

# The Number of Curves of Genus 2 with a Given Refined Humbert Invariant

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## 1 Introduction

If  $C/K$  is a curve of genus 2 over an algebraically closed field  $K$ , then it comes equipped with a canonical quadratic form  $q_C$  called its *refined Humbert invariant*; cf. [K1], [K4]. This invariant is useful because many geometric properties of  $C$  are reflected in arithmetic properties of the quadratic form  $q_C$ ; cf. [K1], [K3], [K6], [K8], [Ki3] and [GRV].

For example, the property that the Jacobian  $J_C$  of  $C$  is isogeneous to the self-product  $E \times E$  for some elliptic curve  $E/K$  with complex multiplication can be characterized by a property of the form  $q_C$ ; see Theorem 4 below. For convenience, let us call a curve  $C/K$  with this property a *curve of CM product type*. In this case the invariant  $q_C$  can be viewed as an equivalence class of positive integral *ternary* quadratic forms.

For such a curve  $C/K$ , it turns out that there are only finitely many isomorphism classes of curves  $C'/K$  which have the “same” refined Humbert invariant. The purpose of this paper is to determine this number precisely.

To state the result, let  $d_C = d(q_C)$  denote the discriminant of  $q_C$  (in the sense of Brandt [B1] or of Watson [Wa]), and let  $\Delta_C = \Delta_{q_C} := d_C/16$ . Furthermore, let  $\kappa_C = \kappa_{q_C} \geq 1$  be defined as follows. If  $q_C$  is a primitive form, then  $\kappa_{q_C} = -I_1(q_C)/16$ , where  $I_1(q)$  is the genus invariant of a ternary form  $q$  (as defined by Brandt [B1]; see §2), and if  $q_C$  is imprimitive, then  $\kappa_{q_C} = -I_1(q_C/4)$ .

**Theorem 1** *If  $C/K$  is a curve of genus 2 of CM product type, then the number  $N_C$  of isomorphism classes of genus 2 curves  $C'/K$  whose refined Humbert invariant  $q_{C'}$  is equivalent to that of  $C$  is given by the formula*

$$(1) \quad N_C = 2^{\omega(\kappa_C)} h(\Delta_C) \frac{|\text{Aut}(C)|}{|\text{Aut}(q_C)|},$$

where  $\omega(\kappa_C)$  denotes the number of distinct prime divisors of  $\kappa_C$ , and  $h(\Delta_C)$  denotes the number of proper equivalence classes of positive primitive binary forms of discriminant  $\Delta_C$ .

By using standard finiteness results, the above theorem implies the following interesting fact.

**Corollary 2** *For any algebraically closed field  $K$  and any integer  $n \geq 1$ , there are only finitely many isomorphism classes of genus 2 curves  $C/K$  of CM product type such that  $N_C \leq n$ . Moreover, there exist infinitely many isomorphism classes of curves  $C/K$  of CM product type such that  $N_C > n$ .*

It is interesting to observe that the analogue of this corollary is not true if  $C/K$  is a *superspecial curve*, i.e., if  $J_C \simeq E \times E$ , where  $E/K$  is a supersingular elliptic curve. Such a curve can also be characterized by properties of its refined Humbert invariant  $q_C$  (see Theorem 5 below). However, since by [IKO] there are only finitely many isomorphism classes of superspecial curves over  $K$ , the number  $N_C$  is bounded, contrary to the CM product case.

The result of Theorem 1 is a special case of a more general formula which is valid for any principally polarized abelian surfaces  $(A, \theta)$  of CM product type. For this, note that the refined Humbert invariant  $q_{(A, \theta)}$  is defined for any principally polarized abelian surface  $(A, \theta)$  (cf. §2), and that we have by definition that  $q_C := q_{(J_C, \theta_C)}$ , if  $C/K$  is a curve of genus 2.

In order to generalize formula (1) to a principally polarized abelian surface  $(A, \theta)$ , we first observe that all the quantities on the right hand of (1) naturally generalize to more general ternary forms, with the exception of the factor  $|\text{Aut}(C)|$  (cf. §2 below). But in [K8] it was shown that  $|\text{Aut}(C)|$  is given by an expression involving the number  $r_n^*(q_C)$  of primitive representations of an integer  $n$  by  $q_C$ . More precisely, we have by Theorem 25 of [K8] that  $|\text{Aut}(C)| = 2a(q_C)$ , where for any form  $q$  we put

$$(2) \quad a(q) := \max(1, r_1^*(q)) \max(1, r_4^*(q), 3r_4^*(q) - 12).$$

This leads to the following generalization of Theorem 1.

**Theorem 3** *If  $(A, \theta)/K$  is a principally polarized abelian surface of CM product type with refined Humbert invariant  $q := q_{(A, \theta)}$ , then the set  $\mathcal{N}_{(A, \theta)}$  of isomorphism classes of principally polarized abelian surfaces  $(A', \theta')/K$  such that  $q_{(A', \theta')}$  is equivalent to  $q$  is finite. Moreover, if  $N_{(A, \theta)} := |\mathcal{N}_{(A, \theta)}|$  denotes the number of these isomorphism classes, then we have that*

$$(3) \quad N_{(A, \theta)} = 2^{\omega(\kappa_q)+1} h(\Delta_q) \frac{a(q)}{|\text{Aut}(q)|},$$

except when  $q$  is equivalent to  $x^2 + 4\kappa(y^2 + \varepsilon yz + z^2)$ , for some  $\kappa > 1$  and for some  $\varepsilon = 0$  or 1. In this exceptional case we have that

$$(4) \quad N_{(A, \theta)} = (2^{\omega(\kappa)-1} + 1 + \varepsilon) \frac{h(-(4 - \varepsilon)\kappa^2)}{2 + \varepsilon}.$$

Note that the exceptional case of Theorem 3 cannot occur when  $(A, \theta) = (J_C, \theta_C)$  is a Jacobian; cf. §3 below.

In §4 we present some explicit examples which show that in certain cases the curve  $C/K$  is uniquely determined by its refined Humbert invariant  $q_C$ ; cf. Proposition 15 and Examples 17 and 20.

It is perhaps useful to mention that when  $\text{char}(K) = 0$ , the set  $\mathcal{N}_{(A, \theta)}$  of isomorphism classes as in Theorem 3 can be identified with a *generalized Humbert scheme*  $H(q)$ , which is attached to an arbitrary quadratic form  $q$ . (These  $H(q)$ 's were introduced in §3 of [K3] and generalize the usual Humbert surfaces  $H_N$ .) Thus  $\mathcal{N}_{(A, \theta)} = H(q_{(A, \theta)})$ , as was shown in Proposition 7.2.3 of [Ki1]. Using this fact and Theorem 3, one can count (cf. [Ki1]) the number of *CM-points* of a certain type and, in particular, derive a formula for  $|\mathcal{H}(G, m)|$ , where  $\mathcal{H}(G, m)$  consists of the isomorphism classes of curves of CM product type with automorphism group  $G$  admitting an elliptic subcover of degree  $m$ . The set  $\mathcal{H}(G, m)$  has been studied in several cases (cf. [S] and [Ki3]).

## 2 The quadratic forms $q_A$ and $q_{(A, \theta)}$

Let  $A/K$  be an abelian surface, where  $K$  is an arbitrary algebraically closed field and let  $\text{NS}(A) = \text{Div}(A)/\equiv$  denote its Néron-Severi group. (Here  $\equiv$  denotes the numerical equivalence of divisors on  $A$ .) The intersection product  $(D_1.D_2)$  of divisors  $D_1, D_2$  on  $A$  defines an integral quadratic form  $q_A$  on  $\text{NS}(A)$  which is given by the formula  $q_A(D) = \frac{1}{2}(D.D)$ , for  $D \in \text{NS}(A)$ . Since  $\text{NS}(A) \simeq \mathbb{Z}^\rho$ , where  $\rho = \rho(A)$  is the Picard number of  $A$ , the form  $q_A$  is equivalent to an integral quadratic form  $q$  in  $\rho$  variables, i.e. we have an isomorphism  $(\text{NS}(A), q_A) \simeq (\mathbb{Z}^\rho, q)$  of quadratic modules. We then write  $q_A \sim q$ .

Now suppose that  $A$  has a principal polarization  $\theta \in \mathcal{P}(A)$ . (Here and below,  $\mathcal{P}(A) \subset \text{NS}(A)$  denotes the set of principal polarization on  $A$ .) Then the quadratic form  $\tilde{q}_{(A, \theta)}$  on  $\text{NS}(A)$  is defined by the formula

$$(5) \quad \tilde{q}_{(A, \theta)}(D) = (D.\theta)^2 - 2(D.D), \quad \text{for } D \in \text{NS}(A).$$

It is easy to see that  $\tilde{q}_{(A, \theta)}(D + n\theta) = q_{(A, \theta)}(D)$ , for all  $n \in \mathbb{Z}$ , so  $\tilde{q}_{(A, \theta)}$  induces a quadratic form  $q_{(A, \theta)}$  on the quotient module

$$\text{NS}(A, \theta) = \text{NS}(A)/\mathbb{Z}\theta.$$

Moreover, the Hodge Index Theorem shows that  $q_{(A, \theta)}$  is a positive-definite form on  $\text{NS}(A, \theta) \simeq \mathbb{Z}^{\rho-1}$ ; cf. [K1]. The quadratic form  $q_{(A, \theta)}$  or, more correctly, the quadratic module  $(\text{NS}(A, \theta), q_{(A, \theta)})$  is called the *refined Humbert invariant* of the principally polarized abelian surface  $(A, \theta)$ ; cf. [K3], [K4].

We observe that if  $(A, \theta) \simeq (A', \theta')$ , i.e., if we have an isomorphism  $\varphi : A \xrightarrow{\sim} A'$  of abelian surfaces such that  $\varphi^*(\theta') = \theta$ , then we have an induced module isomorphism  $\varphi^* : \text{NS}(A', \theta') \xrightarrow{\sim} \text{NS}(A, \theta)$  such that  $q_{(A', \theta')} = q_{(A, \theta)} \circ \varphi^*$ , and hence the associated quadratic modules are isomorphic. Equivalently, this means that  $q_{(A, \theta)} \sim q_{(A', \theta')}$ .

For later use we mention here that in the case of an abelian product surface  $A = E_1 \times E_2$  we have by [K3], Proposition 23, an isomorphism

$$(6) \quad \mathbf{D} : \mathbb{Z} \times \mathbb{Z} \times \text{Hom}(E_1, E_2) \xrightarrow{\sim} \text{NS}(E_1 \times E_2)$$

of abelian groups such that for all  $a_i, b_i \in \mathbb{Z}$  and  $f_i \in \text{Hom}(E_1, E_2)$ ,  $i = 1, 2$ ,

$$(7) \quad (\mathbf{D}(a_1, b_1, f_1) \cdot \mathbf{D}(a_2, b_2, f_2)) = a_1 b_2 + a_2 b_1 - \beta_d(f_1, f_2),$$

where  $\beta_d(f_1, f_2) = \deg(f_1 + f_2) - \deg(f_1) - \deg(f_2)$  is the bilinear form defined by the degree map.

Suppose now that  $A$  is isogeneous to a product abelian surface  $E \times E$ , where  $E/K$  is an elliptic curve with complex multiplication, i.e.  $\text{End}(E)$  is isomorphic to an order in an imaginary quadratic field. The set of such abelian surfaces may be classified as follows.

