 90,000 times. In any case, you would be spending $450,000 to play 90,000 games in the expectation of a $100,000 prize. This is a loss of $350,000 over 90,000 games. That is, a loss of $350,000 \div 90,000 = $3.89

You may wish to point out to the students that the average loss suggested by this example is low compared to most lotteries. It is not unreasonable to describe government lotteries as a tax on those who do not understand probability.

You can finish the session (if there is time left) to give students a chance to solve some problems on their own or in small groups. The following set of questions is given in the Appendix as Section A.18:

1. Suppose your friend suggests this game: You will both flip a coin. When both of you get heads, he will give you two jelly beans, but if you do not get two heads, you agree to give him one jelly bean. Who is probably going to end up winning more jelly beans, and roughly at what rate?

The chance of both of you getting heads is 1/4. Thus if the game is played four times, only once will you get two jelly beans from your friend. The other three times you have to give him one. Thus in four games, on average, you will suffer a net loss of one jelly bean. You should not play the game unless you are trying to get rid of your jelly beans.
2. You have five coloured balls in a bag: Three red and two white. Your friend suggests the following game. Without looking in the bag you will take a ball out of the bag, look at it and then put it back in. Every time you pick a white ball your friend will give you two of his baseball cards, and every time you pick a red ball, you must give him one of yours.

(a) Is it a good idea to play this game with him?
(b) If you play the game many times over, who is probably going to win more baseball cards, you or your friend?
(c) If you play the game 50 times, how many baseball cards can you expect to win or lose?

In five draws you can expect to get a red ball three times and a white ball twice. Thus you will get four baseball cards from your friend, and you have to give him three. This game is stacked in your favour. You will gain baseball cards in the long run, and in 50 games you can expect to win 10 of them.

3. At a bazaar to raise money for your school, you are asked to organize a game in which the player must roll a pair of dice. If the dice come up as a double six, the player wins a prize. If each prize costs $1.00 and if you are going to charge 10 cents per game, and if you get 72 people to play the game in an evening, how much money would you expect to make for the school?

The chance of getting a double six is $\frac{1}{36}$, so you should expect to have to pay out $\left(\frac{1}{36}\right) \times 72 \times 1.00 = 2.00$ in prizes. Since the cost of each game is 10 cents, you will collect $72 \times 0.10 = 7.20$. Clearly, you can expect to make a profit of $7.20 - 2.00 = 5.20$.

4. There is a booth at the fair where you can play the following game: You roll a pair of dice. If you throw a double number you get a Teddy bear. The Teddy bear is worth $5.40, and it costs one dollar to play the game. Should you play?

The probability of getting a double number is $\frac{1}{6}$. Thus you expect to win (on average, or in the long run) $\left(\frac{1}{6}\right) \times 5.40 = 0.90$ per game. So you would be paying more for the game than you can expect to win on average, but you might play to game for the fun of it.

5. A test has 60 multiple choice questions on it. Each question has a choice of four alternatives. Peppermint Patti, who hasn’t studied, decides to take a wild guess at each of the questions. If there are 3 marks for each correct answer, what total mark can she expect to get (on average)?
Patti has a 25% chance of getting a question right. Thus she will (on average) expect to get the correct answer to 15 of the 60 questions, giving her a final mark of 45 (out of 180).

6. Suppose that on the test mentioned in question 6 it is decided that marks should be subtracted for every wrong answer. How many marks should be subtracted for each wrong answer to make sure that when Peppermint Patti guesses at all the answers she will (on the average) get a zero mark?

This time we want to make sure that by random guessing she should expect to get zero on each question (on average). Now there is a $\frac{1}{4}$ chance of getting three marks, for an expectation of $(\frac{1}{4}) \times 3 = \frac{3}{4}$ marks. This should be offset by her $\frac{3}{4}$ chance of getting the wrong answer, multiplied by the penalty. So $\frac{3}{4}$ times the penalty should equal $\frac{3}{4}$. That is, there should be a penalty of one mark for a wrong answer. Alternatively, we could use the answer to question 5, and reason as follows: The 45 marks earned by guessing correctly have to be deleted by penalizing the 45 wrong answers. This is accomplished if we charge a penalty of one mark for each wrong answer.

7. You are the operator of a game booth at a fair. The player flips a coin until he gets a head, or until he has flipped the coin three times, whichever happens first. If he gets heads the first time he flips, he wins one cent. If he gets tails the first time, but heads on the second flip, he gets two cents. If he gets tails the first and second times, but heads on his third flip, he gets eight cents. If after three tries he still has nothing but tails, the game is over and he gets no prize. How much should you charge for the game to break even?