**Theorem 4** *If  $A/K$  is an abelian surface, then the following conditions are equivalent:*

- (i)  *$A$  is isogeneous to a product  $E \times E$ , for some CM elliptic curve  $E/K$ .*
- (ii)  *$A$  is isomorphic to a product  $E_1 \times E_2$ , where  $E_1/K$  and  $E_2/K$  are two isogeneous CM elliptic curves over  $K$ .*
- (iii)  *$\mathcal{P}(A) \neq \emptyset$ , and for one (and hence for any)  $\theta \in \mathcal{P}(A)$ , the refined Humbert invariant  $q_{(A, \theta)}$  is a ternary form such that  $q_{(A, \theta)}(D) = n^2$  is a positive square, for some  $D \in \text{NS}(A, \theta)$ .*

*Proof.* (i)  $\Rightarrow$  (ii): If  $K = \mathbb{C}$ , then this is due to Shioda and Mitani [SM]. For a general ground field  $K$ , this is a special case of Theorem 2 of [K2].

(ii)  $\Rightarrow$  (iii): Let  $\theta_{E_1, E_2}$  denote the product polarization on  $A' := E_1 \times E_2$ , i.e.,  $\theta_{E_1, E_2}$  is the image of the divisor  $E_1 \times \{0\} + \{0\} \times E_2 \in \text{Div}(A')$  in  $\text{NS}(A') = \text{Div}(A')/\equiv$ . Then  $\theta_{E_1, E_2} \in \mathcal{P}(A')$ , and so  $\mathcal{P}(A) \neq \emptyset$  because if  $\varphi : A \xrightarrow{\sim} A'$  is any isomorphism, then  $\varphi^*(\theta_{E_1, E_2}) \in \mathcal{P}(A)$ .

Now let  $\theta \in \mathcal{P}(A)$ . Then  $\text{rank}(\text{NS}(A, \theta)) = \rho(E_1 \times E_2) - 1 = \text{rank}(\text{Hom}(E_1, E_2)) + 1$  by (6). Since  $E_1 \sim E_2$  are isogeneous CM elliptic curves by (ii), it follows that  $\text{rank}(\text{Hom}(E_1, E_2)) = 2$ , so  $\text{rank}(\text{NS}(A, \theta)) = 3$ , and hence  $q_{(A, \theta)}$  is a ternary form.

Moreover, by Theorem 1.5 of [K1] we know that if  $E \leq A$  is an elliptic subgroup on  $A$ , then  $q_{(A, \theta)}([E]) = (E \cdot \theta)^2 > 0$ , where  $[E] \in \text{NS}(A, \theta)$  denotes the image of  $E$  in  $\text{NS}(A, \theta)$ . Since  $E = \varphi^{-1}(E_1 \times \{0\}) \leq A$ , we see that (iii) holds.

(iii)  $\Rightarrow$  (i): Let  $\theta \in \mathcal{P}(A)$ , and let  $D \in \text{NS}(A, \theta)$  be such that  $q_{(A, \theta)}(D) = n^2 > 0$ . Then  $D \neq 0$ , so we can write  $D = mD'$ , where  $m \geq 1$  and  $D'$  is primitive in  $\text{NS}(A, \theta)$ , and hence  $q_{(A, \theta)}(D') = \left(\frac{n}{m}\right)^2$ . By Theorem 1.5 of [K1] there exists an elliptic subgroup  $E \leq A$  such that  $[E] = D'$ . Thus, by Poincaré,  $A$  is isogeneous to  $E \times A'$ , for some abelian subvariety  $A'$  of  $A$  (cf. [Mu], p. 173). Since  $\dim(A) = 2$ , it follows that  $\dim(A') = 1$ , so  $A' = E'$  is an elliptic curve, and hence  $A \sim E \times E'$ .

By hypothesis,  $\text{rank}(\text{NS}(A, \theta)) = 3$ , so  $4 = \rho(A) = \rho(E \times E') = \text{rank}(\text{Hom}(E, E')) + 2$ , i.e.,  $\text{rank}(\text{Hom}(E, E')) = 2$ . This means that  $E$  and  $E'$  are isogeneous CM curves, so  $A$  satisfies (i).

In view of property (ii), we call any surface satisfying these properties a *CM abelian product surface*. (In the introduction these were called abelian surfaces of CM product type.) Note that property (iii) gives a characterization of such surfaces in terms of the refined Humbert invariant  $q_{(A, \theta)}$ .

While the exact analogue of Theorem 4 is not true for superspecial abelian surfaces, we do have the following result.

**Theorem 5** *If  $A/K$  is an abelian surface, and if  $p = \text{char}(K) > 0$ , then the following conditions are equivalent:*

- (i)  *$A$  is isomorphic to a product  $E \times E$ , for some supersingular elliptic curve  $E/K$ .*
- (ii)  *$A$  is isomorphic to a product  $E_1 \times E_2$ , where  $E_1/K$  and  $E_2/K$  are two supersingular elliptic curves over  $K$ .*
- (iii)  *$\mathcal{P}(A) \neq \emptyset$ , and for one (and hence for any)  $\theta \in \mathcal{P}(A)$ , the refined Humbert invariant  $q_{(A, \theta)}$  is a quadratic form of rank 5 such that  $q_{(A, \theta)}(D) = n^2$ , for some  $D \in \text{NS}(A, \theta)$  and some  $n \in \mathbb{Z}$  with  $p \nmid n$ .*

*Proof.* (ii)  $\Rightarrow$  (i): This follows from Deligne's Theorem (see [IKO], p. 145.)

(i)  $\Rightarrow$  (iii): This is partially similar to the proof of (ii)  $\Rightarrow$  (iii) of Theorem 4. Let  $\theta_{E, E}$  denote the product polarization on  $A' := E \times E$ . Then  $\theta_{E, E} \in \mathcal{P}(A')$ , and so  $\mathcal{P}(A) \neq \emptyset$  because we have an isomorphism  $\varphi : A \xrightarrow{\sim} A'$ .

Now let  $\theta \in \mathcal{P}(A)$  be any principal polarization. Then  $\text{rank}(\text{NS}(A, \theta)) = \rho(E \times E) - 1 = \text{rank}(\text{End}(E)) + 1 = 5$  by (6) and by the fact that  $E$  is supersingular.

To verify the last property of (iii), we will use Corollary 24 of [K3] which states that  $\theta = \mathbf{D}(a_1, a_2, f)$ , for some  $a_1, a_2 \in \mathbb{Z}$  and  $f \in \text{End}(E)$  such that  $a_1 > 0$  and  $a_1 a_2 - d = 1$ , where  $d = \deg(f)$ . We now consider two cases:

**Case 1:**  $p \mid \gcd(a_1, a_2)$ .

Then  $p \nmid d$ . Consider  $D = \mathbf{D}(1, d, f)$ . By (7) we have that  $(D, D) = 0$  and  $n := (D, \theta) = a_1 d + a_2 - 2d \equiv -2d \not\equiv 0 \pmod{p}$  because  $p \neq 2$  by [IKO], Theorem 3.3 (III). Thus by (5) we have that  $q_{(A, \theta)}(D) = n^2$  and  $p \nmid n$ , so (iii) holds in this case.

**Case 2:**  $p \nmid \gcd(a_1, a_2)$ .

In this case either  $p \nmid a_1$  or  $p \nmid a_2$ . Then put  $D_1 = \mathbf{D}(0, 1, 0)$ , if  $p \nmid a_1$  and put  $D_2 = \mathbf{D}(1, 0, 0)$ , if  $p \nmid a_2$ . Then by (7) we have that  $(D_i, D_i) = 0$ , and  $n_i := (D_i, \theta) = a_i \not\equiv 0 \pmod{p}$ . Thus  $q_{(A, \theta)}(D_i) = n_i^2$ , and so (iii) holds in this case as well.

(iii)  $\Rightarrow$  (ii): Let  $\theta \in \mathcal{P}(A)$ , and let  $D \in \text{NS}(A, \theta)$  be such that  $q_{(A, \theta)}(D) = n^2 \not\equiv 0 \pmod{p}$ . Then  $D \neq 0$ , so we can write  $D = mD'$ , where  $m \geq 1$  and  $D'$  is primitive in  $\text{NS}(A, \theta)$ , and hence  $q_{(A, \theta)}(D') = N^2$  with  $N := (\frac{n}{m})^2$ . By Theorem 18 of [K5] there exists an  $N$ -presentation  $(E_1, E_2, \psi, \pi)$  of  $A/K$ . This means that we have an isogeny  $\pi : E_1 \times E_2 \rightarrow A$  and an anti-isometry  $\psi : E_1[N] \rightarrow E_2[N]$  such that  $\text{Ker}(\pi) = \text{Graph}(-\psi)$ .

By hypothesis,  $\rho(A) = \text{rank}(\text{NS}(A, \theta)) + 1 = 6$ . Since  $\pi$  is an isogeny, we have that  $\rho(A) = \rho(E_1 \times E_2) = \text{rank}(\text{Hom}(E_1, E_2)) + 2$  by (6), and so  $\text{rank}(\text{Hom}(E_1, E_2)) = 4$ . This means that  $E_1$  and  $E_2$  are supersingular elliptic curves.

Since  $p \nmid N$ , we have that  $|\text{Hom}(E_1[N], E_2[N])| = N^4$ . Moreover, since  $\mathcal{H} := \text{Hom}(E_1, E_2) \simeq \mathbb{Z}^4$ , we also have that  $|\mathcal{H}/N\mathcal{H}| = N^4$ , and so Lemma 22 of [K5] shows that the map  $\rho_N : \text{Hom}(E_1, E_2) \rightarrow \text{Hom}(E_1[N], E_2[N])$  (given by  $\rho_N(h) = h|_{E_1[N]}$ ) is surjective. Thus  $\psi = h|_{E[N]}$ , for some  $h \in \text{Hom}(E_1[N], E_2[N])$ . This means that the isogeny defect  $m_\psi = 1$ , and hence  $A \simeq E_1 \times E_2$  by Theorem 4 of [K5]. (Note that the proof of Theorem 4 is valid for all principally polarized abelian surfaces  $(A, \theta)$  and not just Jacobians.) This shows that (ii) holds.

We now come back to the case of a CM product surface, and recall from [K3], [K4] and [Ki2] some basic arithmetic facts about the forms  $q_A$  and  $q_{(A, \theta)}$  for such a surface. To state these, we will use discriminant  $\text{disc}(f) = d(f)$  of an integral quadratic form  $f$  as defined in Watson [Wa], p. 2. Moreover, the *content*  $\text{cont}(f)$  of such a form is the gcd of its values. (It is easy to see that when  $f$  is a ternary form, then  $\text{cont}(f) = t$  is the “coefficient-divisor” of [B1].) If  $\text{cont}(f) = 1$ , then  $f$  is said to be *primitive*; cf. [Wa], p. 4. In addition, the basic invariants  $I_k(f)$  for  $k = 1, 2$  of an integral primitive ternary form  $f$  are defined as in Brandt [B1]. (These invariants are closely related to the invariants  $\Omega$  and  $\Delta$  as defined in Dickson [Di].)

More precisely, the genus invariant  $|I_1(q)|$  of a primitive form  $q$  is the content of the *adjoint form*  $\text{adj}(q)$ , which is defined by the formula (as in Watson [Wa], p. 25)

$$(8) \quad M(\text{adj}(q)) = -2\text{adj}(M(q)) = -2\det(M(q))M(q)^{-1}.$$

The sign is assigned so that the *reciprocal*  $F_f := \text{adj}(f)/I_1(f)$  of  $f$  is a positive definite form when  $f$  is positive definite. Furthermore,  $I_2(f) := I_1(F_f)$ .

**Proposition 6** *Let  $A \simeq E \times E'$  be a CM abelian product surface, and let  $q_{E, E'}$  denote the degree form on  $\text{Hom}(E, E')$ . Let  $\Delta = d(q_{E, E'})$  denote its discriminant and  $\kappa = \text{cont}(q_{E, E'})$  its content. If  $\theta \in \text{NS}(A)$  is a principal polarization, then*

$$(9) \quad d(q_A) = \Delta \quad \text{and} \quad d(q_{(A, \theta)}) = 16\Delta.$$

Moreover, if  $c := \text{cont}(q_{(A,\theta)})$ , then  $c = 1$  or  $4$ , and we have that

$$(10) \quad I_1(q_{(A,\theta)}/c) = -\frac{16\kappa}{c^2} \quad \text{and} \quad I_2(q_{(A,\theta)}/c) = \frac{c\Delta}{\kappa^2}.$$

*Proof.* We have that  $\text{rank}(\text{Hom}(E, E')) = 2$  because  $E$  and  $E'$  are isogeneous CM elliptic curves, and so  $q_{E,E'}$  is a binary quadratic form. It thus follows from Corollary 24 of [K3] that  $\det(q_A) = -\det(q_{E,E'})$ , where (as in [K3])  $\det(f)$  denotes the determinant of the associated Gram matrix  $M(f)$ . Thus, since  $\text{rank}(\text{NS}(A)) = 4$ , it follows from [Wa], p. 2, that  $d(q_A) = \det(q_A) = -\det(q_{E,E'}) = d(q_{E,E'})$ . This proves the first equality of (9).

The second equality of (9) follows from Proposition 9 of [K3]. Indeed, by that result we have that  $\det(q_{(A,\theta)}) = -2^5 \det(q_A)$ , so  $d(q_{(A,\theta)}) = -\frac{1}{2} \det(q_{(A,\theta)}) = 2^4 \det(q_A) = 16d(q_A) = 16\Delta$ . This proves (9).

If  $c = 1$ , i.e., if  $q_{(A,\theta)}$  is primitive, then formula (10) follows from Propositions 25 and 18 of [K7]. If  $c > 1$ , then by Corollary 3.5 of [Ki2] and the proof of Proposition 4.1 of [Ki2], we see that  $c = 4$ , and so in this case (10) follows directly from that proposition.

In view of the above result, it is useful to introduce the following notation (which was already used in Theorem 3).

**Notation.** If  $q$  is a positive ternary quadratic form such that  $c := \text{cont}(q) = 1$  or  $4$ , then put

$$\Delta_q := d(q)/16 \quad \text{and} \quad \kappa_q := -c^2 I_1(q/c)/16.$$

It is clear from the definition that the form  $q_{(A,\theta)}$  is determined by the form  $q_A$  and an element  $\theta$  with  $q_A(\theta) = 1$ ; this is the so-called  $\theta$ -construction in [K4]. We now show conversely that the ternary form  $q_{(A,\theta)}$  determines the quaternary form  $q_A$  (up to equivalence). More precisely, we prove the following result.

**Theorem 7** *Let  $A_1$  and  $A_2$  be two CM abelian product surfaces, and let  $\theta_i \in \text{NS}(A_i)$  be a principal polarization on  $A_i$ , for  $i = 1, 2$ . If  $q_{(A_1,\theta_1)}$  is equivalent to  $q_{(A_2,\theta_2)}$ , then  $q_{A_1}$  is equivalent to  $q_{A_2}$ .*

To prove this result, we will use several facts which were proven elsewhere. The first concerns the  $p$ -adic equivalence class of the form  $q_{(A,\theta)}$ , where the  $p$ -adic equivalence of two integral forms  $f_1$  and  $f_2$  is defined as in Jones [Jo], p. 82, and is denoted by  $f_1 \sim_p f_2$ . Recall also that  $f_1$  and  $f_2$  are *genus-equivalent* if  $f_1 \sim_p f_2$ , for all primes  $p$  (including  $p = \infty$ ). We let  $\text{gen}(f_1)$  denote the set of (equivalence classes of) integral forms  $f_2$  which are genus-equivalent to  $f_1$ .

**Proposition 8** *Let  $A = E \times E'$  be a CM abelian product surface, and let  $\theta$  be a principal polarization on  $A$ . If  $f_q := x^2 \perp 4q$ , where  $q = q_{E,E'}$  is the degree form, then*

$$(11) \quad q_{(A,\theta)} \sim_p f_q, \quad \text{for all odd primes } p.$$

*Furthermore, if  $q_{(A,\theta)}$  is primitive, then (11) also holds for  $p = 2$ , and so in this case  $q_{(A,\theta)}$  is genus-equivalent to  $f_q$ .*

*Proof.* The first assertion (11) follows immediately from Corollary 19 of [K4] because by equation (29) of [K7] we know that  $f_q \sim q_{(A,\theta_{E,E'})}$ , where  $\theta_{E,E'}$  is the product polarization on  $A$ . The second assertion follows from Theorem 20 of [K4], as is explained in the proof of Proposition 25 of [K7].

In order to state the next result, we recall from [K7] and [Ki2] that if  $f : M \rightarrow \mathbb{Z}$  is a quadratic form on a module  $M$ , then  $R(f) := \{f(x) : x \in M\}$  denotes the set of values represented by  $f$ , and if  $a, m$  are integers, then

$$R_{a,m}(f) := \{n \in R(f) : n \equiv a \pmod{m}\}.$$

**Proposition 9** *Let  $A = E \times E'$  be a CM abelian product surface, and let  $\theta$  be a principal polarization on  $A$ . If  $q_{(A,\theta)}$  is imprimitive, then  $d(q_{E,E'}) \equiv 0 \pmod{4}$  and  $R_{3,4}(q_{E,E'}) \neq \emptyset$ . In particular,  $\kappa := \text{cont}(q_{E,E'})$  is odd. Furthermore, there exists  $n \in R_{3,4}(q_{E,E'})$  such that  $\frac{n}{\kappa} \in R(F_{q_{(A,\theta)}/4})$ .*

*Proof.* The first assertions follow immediately from Proposition 3.6 of [Ki2]. Moreover, the last assertion follows from Propositions 3.7 and 4.1 of [Ki2]. To see this, put  $n = q_{E,E'}(h)$ , where  $h \in \text{Hom}(E, E')$  is as in Proposition 4.1 of [Ki2]. Then by Propositions 3.7 and 4.1 of [Ki2] we have that  $n \in R_{3,4}(q_{E,E'})$  and that  $\frac{n}{\kappa} \in R(F_f)$ , where  $F_f$  denotes the reciprocal of the primitive form  $f := q_{(A,\theta)}/4$ .

We are now ready to prove Theorem 7.

*Proof of Theorem 7.* For  $i = 1, 2$ , let  $E_i/K$  and  $E'_i/K$  be two (isogeneous) CM elliptic curves such that  $A_i \simeq E_i \times E'_i$ . We first observe that it suffices to show that the given hypothesis implies that the binary forms  $q_1 := q_{E_1,E'_1}$  and  $q_2 := q_{E_2,E'_2}$  are genus-equivalent, i.e., that

$$(12) \quad \text{gen}(q_1) = \text{gen}(q_2).$$

Indeed, since we have that

$$(13) \quad q_{A_i} \sim xy \perp (-q_i), \quad \text{for } i = 1, 2,$$

by [K3], Proposition 23 (or by [K4], formula (6)), it follows from Remark 27 of [K4] that (12) implies that  $q_{A_1} \sim q_{A_2}$ , as desired.

To prove that (12) holds, we will distinguish two cases.

**Case 1.**  $q := q_{(A_1, \theta_1)}$  is primitive.

Since  $q$  is primitive, so is  $q_{(A_2, \theta_2)} \sim q$ . Thus, by Proposition 8 we have that  $q_{(A_i, \theta_i)} \in \text{gen}(f_{q_i})$ , for  $i = 1, 2$ . Thus,  $\text{gen}(f_{q_1}) = \text{gen}(f_{q_2})$  because  $q_{(A_1, \theta_1)} \sim q_{(A_2, \theta_2)}$ , and so  $x^2 + 4q_1 \sim_p x^2 + 4q_2$ , for all primes  $p \geq 2$ . Since  $x^2$  is a form of unit determinant in the sense of Jones [Jo], it follows from the Cancellation Theorem 37 of [Jo] that  $2q_1 \sim_p 2q_2$ , for all primes  $p \geq 2$ , and so  $\text{gen}(2q_1) = \text{gen}(2q_2)$  because  $2q_1$  and  $2q_2$  are both positive binary forms. Then also  $\text{gen}(q_1) = \text{gen}(q_2)$ , which proves (12) in this case.

**Case 2.**  $q = q_{(A_1, \theta_1)}$  is imprimitive.

Since  $q$  is imprimitive, so is  $q_{(A_2, \theta_2)} \sim q$ , and hence both have content  $c = 4$  by Proposition 6. Since  $I_k(q_{(A_1, \theta_1)}/4) = I_k(q_{(A_2, \theta_2)}/4)$ , for  $k = 1, 2$ , it therefore follows from (10) that  $q_1$  and  $q_2$  have the same content  $\kappa$  and the same discriminant  $\Delta$ . Thus,  $q'_i := q_i/\kappa$  are two primitive binary forms of the same discriminant  $\Delta' = \Delta/\kappa^2$ . It is clear that (12) follows once we have shown that  $\text{gen}(q'_1) = \text{gen}(q'_2)$ .

To prove this, we will use Theorem 3.21 and Lemma 3.20 of Cox [Co] which state that  $\text{gen}(q'_1) = \text{gen}(q'_2)$  if and only if  $q'_1$  and  $q'_2$  have the same “complete character”.

This latter condition is defined as follows. For an odd prime  $p$ , let  $\chi_p$  denote the Legendre character which is defined by  $\chi_p(a) = \left(\frac{a}{p}\right)$ , for  $a \in \mathbb{Z}$ , and put  $\chi_{-4}(n) = \delta(n) = (-1)^{(n-1)/2}$  and  $\chi_8(n) = \epsilon(n) = (-1)^{(n^2-1)/8}$ , for  $n \equiv 1 \pmod{2}$ . (Here and below we will use the  $\chi_n$  notation of [B1] instead of the  $\delta, \epsilon$  notation of [Co].) Then the list of *assigned characters* of discriminant  $\Delta'$  is  $X(\Delta') := \{\chi_p : p|\Delta'\} \cup X_s(\Delta')$ , where  $X_s(\Delta') \subset X_s := \{\chi_{-4}, \chi_8, \chi_{-4}\chi_8\}$  is the set of *supplementary characters* which is given in the table on p. 55 of [Co]. For each  $\chi \in X(\Delta')$ , the value  $\chi(q'_i) := \chi(r_i)$  does not depend on the choice of  $r_i \in R(q'_i)$ , provided that  $\gcd(r_i, \Delta') = 1$ . Thus, by Cox [Co], loc. cit., we have that  $\text{gen}(q'_1) = \text{gen}(q'_2)$  if and only if

$$(14) \quad \chi(q'_1) = \chi(q'_2), \quad \text{for all } \chi \in X(\Delta').$$

It thus suffices to verify (14). For this, we recall that there is a similar theory of assigned characters for primitive ternary forms; cf. Smith [Sm] and Brandt [B1], [B2].

We will first apply this theory to the ternary forms  $f_{q_i}$ , for  $i = 1, 2$ . Since  $f_{q_i}$  is primitive and  $f_{q_i} \sim q_{(A_i, \theta_{E_i, E'_i})}$  (cf. the proof of Proposition 8), it follows from Proposition 6 that

$$(15) \quad d(f_{q_i}) = 16\Delta, \quad I_1(f_{q_i}) = -16\kappa \quad \text{and} \quad I_2(f_{q_i}) = \Delta', \quad \text{for } i = 1, 2.$$

Let  $p$  be an odd prime. Since  $q_{(A_1, \theta_1)} \sim q_{(A_2, \theta_2)}$  by hypothesis, it follows from (11) that  $f_{q_1} \sim_p q_{(A_1, \theta_1)} \sim_p q_{(A_2, \theta_2)} \sim_p f_{q_2}$ . This (together with (15)) implies that their reciprocals  $F_i := F_{f_{q_i}}$  are also  $p$ -adically equivalent (for  $p$  odd), as is well known; cf.