There is a $\frac{1}{2}$ chance of getting heads the first time. This corresponds to an expected pay-off of half a cent. There is a $\frac{1}{4}$ chance of getting tails first and then heads. This corresponds to an expected pay-off of $(\frac{1}{4}) \times 2 = \frac{1}{2}$ cent. There is a $\frac{1}{8}$ chance of getting tails, tails and then heads. This corresponds to a pay-off of $\frac{1}{8} \times 8 = 1$ cent. Thus, the total pay-off expected, per game, is 2 cents. This is what you should charge if you want to break even.
Appendix A

Problem Sets and Templates

The problem sets and templates used in the various chapters are collected here without interrupting advice for the instructor. Each begins on a new page to facilitate photocopying.
A.1 Templates for Section 1.4
A.2 Templates for Section 1.5

Count the chairs

\[ c(n) = \ldots \]
Count the white flagstones

\[
f(n) = \ldots
\]
Count the white flagstones

\[ g(n) = \ldots \]
A.3 Templates for Section 2.2
### Template for Section 3.1

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The table above represents the template for Section 3.1.
A.5 Problems for Section 3.2

1. Which of the following numbers are prime?

\[
\begin{array}{ccc}
273 & 37 & 251 \\
83 & 272 & 543 \\
\end{array}
\]

2. Write down, in order, all the prime numbers under 100.

3. Find the prime number factors of the following numbers:

\[
\begin{array}{ccc}
28 & 39 \\
63 & 34 \\
\end{array}
\]

4. Is 5477 a prime number? At worst (if it turns out to be a prime number) how many calculations do you have to perform to determine that it is a prime number?

5. Note that the prime number factors of the number 420 are given by the equation \(420 = 2 \times 2 \times 3 \times 5 \times 7\).

(a) If you multiply 420 by 3 you get 1260. Can you decide in your head simply by looking at the list of prime number factors of 420 what the prime number factorization of 1260 is?

(b) If you divide 420 by 2 you get 210 of course. Can you decide in your head by looking at the list of prime number factors of 420 what the prime number factors of 210 are?

(c) If you divide 420 by 3 you get 140. Decide in your head what the prime number factors of 140 are.

(d) If you multiply 420 by 11 you get 4620. Decide in your head what the prime number factors of 4620 are.

(e) If you multiply 420 by 10 you get 4200. Decide in your head what the prime number factors of 4200 are.

6. A number \(n\) has the following prime number factors:

\[
n = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 5 \times 7 \times 11 \times 11 \times 13 \times 13 \times 19.
\]

Without finding the value of the number \(n\), answer the following questions:
(a) I want to divide $n$ by 2, then divide the answer by 2, then divide the answer by 2 again, and so on. How many times can I do this before I stop getting a whole number?

(b) Can the number $n$ be divided by 12? Write down a reason for your answer.

7. I have a whole number $x$ in my mind. I multiply the number by 5, then I divide the result by 2, then I multiply the result by 7, then I divide the result by 5. When I have done all this I am left with a prime number. What is the value of $x$?

8. I have a number $a$ in my mind. If I square the number and then divide the result by 45 I am left with a prime number. What is the value of $a$?

9. I have a number $b$ in my mind. I multiply the number by 3, then I square the result, and then I divide that by 99. If the final answer is a prime number, what is the value of $b$?

10. I have a number $c$ in my mind. I multiply the number by 3, then I square the result, then I divide that by 21, and finally I divide by 12. If I am left with a prime number, what was the value of $c$?
A.6 Template for Section 3.4

On the next page you will find an enlarged copy of Figure 3.1 printed sideways and suitable for reproduction on an overhead transparency.
A.7  Problems for Section 3.7

1. Find the greatest common divisor of 230 and 180.
2. Find the greatest common factor of 128 and 72.
3. Find two numbers whose greatest common factor is 12.
4. Find two numbers whose greatest common factor is 1234.
5. Note that
6. Find the l.c.m. of 65 and 26 by first finding their prime factorizations.
7. Find the l.c.m. and the g.c.d. of 84 and 35.
8. Find the l.c.m. of each of the following pairs of numbers:
   (a) 22 and 33 (b) 24 and 36 (c) 25 and 35
9. Find the l.c.m. and g.c.d. of 72, 180 and 300.
10. Which of the following numbers is equal to \( n \times n \) for some whole number \( n \)?
    (a) \( 2 \times 2 \times 3 \times 3 \times 3 \times 3 \)
    (b) \( 2 \times 5 \times 5 \times 11 \times 11 \)
    (c) \( 13 \times 13 \times 17 \times 17 \)