Proposition 3.2.10(iii) of [Ki1]. Furthermore, from that proposition we also obtain that

$$(16) \quad \chi_p(n_1) = \chi_p(n_2), \quad \forall \text{ odd } p \mid \Delta', \quad \forall n_i \in R(F_i) \text{ with } \gcd(n_i, \Delta') = 1, \quad i = 1, 2.$$

Now by formula (44) of [K4] we have that  $F_i \sim (-\Delta'\kappa)x^2 \perp q'_i$ , so  $R(q'_i) \subset R(F_i)$ , for  $i = 1, 2$ . Thus, if  $p$  is an odd prime with  $p \mid \Delta'$ , then  $\chi_p \in X(\Delta')$ , so if  $n_i \in R(q'_i)$  with  $\gcd(n_i, \Delta') = 1$ , then by (16) we obtain that  $\chi_p(q'_1) = \chi_p(n_1) = \chi_p(n_2) = \chi_p(q'_2)$ . This means (14) holds for all  $\chi \in X(\Delta') \setminus X_s(\Delta')$ .

It thus remains to show that (14) also holds for  $\chi \in X_s(\Delta')$ . For this, note first that by Proposition 9 we have that  $\Delta' \equiv \Delta \equiv 0 \pmod{4}$ . As in [Co], p. 55, put  $n := -\frac{\Delta'}{4}$ .

Suppose first that  $n \equiv 3 \pmod{4}$ . Then  $X_s(\Delta') = \emptyset$  by [Co], so there is nothing to prove. Next, suppose that  $n \equiv 1 \pmod{4}$ , so  $X_s(\Delta') = \{\chi_{-4}\}$  by [Co]. Now by Proposition 9 there exists  $n_i \in R_{3,4}(q_i)$ , so  $n'_i := \frac{n_i}{\kappa} \in R_{1,2}(q_i)$  because  $\kappa$  is odd. Thus, for  $i = 1, 2$ , we have that  $\chi_{-4}(q'_i) = \chi_{-4}(n'_i) = \chi_{-4}(\kappa)^{-1} \chi_{-4}(n_i) = -\chi_{-4}(\kappa)$  because  $n_i \equiv 3 \pmod{4}$ , and so  $\chi_{-4}(q'_1) = \chi_{-4}(q'_2)$ . Thus, (14) holds in this case as well.

We are thus left with the case that  $n \equiv 0 \pmod{2}$ , i.e., that  $\Delta' \equiv 0 \pmod{8}$ . In this case we will make use of the supplementary characters of the reciprocals  $F_{f_i}$  of the forms  $f_i = q_{(A_i, \theta_i)}/4$ , for  $i = 1, 2$ . By hypothesis,  $f_1 \sim f_2$ , and by Proposition 6 we see that  $f_i$  is a primitive ternary form with genus invariants  $I_1(f_i) = -\kappa$  and  $I_2(f_i) = 4\Delta'$ . Since  $\Delta' \equiv 0 \pmod{8}$ , we have that  $I_1(F_{f_i}) = I_2(f_i) \equiv 0 \pmod{32}$ . By Brandt [B1], p. 337, this implies that  $\chi_{-4}$ ,  $\chi_8$  and  $\chi_{-8} := \chi_{-4}\chi_8$  are assigned characters of  $F_{f_i}$ . This means that if  $\chi \in X_s = \{\chi_{-4}, \chi_8, \chi_{-8}\}$ , and if  $i = 1, 2$ , then we have that

$$(17) \quad \chi(r) = \chi(r'), \quad \text{for all } r, r' \in R_{1,2}(F_{f_i}).$$

Now let  $\chi \in X_s(\Delta') \subset X_s$ . Then  $\chi(q'_i) = \chi(n_i)$ , for any  $n_i \in R_{1,2}(q'_i)$ . By Proposition 9 there exists  $n_i \in R_{1,2}(q'_i) \cap R_{1,2}(F_{f_i})$ . Now since  $f_1 \sim f_2$ , we also have that  $F_{f_1} \sim F_{f_2}$  and so  $R_{1,2}(F_{f_1}) = R_{1,2}(F_{f_2})$ , and so  $n_1, n_2 \in R_{1,2}(F_{f_1})$ . Thus, by (17) we obtain that  $\chi(q'_1) = \chi(n_1) = \chi(n_2) = \chi(q'_2)$ , for all  $\chi \in X_s(\Delta)$ . This proves that (14) holds in all cases, and hence also (12), as desired.

### 3 The number $N_{(A, \theta)}$ of isomorphism classes

The purpose of this section is to prove the main results stated in the introduction. Let  $\langle A, \theta \rangle$  denote the isomorphism class of a principally polarized abelian surface  $(A, \theta)$ , and let  $\mathcal{A}_2(K)$  denote the set of isomorphism classes of principally polarized abelian surfaces  $(A, \theta)/K$ . When  $(A, \theta)/K$  is a fixed principally polarized abelian surface, put

$$\mathcal{N}_{(A, \theta)} := \{ \langle A', \theta' \rangle \in \mathcal{A}_2(K) : q_{(A', \theta')} \sim q_{(A, \theta)} \}.$$

By Theorem 4, it is clear that if  $A$  is a CM abelian product surface, then so is  $A'$ , for any  $\langle A', \theta' \rangle \in \mathcal{N}_{(A, \theta)}$ . As in the introduction, let us put

$$N_{(A, \theta)} = |\mathcal{N}_{(A, \theta)}|.$$

The first step is to count the number of non-isomorphic abelian surfaces  $A'/K$ , which appear in the set  $\mathcal{N}_{(A, \theta)}$ . To formulate this, let  $\langle A \rangle$  denote the isomorphism class of an abelian surface  $A/K$ . For a given quadratic form  $q$ , let us define

$$\mathcal{N}(q) := \{ \langle A \rangle : q_{(A, \theta)} \sim q, \text{ for some } \theta \in \mathcal{P}(A) \}$$

as the set of isomorphism classes of abelian surfaces  $A/K$  such that  $q_{(A, \theta)}$  is equivalent to  $q$ , for some  $\theta \in \mathcal{P}(A)$ . Note that if  $q_{(A, \theta)}$  is a ternary form which represents a square, then we see that  $\mathcal{N}(q_{(A, \theta)})$  is a subset of the set of isomorphism classes of CM abelian product surfaces by Theorem 4.

In order to find  $|\mathcal{N}(q_{(A, \theta)})|$  in this case, recall from the previous section that the form  $q_{(A, \theta)}$  determines the form  $q_A$  (up to equivalence); cf. Theorem 7. This fact is the key tool to complete the first step because it provides the relation between the set  $\mathcal{N}(q_{(A, \theta)})$  and the set  $\mathcal{N}(A)$ , where

$$\mathcal{N}(A) := \{ \langle A' \rangle : A' \sim A, q_{A'} \sim q_A \}$$

denotes the set of isomorphism classes of abelian surfaces  $A'/K$  which are isogeneous to a given  $A$  and whose intersection form  $q_{A'}$  is equivalent to  $q_A$ . The advantage of studying this set is that [K2] gives an explicit formula for the cardinality  $|\mathcal{N}(A)|$  when  $A$  is a CM abelian product surface. We now prove the following result, which will complete the first step.

**Proposition 10** *Let  $A = E_1 \times E_2$  be a CM abelian product surface, and let  $\theta \in \mathcal{P}(A)$ . Put  $q := q_{(A, \theta)}$ , and let  $q_1 := q_{E_1, E_2}$ . Then we have that*

$$(18) \quad \mathcal{N}(q) = \mathcal{N}(A).$$

Moreover, if we put  $\Delta' = d(q_1) / \text{cont}(q_1)^2$ , then

$$(19) \quad |\mathcal{N}(q)| = \frac{h(\Delta')}{g(\Delta')},$$

where  $g(\Delta')$  denotes the number of genera of discriminant  $\Delta'$ .

*Proof.* We first observe that

$$(20) \quad \text{End}^0(E_i) \simeq \mathbb{Q}(\sqrt{d(q_1)}), \quad \text{for } i = 1, 2.$$

Indeed, since  $E_1$  and  $E_2$  are two isogeneous CM elliptic curves, we have that  $F := \text{End}^0(E_1) \simeq \text{End}^0(E_2)$  is an imaginary quadratic field, and by Corollary 42 of [K2] we have that  $d(q_1) = \text{lcm}(f_{E_1}, f_{E_2})^2 \Delta_F$ , where  $\Delta_F$  is the discriminant of  $F$  and  $f_{E_i}$  is the *endomorphism conductor* of  $E_i$  as defined in [K2]. From this, (20) follows because  $F \simeq \mathbb{Q}(\sqrt{\Delta_F})$ .

To prove (18), suppose first that  $\langle A' \rangle \in \mathcal{N}(q)$ , so by definition there is a  $\theta' \in \mathcal{P}(A')$  such that  $q_{(A', \theta')} \sim q$ . Since  $A$  is a CM abelian product surface, so is  $A'$  by Theorem 4, and hence we have by Theorem 7 that  $q_A \sim q_{A'}$ . Thus, in order to show that  $\langle A' \rangle \in \mathcal{N}(A)$ , it suffices to prove that  $A \sim A'$ .

Now since  $A'$  is a CM product surface, we have that  $A' \simeq E'_1 \times E'_2$ , for some isogeneous CM elliptic curves  $E'_1/K$  and  $E'_2/K$ . If we put  $q'_1 := q_{E'_1, E'_2}$ , then by (9) we see that  $d(q_1) = d(q)/16 = d(q_{(A', \theta')})/16 = d(q'_1)$ . Thus, equation (20) (applied to  $E'_i$ ) shows that  $\text{End}^0(E'_i) \simeq \mathbb{Q}(\sqrt{d(q'_1)}) = \mathbb{Q}(\sqrt{d(q_1)}) \simeq \text{End}^0(E_i)$ , for  $i = 1, 2$ , and so it follows that  $E_i \sim E'_i$ ; cf. Proposition 36 of [K2]. It thus follows that  $A' \simeq E'_1 \times E'_2 \sim E_1 \times E_2 = A$ , and hence  $\langle A' \rangle \in \mathcal{N}(A)$ .

To prove the opposite inclusion, let  $\langle A' \rangle \in \mathcal{N}(A)$ , so we have that  $q_{A'} \sim q_A$ . Hence, since  $\theta \in \mathcal{P}(A)$ , it follows from Proposition 29 of [K7] that  $q \sim q_{(A', \theta')}$ , for some  $\theta' \in \mathcal{P}(A')$ , which proves that  $\langle A' \rangle \in \mathcal{N}(q)$ , and so (18) follows.

To prove (19), we will use Corollary 70 of [K2]. To apply it, note first that there exists an elliptic curve  $E/K$  with  $E \sim E_i$  such that the discriminant  $\Delta_E$  of the order  $\text{End}(E)$  equals  $d(q_1)$ . Indeed, by Proposition 29 of [K2] there exists an  $E \sim E_i$  such that  $f_E = \text{lcm}(f_{E_1}, f_{E_2})$ , and so by Corollary 42 of [K2] we have that  $d(q_1) = \text{lcm}(f_{E_1}, f_{E_2})^2 \Delta_F$ , where  $F = \text{End}^0(E) \simeq \text{End}^0(E_i)$ , and so  $d(q_1) = f_E^2 \Delta_F = \Delta_E$ .

We thus have that  $A \simeq E_1 \times E_2 \sim E \times E$ . Moreover, by (13) we know that  $q_A \sim xy \perp (-q_1)$ , so it follows that  $\mathcal{N}(A) = \mathcal{N}_{q_1} := \{A' \sim E \times E : q_{A'} \sim xy \perp (-q_1)\} / \simeq$ . We thus see that formula (19) follows directly from that given in Corollary 70 of [K2].