The next questions are very challenging questions Do not expect to do these questions quickly, one after the other. Find one that you think you can do. Spend some time thinking about it carefully and \textit{patiently}. If you can think of a way of solving it, write down a very careful and detailed answer indicating how you solved the problem. If you cannot think of an answer, there is no shame in it. Good questions take a long time to solve. Go to another question and try it first. Take the remaining questions home. Ask your mom or dad or older brother or sister to solve them with you.

10. Is 
    \[
    2 \times 2 \times 2 \times 3 \times 7 \times 7 \times 11 \times 19 \times 23 \times 23 \times 29 \times 31 =
    \]
    \[ 116887742667 \] ?
    Explain your answer.
11. How many zeros are there at the end of the number
\[2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 5 \times 5 \times 5 \times 7 \times 7 \times 11 \times 13 \times 13 \times 17 \times 17\]?

Give a clear explanation of the reason for your answer.

12. If 123123123122000 is multiplied by 456456456455000, how many zeros will there be at the end of the answer, and why?

13. How many zeros are there at the end of the greatest common factor of
\[2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 7 \times 7 \times 11 \times 13 \times 13 \times 13 \times 19\]

and
\[2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 5 \times 5 \times 5 \times 11 \times 11 \times 17 \times 17 \times 17 \times 17 \times 19\]?

Explain how you got your answer.

14. When I multiplied
\[2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 7 \times 11 \times 11 \times 17 \times 19 \times 19 \times 19\]

I wrote down 36782351000 as the answer. My friend pointed out that I wrote the first digit down incorrectly. Can you decide what the first digit should be?

15. How many zeros are there at the end of the number
\[1 \times 2 \times 3 \times 4 \times \cdots \times 19 \times 20\]?

16. Suppose we have three cog wheels, one with 42 cogs, one with 34 cogs, and one with 39 cogs. The wheels are touching each other as shown.

Each wheel has an arrow painted on it that points up. How many times does the wheel with 34 cogs have to go around before all arrows point up again?
A.8 Problems for Section 4.3

1. Write the following numbers as decimals:

\[
\frac{1}{5}, \frac{31}{125}, \frac{1102}{250}.
\]

2. Express the following numbers in decimal form:

\[
\frac{33}{14}, \frac{1}{11}.
\]

3. Each of the following numbers is written in decimal form. Find the fraction to which the number is equal. Your answer must be a fraction in simplest form.

\[
0.25, 1.024, 0.525
\]

4. First write the following number in decimal form, and then express it as a fraction.

\[
2 + \frac{1}{10} + \frac{7}{100} + \frac{6}{1000}
\]

5. Without actually finding the decimal expressions for the following fractions, decide which of them will produce a terminating decimal and which will produce a repeating decimal:

\[
\frac{377}{160}, \frac{87}{620}, \frac{251}{1325}, \frac{3291}{625}
\]

6. Convert the following decimal expressions to fractions:

\[
3.\overline{781}, 0.1\overline{2}, 3.2\overline{38}, 7.21\overline{23}
\]
A.9 Template for Section 4.5

The template is found on the next page.
A.10 Problems for Section 5.4

1. Finish the following calculations in arithmetic (mod 3):
   \[1 + 2 \times 1 + 2 \equiv \? \pmod{3}\]
   \[2 \times (1 + 0 + 2) + 2 \equiv \? \pmod{3}\]
   \[245 \equiv \? \pmod{3}\]
   \[12 \times 13 + 36 \times 38 \equiv \? \pmod{3}\]

2. Make addition and multiplication tables for arithmetic (mod 5); that is, the arithmetic of remainders you get when you divide numbers by 5.

3. When you try to divide a bag of 237 apples equally among five friends, how many apples will be left over? How would you write this fact as an equation?

4. When you have a bag of 13 oranges which you want to distribute equally among 5 friends, how many oranges will be left over? Express this as an equation.

5. When you have 24 bags of 13 oranges each, and you try to divide them equally among five friends, describe how you would distribute the oranges if you want to leave as many bags undisturbed as possible, and avoid unnecessary calculations. How many oranges will be left over at the end? Express this sentence by means of an equation.