The second step in finding a formula for  $N_{(A, \theta)}$  is to determine the number of isomorphism classes of principal polarizations  $\theta'$  on a given abelian surface  $A'$  with  $\langle A' \rangle \in \mathcal{N}(q_{(A, \theta)})$ . For this, we will use the results of [K8] and [K9]. As in those papers, let

$$\mathcal{P}(A, q) := \{\theta \in \mathcal{P}(A) : q_{(A, \theta)} \sim q\}$$

denote the set of principal polarizations  $\theta$  on  $A$  such that  $q_{(A, \theta)}$  is equivalent to a given quadratic form  $q$ . By Theorem 9 of [K8] we know that the automorphism group  $\text{Aut}(A)$  of  $A$  acts on  $\mathcal{P}(A, q)$ , so we can consider the set  $\overline{\mathcal{P}}(A, q) := \text{Aut}(A) \backslash \mathcal{P}(A, q)$  of orbits of  $\mathcal{P}(A, q)$  under this action. We now show that the number  $N_{(A, \theta)}$  can be expressed in terms of a sum of the cardinalities of suitable sets  $\overline{\mathcal{P}}(A_i, q)$ .

**Proposition 11** *Let  $(A, \theta)$  be a principally polarized abelian surface, where  $A$  is a CM abelian product surface, and let  $q = q_{(A, \theta)}$ . If  $A_1, \dots, A_n$  is a system of represen-*

tatives of the finite set  $\mathcal{N}(q)$ , then

$$(21) \quad \mathcal{N}_{(A,\theta)} = \{ \langle A_i, \theta_i \rangle : \theta_i \in \mathcal{P}(A_i, q), \text{ for } 1 \leq i \leq n \},$$

and hence

$$(22) \quad N_{(A,\theta)} = \sum_{i=1}^n |\overline{\mathcal{P}}(A_i, q)|.$$

In particular,  $N_{(A,\theta)} < \infty$ .

*Proof.* If  $\langle A', \theta' \rangle \in \mathcal{N}_{(A,\theta)}$ , then by definition  $\langle A' \rangle \in \mathcal{N}(q)$  and  $\theta' \in \mathcal{P}(A', q)$  because  $q_{(A', \theta')} \sim q_{(A, \theta)} = q$ . Thus,  $A' \simeq A_i$ , for some  $1 \leq i \leq n$ , and hence the left hand side of (21) is contained in the right hand side.

To prove the opposite inclusion, let  $\langle A_i, \theta_i \rangle$  be an element contained in the right hand side of (21), so  $\langle A_i \rangle \in \mathcal{N}(q)$  and  $\theta_i \in \mathcal{P}(A_i, q)$ . This implies that  $q_{(A_i, \theta_i)} \sim q = q_{(A, \theta)}$ , and hence  $\langle A_i, \theta_i \rangle \in \mathcal{N}_{(A, \theta)}$ , which verifies (21).

To verify (22), recall first from Proposition 10 that  $\mathcal{N}(q)$  is a finite set, so the sum on the right hand side of (22) is a finite sum.

Next we observe that if  $\langle A_i, \theta_i \rangle = \langle A_j, \theta_j \rangle$ , then in particular  $A_i \simeq A_j$  and so  $i = j$ . Thus,  $\langle A_i, \theta_i \rangle = \langle A_j, \theta_j \rangle$  if and only if  $i = j$  and  $\theta_i$  and  $\theta_j$  lie in the same  $\text{Aut}(A_i)$ -orbit of  $\mathcal{P}(A_i, q)$ . This means that for each  $i$ , the number of  $\langle A', \theta' \rangle \in \mathcal{N}_{(A, \theta)}$  with  $A' \simeq A_i$  is equal to  $|\overline{\mathcal{P}}(A_i, q)|$ , and so (22) follows from (21).

Since  $\overline{\mathcal{P}}(A_i, q)$  is always a finite set (cf. Theorem 1 of [K8], together with formula (7) of [K8]), it follows from (22) that  $N_{(A, \theta)} = |\mathcal{N}_{(A, \theta)}|$  is also finite.

To analyze  $N_{(A, \theta)}$  further, we will make use of a formula for the cardinality of the set  $\overline{\mathcal{P}}(A, q)$  found in [K8] and [K9]. One of the ingredients of this formula is the quantity  $a(q)$  which was mentioned in (2), and which used certain “representation numbers”  $r_n^*(q)$ . These numbers are defined as follows. If  $n > 0$  is an integer, and if  $q$  is a positive ternary form, then the *number of primitive representations* of  $n$  by  $q$  is defined by

$$r_n^*(q) := |\{(x_1, x_2, x_3) \in R_n(q) : \gcd(x_1, x_2, x_3) = 1\}|,$$

where  $R_n(q) := \{(x_1, x_2, x_3) \in \mathbb{Z}^3 : q(x_1, x_2, x_3) = n\}$  denotes the set of all representations of an integer  $n$  by  $q$ .

**Proposition 12** *Let  $A = E \times E'$  be a CM abelian product surface, and suppose that  $\mathcal{P}(A, q)$  is nonempty. Let  $\Delta = d(q_{E, E'})$  denote the discriminant of the degree map  $q_{E, E'}$ , and let  $\kappa = \text{cont}(q_{E, E'})$  be its content, and put  $\Delta' = \Delta/\kappa^2$ . If  $q$  is not equivalent to  $x^2 + 4\kappa(y^2 + \varepsilon yz + z^2)$ , for any  $\kappa > 1$  and  $\varepsilon = 0$  or  $1$ , then we have that*

$$(23) \quad |\overline{\mathcal{P}}(A, q)| = \frac{2^{\omega(\kappa)+1} g(\Delta') h(\Delta) a(q)}{|\text{Aut}(q)| h(\Delta')}.$$

On the other hand, if  $q$  is equivalent to  $x^2 + 4\kappa(y^2 + \varepsilon yz + z^2)$ , for some  $\kappa > 1$  and for some  $\varepsilon = 0$  or  $1$ , then we have that

$$(24) \quad |\overline{\mathcal{P}}(A, q)| = (2^{\omega(\kappa)-1} + 1 + \varepsilon) \frac{h((\varepsilon - 4)\kappa^2)}{2 + \varepsilon}.$$

*Proof.* Since  $\mathcal{P}(A, q)$  is non-empty, there is a  $\theta \in \mathcal{P}(A)$  with  $q_{(A, \theta)} \sim q$ . Note that  $q$  is a ternary form by Theorem 4.

To prove (23), suppose first that  $q$  does not represent 1. Then (23) follows directly from Theorem 3 and formula (7) of [K8]. (Note that in this case (2) reduces to the formula (3) of [K8]; cf. Corollary 18 of [K9].)

Next, if  $q$  represents 1 but is not equivalent to  $x^2 + 4\kappa(y^2 + \varepsilon yz + z^2)$ , for any  $\kappa > 1$  and  $\varepsilon \in \{0, 1\}$ , then from the proof of Corollary 18 of [K9] we see that  $a(q) = 2 \max(1, r_4^*(q))$ , and hence (23) follows from Theorem 1 of [K9] together with formula (7) of [K8].

We are thus left with the case that  $q \sim x^2 + 4\kappa(y^2 + \varepsilon yz + z^2)$ , for some  $\kappa > 1$  and  $\varepsilon \in \{0, 1\}$ . Put  $q_{\varepsilon, \kappa} := \kappa(y^2 + \varepsilon yz + z^2)$ , so  $\text{cont}(q_{\varepsilon, \kappa}) = \kappa$  and  $d(q_{\varepsilon, \kappa})/\kappa^2 = \varepsilon - 4$ . Since  $x^2 + 4\kappa(y^2 + \varepsilon yz + z^2) = f_{q_{\varepsilon, \kappa}}$  in the notation of [K4], the formula given in the proof of Proposition 53 of [K4] shows that  $I_1(q) = I_1(f_{q_{\varepsilon, \kappa}}) = -16\kappa$  and  $d(q) = d(f_{q_{\varepsilon, \kappa}}) = 16\kappa^2(\varepsilon - 4)$ . It thus follows from Proposition 6 that in this case  $\Delta = d(q_{E, E'}) = \kappa^2(\varepsilon - 4)$  and  $\Delta' = \varepsilon - 4$ .

Put  $u(\Delta') := |\text{Aut}^+(q_{\varepsilon, \kappa}/\kappa)|$  (as in [K9]). Thus,  $u(\Delta') = 4 + 2\varepsilon$ , as is well-known; cf. [Jo], Theorem 51a. Moreover, since  $h(\Delta') = 1$  (cf. Theorem 7.30 of [Co]), we also have that  $g(\Delta') = 1$ , and so we obtain from Proposition 20 of [K9] that

$$|\overline{\mathcal{P}}(A, q)| = (2^{\omega(\kappa)} + u(\Delta') - 2) \frac{2g(\Delta')h(\Delta)}{|\text{Aut}(q_{\varepsilon, \kappa})| h(\Delta')} = (2^{\omega(\kappa)} + 2 + 2\varepsilon) \frac{2h(\Delta)}{|\text{Aut}(q_{\varepsilon, \kappa})|}.$$

This shows that (24) holds because  $|\text{Aut}(q_{\varepsilon, \kappa})| = 2u(\Delta') = 8 + 4\varepsilon$ .

It follows from the above proposition together with Proposition 6 that  $|\overline{\mathcal{P}}(A_i, q)|$  does not depend on the choice of  $\langle A_i \rangle \in \mathcal{N}(q)$ .

**Corollary 13** *Let  $A$  be a CM abelian product surface. If  $\mathcal{P}(A, q)$  is nonempty, then*

$$(25) \quad |\overline{\mathcal{P}}(A, q)| = |\overline{\mathcal{P}}(A', q)|, \text{ for every } \langle A' \rangle \in \mathcal{N}(q).$$

*Proof.* By hypothesis,  $A \simeq E_1 \times E_2$ , for some isogeneous CM elliptic curves  $E_i/K$ . Since  $\mathcal{P}(A, q) \neq \emptyset$ , there is a  $\theta \in \mathcal{P}(A)$  such that  $q_{(A, \theta)} \sim q$ , and so  $\langle A \rangle \in \mathcal{N}(q)$ .

Let  $\langle A' \rangle \in \mathcal{N}(q)$ . Then by definition there exists a  $\theta' \in \mathcal{P}(A')$  such that  $q_{(A', \theta')} \sim q \sim q_{(A, \theta)}$ , and hence  $A'$  is again a CM abelian product surface by Theorem 4. Thus,  $A' \simeq E'_1 \times E'_2$ , for some isogeneous CM elliptic curves  $E'_i/K$ . Since  $q_{(A', \theta')} \sim q_{(A, \theta)}$ ,

we have that  $d(q_{(A',\theta')}) = d(q_{(A,\theta)})$ ,  $\text{cont}(q_{(A',\theta')}) = \text{cont}(q_{(A,\theta)})$  and that  $I_1(q_{(A',\theta')}) = I_1(q_{(A,\theta)})$ . It thus follows from Proposition 6 that

$$(26) \quad d(q_{E'_1, E'_2}) = d(q_{E_1, E_2}) \quad \text{and} \quad \text{cont}(q_{E'_1, E'_2}) = \text{cont}(q_{E_1, E_2}).$$

Now if  $q_{(A',\theta')}$  is not equivalent to  $x^2 + 4\kappa(y^2 + \varepsilon yz + z^2)$ , for any  $\kappa > 1$  and  $\varepsilon \in \{0, 1\}$ , then the same is true for  $q_{(A,\theta)} \sim q_{(A',\theta')}$ , and then formula (23), together with (26), shows that  $|\overline{\mathcal{P}}(A, q)| = |\overline{\mathcal{P}}(A', q)|$ . On the other hand, if  $q_{(A',\theta')} \sim x^2 + 4\kappa(y^2 + \varepsilon yz + z^2)$ , for some  $\kappa > 1$  and  $\varepsilon \in \{0, 1\}$ , then the assertion follows from (24).

We are now ready to prove Theorem 3.

*Proof of Theorem 3.* By hypothesis,  $A$  satisfies condition (i) of Theorem 4, so by that theorem there exist two isogeneous CM elliptic curves  $E_i/K$  such that  $A \simeq E_1 \times E_2$ . In addition,  $q := q_{(A,\theta)}$  is a ternary form. Thus  $N_{(A,\theta)} < \infty$  by Proposition 11.

Let  $\Delta = d(q_{E_1, E_2})$  and  $\kappa = \text{cont}(q_{E_1, E_2})$ . Then from (9), (10) and the definitions we see that  $\Delta_q = \Delta$  and that  $\kappa_q = \kappa$ . Put  $\Delta' = \Delta/\kappa^2$ .

Now if  $q$  is not equivalent to  $f_{q_{\varepsilon, \kappa}} := x^2 + 4\kappa(y^2 + \varepsilon yz + z^2)$ , for any  $\kappa > 1$  and  $\varepsilon \in \{0, 1\}$ , then (3) follows from (22), (25), (19) and (23) because

$$\begin{aligned} N_{(A,\theta)} &= \sum_{\langle A_i \rangle \in \mathcal{N}(q)} |\overline{\mathcal{P}}(A_i, q)| = |\mathcal{N}(q)| |\overline{\mathcal{P}}(A, q)| \\ &\stackrel{(19)}{=} \frac{h(\Delta')}{g(\Delta')} |\overline{\mathcal{P}}(A, q)| \stackrel{(23)}{=} \frac{h(\Delta')}{g(\Delta')} \frac{2^{\omega(\kappa)+1} g(\Delta') h(\Delta) a(q)}{|\text{Aut}(q)| h(\Delta')} \\ &= 2^{\omega(\kappa)+1} h(\Delta) \frac{a(q)}{|\text{Aut}(q)|} = 2^{\omega(\kappa_q)+1} h(\Delta_q) \frac{a(q)}{|\text{Aut}(q)|}. \end{aligned}$$

On the other hand, if  $q \sim f_{q_{\varepsilon, \kappa}}$ , for some  $\kappa > 1$  and some  $\varepsilon \in \{0, 1\}$ , then  $h(\Delta') = g(\Delta') = 1$ , as we saw in the proof of Proposition 12. This implies by (19) that  $|\mathcal{N}(q)| = 1$ , so  $N_{(A,\theta)} = |\overline{\mathcal{P}}(A, q)|$  by (22), and hence (4) follows from (24).

We will now deduce Theorem 1 from Theorem 3. For this, recall (from [Mi], for example) that if  $C/K$  is a curve of genus 2, then its Jacobian  $J_C$  is an abelian surface and the image  $j(C)$  of  $C$  via the canonical embedding  $j : C \hookrightarrow J_C$  is an ample divisor on  $J_C$ . Furthermore, the image  $\theta_C \in \text{NS}(J_C)$  of  $j(C)$  in the Néron-Severi group is a principal polarization. We then write  $q_C := q_{(J_C, \theta_C)}$  for its associated refined Humbert invariant.

It turns out that whether or not a given principally polarized abelian surface  $(A, \theta)$  is a Jacobian can be determined from its refined Humbert invariant  $q_{(A,\theta)}$ . Indeed, by Proposition 6 of [K3] we have that

$$(27) \quad (A, \theta) \simeq (J_C, \theta_C), \text{ for some curve } C/K \Leftrightarrow q_{(A,\theta)}(D) \neq 1, \forall D \in \text{NS}(A, \theta).$$

This result, together with Torelli's Theorem, implies that the number  $N_C$  defined in the introduction equals the number  $N_{(J_C, \theta_C)}$ , if  $C$  is of CM product type. More precisely:

**Lemma 14** *If  $C/K$  is a curve of genus 2, then the map  $C' \mapsto (J_{C'}, \theta_{C'})$  induces a bijection  $\mathcal{N}_C \xrightarrow{\sim} \mathcal{N}_{(J_C, \theta_C)}$ , where  $\mathcal{N}_C$  denotes the set of isomorphism classes  $\langle C' \rangle$  of curves  $C'/K$  of genus 2 such that  $q_{C'} \sim q_C$ . Thus, if  $C/K$  is of CM product type, then  $N_C := |\mathcal{N}_C| = N_{(J_C, \theta_C)}$ .*

*Proof.* It is clear from the definitions that the given rule induces a map  $\varphi_C : \mathcal{N}_C \rightarrow \mathcal{N}_{(J_C, \theta_C)}$ . This map is injective by Torelli's Theorem; cf. Theorem 12.1 in [Mi]. Moreover,  $\varphi_C$  is surjective because if  $\langle A, \theta \rangle \in \mathcal{N}_{(J_C, \theta_C)}$ , then  $q_{(A, \theta)} \sim q_{(J_C, \theta_C)}$ . Since  $q_{(J_C, \theta_C)}$  does not represent 1 by (27), it follows that also  $q_{(A, \theta)}$  does not represent 1, and so by (27) again we have that  $(A, \theta) \simeq (J_{C'}, \theta_{C'})$ , for some curve  $C'/K$  of genus 2. Since  $q_{C'} = q_{(J_{C'}, \theta_{C'})} \sim q_{(A, \theta)} \sim q_{(J_C, \theta_C)} = q_C$ , we see that  $\langle C' \rangle \in \mathcal{N}_C$ , so  $\langle A, \theta \rangle = \varphi_C(\langle C' \rangle)$  lies in the image of  $\varphi_C$ . Thus,  $\varphi_C$  is surjective and hence bijective.

If  $C$  is of CM product type, then  $J_C$  is a CM abelian product surface by Theorem 4, and so  $N_{(J_C, \theta_C)} < \infty$  by Proposition 11. Thus, the first assertion shows that  $N_C = |\mathcal{N}_{(J_C, \theta_C)}| = N_{(J_C, \theta_C)}$ .

*Proof of Theorem 1.* Since  $C$  is of CM product type, we have by Lemma 14 that  $N_C = N_{(J_C, \theta_C)}$ . As was mentioned in its proof,  $J_C$  is a CM abelian product surface. By (27) we know that  $q_C = q_{(J_C, \theta_C)}$  cannot represent 1, so  $q_C$  cannot be equivalent to one of the exceptional forms of Theorem 3, and hence it follows from that theorem that

$$N_C = N_{(J_C, \theta_C)} = 2^{\omega(\kappa_{q_C})+1} h(\Delta_{q_C}) \frac{a(q_C)}{|\text{Aut}(q_C)|}.$$

Since  $q_C$  represents a square by Theorem 4, we have by Theorem 25 of [K8] that

$$(28) \quad 2a(q_C) = |\text{Aut}(C)|,$$

and so (1) follows.

*Proof of Corollary 2.* To prove the first assertion, fix an integer  $n \geq 1$  and let

$$Q_n = \{q_C : C/K \text{ is a curve of genus 2 of product CM type with } N_C \leq n\} / \sim$$

denote the set of equivalence classes of those refined Humbert invariants which arise from curves  $C/K$  of genus 2 of product CM type with  $N_C \leq n$ . It clearly suffices to show that  $|Q_n| < \infty$  because by definition the number of isomorphism classes of such curves with a given  $q \in Q_n$  is equal to  $N_C \leq n$ .

Now if  $q \in Q_n$ , then by Theorem 4 we know that  $q$  is a positive ternary form and so  $|\text{Aut}(q)| \leq 48$  because  $|\text{Aut}^+(q)| \leq 24$  by Theorem 105 of [Di] and because

$|\text{Aut}(q)| = 2|\text{Aut}^+(q)|$  since  $-1 \in \text{Aut}(q) \setminus \text{Aut}^+(q)$ . We thus obtain from (1) that if  $q$  is in  $Q_n$ , then

$$\frac{h(\Delta_q)}{48} \leq 2^{\omega(\kappa_q)} h(\Delta_q) \frac{|\text{Aut}(C)|}{|\text{Aut}(q)|} = N_C \leq n.$$

Now if  $q \in Q_n$ , then  $\Delta_q = d(q)/16$  is a negative quadratic discriminant by Proposition 6. But by the celebrated result of Heilbronn [He] we know that there are only finitely many negative discriminants  $\Delta$  such that  $h(\Delta) \leq 48n$ , so there exists a bound  $B_n$  such that  $h(\Delta) \leq 48n \Rightarrow |\Delta| \leq B_n$ . We therefore see that  $Q_n$  is a subset of the set

$$\tilde{Q}_n = \{q : q \text{ is a positive ternary form with } |d(q)| \leq 16B_n\} / \sim.$$

Now by Theorem 11 of [Wa], there are up to equivalence only finitely many forms of a given rank and discriminant, so  $\tilde{Q}_n$  is a finite set, and hence so is its subset  $Q_n$ . This proves the first assertion.

In order to prove the second assertion, it suffices in view of the first assertion to show that there exist infinitely many non-isomorphic genus 2 curves  $C/K$  of CM product type.

To verify this, we first show that the set  $\mathcal{A}_{CM}(K) = \{\langle A \rangle\}$  consisting of the set of isomorphism classes of CM abelian product surfaces  $A/K$  is infinite. If  $\text{char}(K) = 0$ , then this follows immediately from Theorem 71 of [K2] (and its proof) because there are clearly infinitely many negative discriminants  $\Delta \equiv 0, 1 \pmod{4}$ , and hence there are infinitely many equivalence classes of positive binary quadratic forms. If  $\text{char}(K) = p > 0$ , then this follows from Remark 73 of [K2] because there exist infinitely many negative numbers  $\Delta$  with  $\Delta \equiv 1 \pmod{4p}$ , so there exist infinitely many positive binary quadratic forms whose discriminant  $\Delta$  satisfies  $(\frac{\Delta}{p}) = 1$ .

Next, let  $\mathcal{A}_{CM}^*(K)$  denote the subset of  $\mathcal{A}_{CM}(K)$  consisting of the isomorphism classes of those CM abelian product surfaces  $A/K$  which contain a smooth genus 2 curve  $C/K$ . By Theorem 2 of [K4] we know that  $|\mathcal{A}_{CM}(K) \setminus \mathcal{A}_{CM}^*(K)| \leq 15$ , so  $\mathcal{A}_{CM}^*(K)$  is also an infinite set.

Now if  $C/K$  is a smooth genus 2 curve lying on an abelian surface  $A/K$ , then  $A \simeq J_C$  (by the universal property of the Jacobian), and so if  $C \subset A$ , where  $\langle A \rangle \in \mathcal{A}_{CM}^*(K)$ , then  $C$  is a curve of CM product type. Furthermore, since  $C \simeq C'$  implies that  $J_C \simeq J_{C'}$ , it therefore follows that the infinitely many non-isomorphic abelian surfaces in  $\mathcal{A}_{CM}^*(K)$  give rise to infinitely many non-isomorphic curves  $C/K$  of CM product type, as claimed. This proves the second assertion.

## 4 Examples

In this section, we want to illustrate how Theorem 1 can be applied to some explicit cases. In this context, the following result discusses some cases satisfying  $N_C = 1$ .