6. Suppose you have a lot of bags of oranges, each with 14 oranges in them. How many bags would you have to use, as a minimum, if you want to distribute the contents of the bags to six friends and have nothing left over?
A.11 Problems for Section 5.5

1. How many numbers between 1 and 100 are congruent to 1 (mod 3)? How many are there between 1 and 1000?
2. Find a number which is congruent to 1 (mod 3) and congruent to 2 (mod 4).
3. How many numbers are there between 1 and 100 satisfying the description of question 2?
4. If you add all the odd numbers from 1 to 99, what is the sum congruent to (mod 3)?
5. Suppose $m \equiv 2 \pmod{3}$. Can $m$ be a square number?
A.12 Templates and Problems for Section 6.1
1. How many different two letter words are there if the first letter has to be a consonant and the second one a vowel? Nonsense words are allowed, and it is agreed that the letter Y is to be used only as a vowel.

2. How is the total changed if you allow Y to be used both as a vowel and as a consonant?

3. How many different three letter words are there if the first letter must be a vowel? Once again, nonsense words are allowed, and Y may be used as a vowel only.

4. How many edges does a dodecahedron have, given that a dodecahedron is a regular surface made up of 12 pentagons? Also, how many vertices does it have?

5. How many different pairs of whole numbers add up to 40? (The whole numbers are defined to be the numbers 1, 2, 3, 4, ...).

6. If there are seven puppies in a pet store, and you have to choose one for yourself and one for your friend, in how many ways can you do this?

7. If there are seven puppies in a pet store, and you have to choose two for yourself, in how many ways can it be done?

8. If there are seven puppies in a pet store, and you have to decide which one is the cutest and also which one is the friendliest, in how many ways can you do it?
A.13 Problems for Section 6.2

1. A class has 20 students in it. Both MacDonalds and Burger King want to give a free burger to the student whose essay they think is best. In how many different ways can the two burgers be awarded?

2. A class has 16 students in it, 7 boys and 9 girls. The teacher wants to pick one boy and one girl to play the lead roles in a performance of Romeo and Juliet. In how many different ways can the teacher choose the pair?

3. Suppose instead there were 8 boys and 8 girls in the class. Now how many ways are there to choose Romeo and Juliet?

4. Once again, suppose we have a class with 8 boys and 8 girls. The teacher wants to pick two students to clean the blackboards. In how many ways can the teacher choose?

5. Suppose instead that the teacher wants to award one prize for the best poem written by a boy and one for the best poem written by a girl. This time, how many different ways are there to award these two prizes?

6. Four married couples get together for an evening. When they arrive they shake hands. Of course married people do not shake each other’s hands. How many handshakes will there be?

7. Three girls in one family are going to get married to three boys in another family. In how many ways can this be done?

8. You have two candy bars, and you would like to give them to your friends. If you have four friends, in how many ways can you do this, assuming that you do not want to give them both to the same friend?

9. Suppose you decide that you will allow yourself to give both candy bars to the same friend. Now in how many ways can it be done?

10. Five people are competing in a chess tournament. Each player has to play every other player exactly once. How many games will be played?

11. Twenty people are competing in a badminton competition. After every game, the loser is eliminated from the tournament, so he is not allowed to play any more. The tournament continues until only one competitor is left. How many games have to be played?
1. Suppose we throw two dice, one coloured red and the other coloured black. How many different outcomes are there?

2. Suppose we throw two dice which look identical. How many different outcomes are there?

3. Suppose we throw a single die three times and write down the numbers we get.
   (a) How many different outcomes are there if we pay attention to the order in which the numbers come up?
   (b) How many of these outcomes have the same number repeated three times?
   (c) How many of the outcomes have a total of 12?

4. Suppose we flip a coin 3 times, and write down the total numbers of heads and tails at the end. How many different outcomes are there?

5. What if we flip the coin 4 times?

6. Suppose we flip the coin 5 times and write down the H’s and T’s in the order in which they occur.
   (a) How many possible outcomes are there?
   (b) How many of these outcomes will have three heads and two tails?
   (c) How many of the outcomes will have four heads and one tail?
   (d) How many of them will result in an even number of heads?
A.15 Template for Section 7.1

\begin{center}
\begin{tabular}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
\end{tabular}
\end{center}

Shooting Gallery

\begin{center}
\begin{tabular}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
\end{tabular}
\end{center}

Shooting Gallery
A.16 Problems for Section 7.2

1. There are four coloured balls in a bag, a red one, a white one, a green one and a black one. If you reach into the bag without looking, what is the chance you will choose the red ball?