**Proposition 15** *Let  $A/K$  be a CM abelian product surface, and let  $\theta \in \mathcal{P}(A)$  be a principal polarization on  $A$ . Suppose that  $\kappa_{q_{(A,\theta)}} = 1$  and that  $h(\Delta) = 1$ , where  $\Delta = \Delta_{q_{(A,\theta)}}$ . Then  $A \simeq E \times E$ , where  $E/K$  is up to isomorphism the unique CM elliptic curve such that  $\text{End}(E)$  has discriminant  $\Delta$ .*

*Conversely, if  $E/K$  is a CM elliptic curve such that  $h(\Delta) = 1$ , where  $\Delta$  is the discriminant of  $\text{End}(E)$ , then  $A = E \times E$  is a CM abelian product surface, and  $\Delta_{q_{(A,\theta)}} = \Delta$ ,  $\kappa_{q_{(A,\theta)}} = 1$ , for every principal polarization  $\theta \in \mathcal{P}(A)$  on  $A$ . In addition, we have that  $N_{(A,\theta)} = 1$  and  $|\text{Aut}(q_{(A,\theta)})| = 2a(q_{(A,\theta)})$ .*

*In particular, if  $C$  is a curve of CM product type with  $\kappa_C = 1$  and  $h(\Delta_C) = 1$ , then  $N_C = 1$ ,  $|\text{Aut}(C)| = |\text{Aut}(q_C)|$  and  $J_C \simeq E \times E$ , where  $E/K$  is up to isomorphism the unique elliptic curve such that  $\text{End}(E)$  has discriminant  $\Delta_C$ .*

*Proof.* Since  $A/K$  is a CM abelian product surface, we have that  $A \simeq E_1 \times E_2$ , where  $E_i/K$  are two isogeneous CM elliptic curves  $E_i/K$ . By Proposition 6 we have that  $d(q_{E_1,E_2}) = d(q_{(A,\theta)})/16 = \Delta_{q_{(A,\theta)}} = \Delta$  and  $\text{cont}(q_{E_1,E_2}) = \kappa_{q_{(A,\theta)}} = 1$ . Thus,  $q_{E_1,E_2}$  is a primitive positive binary quadratic form of discriminant  $\Delta$ . Since  $h(\Delta) = 1$  by hypothesis,  $q_{E_1,E_2}$  is equivalent to the principal form  $1_\Delta$  of discriminant  $\Delta$ , and hence  $q_{E_1,E_2}$  represents 1. Thus, there exists an  $h \in \text{Hom}(E_1, E_2)$  with  $\deg(h) = 1$ , so  $E := E_1 \simeq E_2$ , and hence  $q_{E_1,E_2} \sim q_{E,E}$ . This means that  $\text{End}(E)$  has discriminant  $\Delta$ , and that  $A \simeq E \times E$ .

Since  $h(\Delta) = 1$ , then, as is well-known, the isomorphism class of the elliptic curve  $E/K$  is uniquely determined by the fact that the endomorphism ring of  $E/K$  has discriminant  $\Delta$ ; cf. [K2], equation (55).

Conversely, suppose that  $A = E \times E$  where  $E/K$  is a CM elliptic curve, and let  $\Delta$  be the discriminant of  $\text{End}(E)$ . Then  $\Delta = d(q_{E,E})$  (by definition), and clearly  $\text{cont}(q_{E,E}) = 1$  because  $1_E \in \text{End}(E)$ . Thus, by Proposition 6 (and the definitions) we have that  $\Delta_{q_{(A,\theta)}} = \Delta$  and  $\kappa_{q_{(A,\theta)}} = 1$ , for every  $\theta \in \mathcal{P}(A)$ .

It remains to show that  $N_{(A,\theta)} = 1$  and that  $2a(q) = |\text{Aut}(q)|$ , where  $q = q_{(A,\theta)}$ . Now since  $\kappa_q = 1$ , we see that  $q$  is not one of the exceptional forms of Theorem 3, and so it follows from (3) that  $N_{(A,\theta)} = \frac{2a(q)}{|\text{Aut}(q)|}$ . Since  $a(q) = a(\theta)$  by Corollary 17 of [K9] (where  $a(\theta)$  is defined as in [K9]) and since  $a(\theta) \mid |\text{Aut}^+(q)| = \frac{1}{2}|\text{Aut}(q)|$  by Proposition 16 of [K8], we see that  $N_{(A,\theta)} \leq 1$ . But since  $N_{(A,\theta)} \geq 1$  (because  $\langle A, \theta \rangle \in \mathcal{N}_{(A,\theta)}$ ), it follows that  $N_{(A,\theta)} = 1$  and  $2a(q) = |\text{Aut}(q)|$ , as claimed.

The last assertion follows directly from the first part by using equation (28).

**Remark 16** (a) If  $\text{char}(K) = 0$  and if  $E/K$  is a CM elliptic curve such that  $h(\Delta_E) = 1$ , where  $\Delta_E$  is the discriminant of  $\text{End}(E)$ , then  $E/K$  comes from an elliptic curve defined over  $\mathbb{Q}$ . An equation of such a curve, together with its  $j$ -invariant, is given in the tables on p. 483 of Silverman [Si].

Similarly, if  $\text{char}(K) = p > 0$ , then the CM elliptic curves with  $h(\Delta_E) = 1$  are obtained by reduction mod  $p$  from the corresponding CM curve over  $\mathbb{Q}$ , provided that  $(\frac{\Delta_E}{p}) = 1$ . On the other hand, if  $(\frac{\Delta_E}{p}) \neq 1$ , then there is no such CM elliptic curve.

(b) In the situation of Proposition 15 there may not be a curve  $C/K$  satisfying the given conditions. Indeed, if  $\Delta = -3, -4$  or  $-7$ , then there is no such curve; cf. Hayashida/Nishi [HN], Theorem 1, or [K4], Theorem 2, or the table in [K8].

We now want to show how Proposition 15 can be used to deduce the uniqueness of a curve  $C/K$  when we start with a fixed ternary quadratic form.

**Example 17** *Consider the following ternary quadratic form*

$$q(x, y, z) = 4x^2 + 4y^2 + 4z^2 + 4yz + 4xz + 4xy.$$

*If  $\text{char}(K) = 0$ , then up to isomorphism there is a unique genus 2 curve  $C/K$  such that  $q_C \sim q$ . Moreover,  $|\text{Aut}(C)| = 48$  and the Jacobian  $J_C$  of  $C$  is isomorphic to  $E \times E$ , where  $E$  is given by the equation  $y^2 = x^3 + 4x^2 + 2x$  and has  $j$ -invariant  $j(E) = 2^6 \cdot 3^5$ .*

*Proof.* We first note that  $q/2$  is an *improperly primitive* form (in the sense of Dickson [Di]) because  $q/4$  is a primitive form and the coefficient of the term  $yz$  (or of  $xy$ , or of  $xz$ ) of  $q/4$  is odd. Thus,  $q$  satisfies condition (1.1) in Theorem 1.2 of [Ki2], and also condition (1.2) holds because  $q(1, 0, 0) = 4$  is a square. Since  $\text{char}(K) = 0$ , it thus follows from that theorem that there exists a principally polarized abelian surface  $(A, \theta)$  over  $K$  such that  $q_{(A, \theta)} \sim q$ .

It is clear that  $q$  cannot represent 1, so it follows from (27) that  $(A, \theta) \simeq (J_C, \theta_C)$ , for some genus 2 curve  $C/K$ . Note that since  $A/K$  is a CM abelian product surface by Theorem 4, it follows that  $C$  is a curve of CM product type, so  $J_C \simeq E_1 \times E_2$ , for some isogeneous CM elliptic curves  $E_i/K$ .

In order to apply Proposition 15, we need to show that  $C$  satisfies the required conditions. For this, we first compute the discriminant  $d(q)$ . By definition (cf. [Wa], p. 2) we have that  $d(q) = -\frac{1}{2} \det(M(q))$ , where  $M(q)$  is the coefficient matrix of  $q$ . In our case we have that

$$M(q) = \begin{pmatrix} 8 & 4 & 4 \\ 4 & 8 & 4 \\ 4 & 4 & 8 \end{pmatrix},$$

so we see that  $d(q) = -8 \cdot 16$ . Thus, by Proposition 6 we obtain that  $d(q_{E_1, E_2}) = -8$ . Since  $-8$  is a fundamental discriminant, it follows that  $q_{E_1, E_2}$  is primitive, so  $\kappa_q = \text{cont}(q_{E_1, E_2}) = 1$ . Thus,  $C$  satisfies the hypotheses of Proposition 15 because  $h(-8) = 1$  (cf. Theorem 7.30 of [Co]), and so  $J_C \simeq E \times E$ , where  $E$  is a CM elliptic curve such that  $\text{End}(E)$  has discriminant  $-8$ . Since this determines  $E/K$  uniquely up to isomorphism (cf. Proposition 15), we see from the first table on p. 483 of [Si] that  $j(E) = 2^6 \cdot 5^3$  and from the second table there that  $E/K$  is given by the equation  $y^2 = x^3 + 4x^2 + 2x$ .

To determine  $\text{Aut}(C)$ , we can use either of two methods. The first consists of computing  $r_4^*(q)$ . For this, we observe that clearly

$$\{\pm(1, 0, 0), \pm(0, 1, 0), \pm(0, 0, 1), \pm(1, -1, 0), \pm(1, 0, -1), \pm(0, 1, -1)\} \subset R_4(q),$$

so  $r_4^*(q) \geq 12$ . We thus see from (2) that  $a(q) = 3r_4^* - 12 \geq 24$ , so by (28) we obtain that  $|\text{Aut}(C)| = 2a(q) \geq 48$ . But since  $|\text{Aut}(C)| \leq 48$ , for all such curves  $C/K$  (cf. Theorem 25 of [K8]), it follows that  $|\text{Aut}(C)| = 48$  (and that  $r_4^*(q) = 12$ ).

Alternately, we can determine  $|\text{Aut}(C)|$  by calculating  $|\text{Aut}(q)| = |\text{Aut}(q/2)|$ , and for this we can use the method and tables of Dickson [Di], §82-83. Indeed, we see that  $q/2$  is the first entry of Table II on p. 185 of [Di], so we have by that table that  $|\text{Aut}^+(q/2)| = 24$ . (The method for computing this table is given in Theorem 105 of [Di].) We thus have that  $|\text{Aut}(q)| = 2|\text{Aut}^+(q/2)| = 48$ , and hence also  $|\text{Aut}(C)| = 48$  by Proposition 15.

**Remark 18** One can say more about the curve  $C/K$  which was constructed in Example 17. Indeed, since here  $\text{char}(K) = 0$ , it is known that up to isomorphism there is only one curve  $C/K$  with  $|\text{Aut}(C)| = 48$ , and so it follows that  $C$  is given by the equation  $y^2 = x(x^4 - 1)$ ; cf. [Ig], §8. Furthermore, we have that  $\text{Aut}(C) \simeq \text{GL}_2(3)$  by Theorem 2 of [SV]. In [AP], this curve is called the *Burnside curve*.

The next Example 19 uses again Proposition 15 to deduce the uniqueness of a curve  $C/K$  for a given ternary quadratic form. This example also illustrates that there may be two non-isomorphic curves of genus 2 on the same abelian surface.

**Example 19** Consider the following two ternary quadratic forms

$$q_1(x, y, z) = 4x^2 + 5y^2 + 8z^2 - 4yz - 4xy \text{ and } q_2(x, y, z) = 4x^2 + 4y^2 + 8z^2 - 4xz.$$

If  $\text{char}(K) = 0$ , then up to isomorphism there is a unique genus 2 curve  $C_i/K$  such that  $q_{C_i} \sim q_i$ , for  $i = 1, 2$ . Moreover,  $|\text{Aut}(C_1)| = 4$  and  $|\text{Aut}(C_2)| = 8$ , and the Jacobian  $J_{C_i}$  of  $C_i$  is isomorphic to  $E \times E$ , where  $E$  is given by the equation  $y^2 = x^3 - 595x + 5586$  and has  $j$ -invariant  $j(E) = 3^3 \cdot 5^3 \cdot 17^3$ . Furthermore,  $C_1 \not\sim C_2$ .

*Proof.* For the first form  $q_1$ , note first that  $q_1(x, y, z) = (2x - y)^2 + 4(y^2 - yz + 2z^2)$  is the sum of two positive forms, and so  $q_1$  is a positive form. In fact, we easily see that  $q_1 \geq 4$ , if  $(x, y, z) \neq (0, 0, 0)$ , so  $q_1$  cannot represent 1.

As in Example 17, we can calculate the discriminants  $d(q_i)$ , for  $i = 1, 2$  by computing the determinant of  $M(q_i)$ . This gives  $d(q_i) = -16 \cdot 28$ , for  $i = 1, 2$ . Since  $q_1(1, 1, 1) = 3^2$ , the form  $q_1$  represents a square which is relatively prime to its discriminant. Furthermore, since  $q_1 \equiv y^2 \pmod{4}$ , we see that  $q_1 \equiv 0, 1 \pmod{4}$ . If  $\text{char}(K) = 0$ , then it follows from Theorem 1 of [K7] that  $q_1 \sim q_{(A, \theta)}$ , for some principally polarized abelian surface  $(A, \theta)$  over  $K$ . Moreover, as was mentioned above,

$q_1$  cannot represent 1, so it follows from (27) that  $(A, \theta) \simeq (J_{C_1}, \theta_{C_1})$ , for some genus 2 curve  $C_1/K$ . As in Example 17, we conclude that  $C_1$  is of CM product type.

For the second form  $q_2$ , note that  $q_2/2$  is an improperly primitive form because  $q_2/4$  is a primitive form and the coefficient of the term  $xz$  of  $q_2/4$  is odd. Moreover,  $q_2$  is a positive form because  $q_2(x, y, z) = 4(x^2 - xz + 2z^2) + 4y^2$  is the sum of two positive forms. Thus,  $q_2$  satisfies condition (1.1) in Theorem 1.2 of [Ki2], and also condition (1.2) holds because  $q_2(1, 0, 0) = 2^2$ . If  $\text{char}(K) = 0$ , then it follows from that theorem that there exists a principally polarized abelian surface  $(A', \theta')$  over  $K$  such that  $q_2 \sim q_{(A', \theta')}$ . Furthermore, it is clear that  $q_2$  cannot represent 1, so it again follows from (27) that  $(A', \theta') \simeq (J_{C_2}, \theta_{C_2})$ , for some genus 2 curve  $C_2/K$ . Similarly, we conclude that  $C_2$  is of CM product type.

In order to apply Proposition 15, we need to show that the  $C_i$  satisfy the required conditions.

To begin, we have that  $\Delta_{C_i} = d(q_i)/16 = -28$ , and so  $h(\Delta_{C_i}) = 1$  by Theorem 7.30 of [Co]. Next, to compute  $\kappa_{C_i}$ , recall from §2 that the genus invariant  $|I_1(q)|$  of a primitive form  $q$  is the content of the adjoint form  $\text{adj}(q)$  (see (8)). Hence, we see that  $|I_1(q_1)| = \gcd(16 \cdot 9, 16 \cdot 8, 16 \cdot 4, 16 \cdot 4, 16 \cdot 2, 16 \cdot 8) = 16$  and that  $|I_1(q_2/4)| = \gcd(8, 7, 4, 0, 4, 0) = 1$ , and thus  $\kappa_{C_i} = 1$ , for  $i = 1, 2$  (by definition). This shows that the  $C_i$  satisfy the hypotheses of Proposition 15, and so  $J_{C_i} \simeq E_i \times E_i$ , where  $E_i$  is a CM elliptic curve such that  $\text{End}(E_i)$  has discriminant  $-28$ , for  $i = 1, 2$ . Since this determines  $E_i/K$  uniquely up to isomorphism (cf. Proposition 15), we obtain that  $E_1 \simeq E_2 := E$ , and hence  $J_{C_i} \simeq E \times E$ , for  $i = 1, 2$ . Furthermore, we see from the first table on p. 483 of [Si] that  $j(E) = 3^3 \cdot 5^3 \cdot 17^3$  and from the second table there that  $E/K$  is given by the equation  $y^2 = x^3 - 595x + 5586$ , as asserted.

To determine  $|\text{Aut}(C_i)|$ , we can use two different methods as was mentioned in Example 17. We use the first method to find  $|\text{Aut}(C_1)|$ , and the second one to calculate  $|\text{Aut}(C_2)|$ .

The first method consists of computing  $a(q_1)$ , and this can be achieved by calculating  $r_4^*(q_1)$ . But, this is already done in Corollary 30 of [K8], for primitive ternary forms  $q_C$ . Indeed, since  $q_1$  satisfies the inequalities of Theorem 103 of [Di],  $q_1$  is an *Eisenstein-reduced* ternary form. Since  $q_1$  is primitive, we are in the situation of Corollary 30 of [K8], and it follows from there that  $a(q_1) = 2$ . Thus,  $|\text{Aut}(C_1)| = 2a(q_1) = 4$  by equation (28).

To apply the second method to the form  $q_2$ , we determine  $|\text{Aut}(q_2)| = |\text{Aut}(q_2/2)|$ . Note that  $q_2/2$  is the sixth entry of Table II on p. 185 of [Di], and so we have by that table that  $|\text{Aut}^+(q_2/2)| = 4$ , and thus  $|\text{Aut}(q_2)| = 2|\text{Aut}^+(q_2/2)| = 8$ . Therefore, it follows that  $|\text{Aut}(C_2)| = |\text{Aut}(q_2)| = 8$  by Proposition 15.

Finally, we observe that  $C_1 \not\simeq C_2$  because  $q_{C_1} \not\sim q_{C_2}$ . Indeed,  $q_{C_1} \sim q_1$  is primitive, whereas  $q_{C_2} \sim q_2$  is not, so  $q_{C_1} \not\sim q_{C_2}$ .

The following example shows that it is possible to have  $N_C = 1$  in a situation where Proposition 15 does not apply.

**Example 20** *Let  $q$  be the primitive ternary quadratic form defined by*

$$q(x, y, z) = 4x^2 + 4y^2 + 9z^2 - 4xy.$$

*If  $\text{char}(K) = 0$  or if  $\text{char}(K) \equiv 1 \pmod{3}$ , then there exists (up to isomorphism) a unique curve  $C/K$  such that  $q_C \sim q$ . Furthermore,  $|\text{Aut}(C)| = 12$  and  $J_C \simeq E_1 \times E_2$ , where  $E_1$  is the elliptic curve given by  $y^2 + y = x^3$  and  $E_2$  is the elliptic curve given by  $y^2 + y = x^3 - 30x + 63$ . Moreover,  $j(E_1) = 0$  and  $j(E_2) = -2^{15} \cdot 3 \cdot 5^3$ .*

*Proof.* Note first that  $q(x, y, z) = 4(x^2 - xy + y^2) + 9z^2$  is the sum of two positive forms, so  $q$  is also positive. In fact, we see that  $q(x, y, z) \geq 4$ , if  $(x, y, z) \neq (0, 0, 0)$ , so  $q$  cannot represent 1.

As in Example 17, we can calculate its discriminant by computing the determinant of  $M(q)$ . This gives  $d(q) = -16 \cdot 27$ . Since  $q(2, 2, 1) = 5^2$ , the form  $q$  represents a square which is relatively prime to its discriminant. Furthermore, since  $q \equiv z^2 \pmod{4}$ , we see that  $q \equiv 0, 1 \pmod{4}$ . If  $\text{char}(K) = 0$ , then it follows from Theorem 1 of [K7] that  $q \sim q_{(A, \theta)}$ , for some principally polarized abelian surface  $(A, \theta)$  over  $K$ . Similarly, if  $p := \text{char}(K) > 0$ , then  $(\frac{d(q)/16}{p}) = (\frac{-3}{p}) = 1$  by our hypothesis, and so it follows from Theorem 28 of [K7] that  $q \sim q_{(A, \theta)}$ , for some principally polarized abelian surface  $(A, \theta)$  over  $K$ .

As was mentioned above,  $q$  cannot represent 1, so it follows from (27) that  $(A, \theta) \simeq (J_C, \theta_C)$ , for some genus 2 curve  $C/K$ . As in Example 17, we conclude that  $C$  is of CM product type.

To show that  $C/K$  is uniquely determined by  $q$ , it suffices to show that  $N_C = 1$ , and for this we will use the formula (1). Thus, we need to compute the quantities on the right hand side of (1).

To begin, we have that  $\Delta_C = d(q)/16 = -27$ , and so  $h(\Delta_C) = 1$  (by Theorem 7.30 of [Co]). Next, we see that  $|I_1(q)| = \gcd(16 \cdot 9, 16 \cdot 9, 48, 0, 0, 16 \cdot 9) = 48$  as was discussed in Example 19, and thus  $\kappa_C = 3$ .

It remains to compute  $|\text{Aut}(C)|$  and  $|\text{Aut}(q)|$ . For this, observe that  $q$  is an Eisenstein-reduced primitive ternary form, and we are thus in the situation of Corollary 30 of [K8], and so it follows from there that  $a(q) = 6$ . Thus,  $|\text{Aut}(C)| = 2a(q) = 12$  by equation (28).

Next, to compute  $|\text{Aut}(q)|$ , we will use Theorem 105 of [Di]. Since  $q$  is Eisenstein-reduced, we see that the table on p. 180 of [Di] is applicable. We observe that the cases of lines 2, 5 and 9 hold and that the others do not. These cases give us five automorphs, and hence we have six automorphs including the identity. Since the coefficients of the terms  $yz$  and  $xz$  are 0, we conclude that  $|\text{Aut}^+(q)| = 6 \cdot 2 = 12$  by the footnote on p. 180 of [Di], and so  $|\text{Aut}(q)| = 24$ .

It therefore follows from (1) that  $N_C = 2^{\omega(3)}h(-27)\frac{|\text{Aut}(C)|}{|\text{Aut}(q_C)|} = \frac{2 \cdot 12}{24} = 1$ , so  $C/K$  is uniquely determined by  $q$  (up to isomorphism).

It remains to determine the structure of  $J_C$ . Since  $C$  is of CM product type, we have that  $J_C \simeq E_1 \times E_2$ , for some isogeneous CM elliptic curves  $E_1$  and  $E_2$ . By (9) and (10) and the above computations, we see that  $q_{E_1, E_2}$  is a binary quadratic form of content 3 and discriminant  $-27$ . Moreover, by (20) we have that  $\text{End}^0(E_1) \simeq \text{End}^0(E_2) = \mathbb{Q}(\sqrt{-27}) =: F$ , and so  $\Delta_F = -3$ .

As in the proof of Proposition 10, let  $f_i = f_{E_i}$  denote the conductor of the order  $\text{End}(E_i)$  of  $F$ . Then by the first formula of (79) in [K2] we see that  $\text{lcm}(f_1, f_2)^2 = \frac{d(q_{E_1, E_2})}{\Delta_F} = \frac{-27}{-3} = 9$ , and so it follows from the second formula of (79) in [K2] that  $\text{gcd}(f_1, f_2) = \frac{\text{lcm}(f_1, f_2)}{\text{cont}(q_{E_1, E_2})} = \frac{3}{3} = 1$ . Thus, by interchanging  $E_1$  and  $E_2$ , if necessary, we may assume without loss of generality that  $f_1 = 1$  and  $f_2 = 3$ . This means that  $\Delta_{E_1} = -3$  and  $\Delta_{E_2} = -27$ . Since  $h(-3) = h(-27) = 1$ , the curves  $E_i/K$  are uniquely determined (up to isomorphism) by their endomorphism discriminants, and so we see from the first table of [Si] on p. 483 that  $j(E_1) = 0$  and that  $j(E_2) = -2^{15} \cdot 3 \cdot 5^3$ , and from the second table that  $E_1$  and  $E_2$  are given by the asserted equations.

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