2. Suppose we have the same bag with four coloured balls, and without looking we take out two. What is the probability that we will choose the black and the white ball?

3. Now suppose we have the same bag again, with the same four balls, but we do the experiment a little differently: We reach in and take one ball, without looking, and then put that ball back into the bag, shake the bag around and then reach in again to take a ball. In this case, what is the chance of picking the black ball and the white ball?

4. Suppose you roll a die. What is the chance of getting more than 4?

5. Suppose you flip two coins. What is the chance of getting two heads?

6. Again suppose you flip two coins. What is the chance of getting at least one head?

7. Suppose you flip three coins.
   (a) What is the chance of getting three tails?
   (b) What is the chance of getting all three coins to come up the same?
   (c) What is the chance of getting one head and two tails?

8. Suppose you roll two dice.
   (a) What is the chance of getting a double six?
   (b) What is the chance of getting a double number of any sort?
   (c) What is the chance of getting a total of seven?

9. Suppose you have a brother and a sister, so that there three of you. At Christmas time you draw names to buy each other a gift. Of course, the name drawing is set up so that if anyone draws his or her own name, the name drawing is done over.
   (a) You hope that your sister will get your name. What is the probability that she will?
   (b) What is the probability that the first time you draw names no-one gets his or her own name?
A.17 Problems for Section 7.3

1. Suppose you ask two friends to pick a number between 1 and 10 (1 and 10 are allowed). Should you be surprised if they pick the same number? What is the probability that they will?

2. Suppose you ask the two friends to pick a number between 1 and 5. What then?

3. Suppose your drawer contains three pairs of socks, a red pair, a white pair, and a blue pair. Unfortunately, it is dark in the room, and the socks are all in a jumble.
   (a) Suppose you pick out two socks. Should you be surprised if they match?
   (b) Suppose you pick out three socks, What is the probability of getting a matching pair?
   (c) Suppose you pick out four socks. What about the probability now?

4. You have two friends over at your house. You mention that you were born on a Sunday. What is the chance that one of your friends was also born on a Sunday?

5. Suppose you have three friends over. What is the chance that two of you are born on the same day of the week?
1. Suppose your friend suggests this game: You will both flip a coin. When both of you get heads, he will give you two jelly beans, but if you do not get two heads, you agree to give him one jelly bean. Who is probably going to end up winning more jelly beans, and roughly at what rate?

2. You have five coloured balls in a bag: Three red and two white. Your friend suggests the following game. Without looking you will take a ball out of the bag, look at it and then put it back in. Every time you pick a white ball your friend will give you two of his baseball cards, and every time you pick a red ball, you must give him one of yours.

   (a) Is it a good idea to play this game with him?

   (b) If you play the game many times over, who is probably going to win more baseball cards, you or your friend?

   (c) If you play the game 50 times, how many baseball cards can you expect to win or lose?

3. At a bazaar to raise money for your school, you are asked to organize a game in which the player must roll a pair of dice. If the dice come up as a double six, the player wins a prize. If each prize costs $1.00 and if you are going to charge 10 cents per game, and if you get 72 people to play the game in an evening, how much money would you expect to make for the school?

4. There is a booth at the fair where you can play the following game. You roll a pair of dice. If you throw a double six you get a Teddy bear. The Teddy bear is worth $7.20, and it costs 50 cents to play the game. Should you play?

5. A test has 60 multiple choice questions on it. Each question has a choice of four alternatives. Peppermint Patti, who hasn’t studied, decides to take a wild guess at each of the questions. If there are 3 marks for each correct answer, what total mark can she expect to get (on average)?

6. Suppose that on the test mentioned in question 6 it is decided that marks should be subtracted for every wrong answer. How many marks should be subtracted for each wrong answer to make sure that when Peppermint Patti guesses at all the answers she will (on the average) get a zero mark?

7. You are the operator of a game booth at a fair. The player flips a coin until he gets a head, or until he has flipped the coin three times, whichever happens first. If he has heads the first time he flips, he wins one cent. If he gets tails the first time, but heads on the second flip, he gets two cents. If he gets tails the first and second times, but heads on his third flip, he gets four cents. If after three tries he still has nothing but tails, the game is over and he gets no prize. How much should you charge for the game to break even?