Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho$$ Coulomb's law

$$\vec{\nabla} \cdot \vec{B} = 0$$ absence of magnetic monopoles

$$\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$ Faraday's law of induction

$$\vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$$ Ampère's law + Maxwell displacement current

Units - Gaussian

$[q]$ statcoulombs $\qquad [\rho] =$ statcoulombs/cm^3

$[E]$ statvolts/cm = statcoulombs/cm^3

$[B]$ = gauss $\qquad [E]/[B] = 1$ $\qquad$ in SI units $[E]/[B] = m/s$

$[J]$ = statamps/cm^2

= statcoulombs/cm^2/s $\qquad [J]/[\rho] = cm/s$

qE = force $\qquad$ 1 dyne = statcoulomb^2/cm^2

$$\frac{1}{8\pi}\left(E^2 + B^2\right) = \text{energy density}$$

$$1\ \text{erg}/\text{cm}^3 = \left(\text{statcoulomb}/\text{cm}^2\right)^2$$

$$= \text{dyne}/\text{cm}^2$$

$$= \text{gauss}^2$$

Ohm's law $\vec{J} = \sigma \vec{E}$ $[\sigma] = s^{-1}$

Lorentz force law

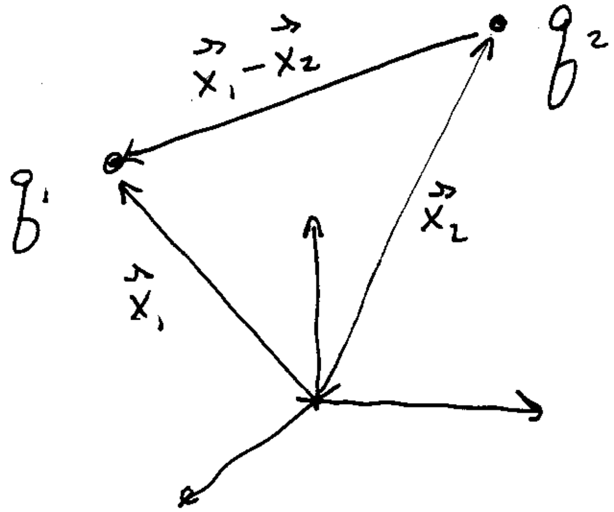
$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

Electrostatics $\quad \vec{E}, \rho$ time independent

$\vec{J}, \vec{B} = 0$

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \qquad \vec{\nabla} \times \vec{E} = 0$$

Begin with Coulomb force law



$$\vec{F}_{\text{on 1 due to 2}} = q_1 q_2 \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3}$$

Electric field

$$\text{field at } \vec{x} \text{ due to 2} = q_2 \frac{\vec{x} - \vec{x}_2}{|\vec{x} - \vec{x}_2|^3} = \vec{E}(\vec{x})$$

$$\vec{F}_{\text{on 1 due to 2}} = q_1 \vec{E}(\vec{x}_1)$$

for system of charges

$$\vec{E}(\vec{x}) = \sum_i q_i \frac{\vec{x} - \vec{x}_i}{|\vec{x} - \vec{x}_i|}$$

$$q_i \rightarrow \rho(\vec{x}')d^3x'$$

$$\vec{E}(\vec{x}) = \int \rho(\vec{x}') \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} d^3x'$$

formally $\rho(\vec{x}) = q\,\delta(\vec{x} - \vec{x}_1)$ for pt charge at $\vec{x} = \vec{x}_1$

We see that

$$\frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} = -\vec{\nabla} \frac{1}{|\vec{x} - \vec{x}'|}$$

Proof $|\vec{x} - \vec{x}'| = \left(x^2 + x'^2 - 2\vec{x}\cdot\vec{x}'\right)^{1/2}$

$$\vec{\nabla}_x |\vec{x} - \vec{x}'| = \frac{1}{2}(\quad)^{-1/2} \times (2\vec{x} - 2\vec{x}')$$

so $\nabla_x |\vec{x} - \vec{x}'| = \dfrac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|}$

$$\vec{\nabla} \frac{1}{|\vec{x} - \vec{x}'|} = -\frac{1}{2} (\quad)^{-3/2} (2\vec{x} - 2\vec{x}') = -\frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} \quad \checkmark$$

So $\vec{E} = -\vec{\nabla} \Phi \qquad \Phi = \displaystyle\int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x'$

Note $\vec{\nabla} \times \vec{E} = \vec{\nabla} \times \vec{\nabla} \Phi = 0$

<u>Alternative</u>

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \qquad \vec{\nabla} \times \vec{E} = 0$$

second equation implies $\vec{E} = -\vec{\nabla} \Phi$

then $\nabla^2 \Phi = -4\pi\rho$

Solve using Green's functions

first solve for δ-fn source

$$\nabla^2 G(\vec{x},\vec{x}') = -4\pi\,\delta^3(\vec{x}-\vec{x}')$$

then $$\Phi(\vec{x}) = \int d^3x'\, G(\vec{x},\vec{x}')\,\rho(\vec{x}')$$

we now show

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|}$$

i.e. $$\nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = -4\pi\,\delta(\vec{x}-\vec{x}')$$

let $\vec{r} = \vec{x}-\vec{x}'$

$$\nabla^2 \frac{1}{r} = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \left(\frac{1}{r}\right) = 0$$ except where $r = 0$

Integrate $\nabla^2 \frac{1}{r}$ over ball of radius $r = R$ centered on origin

$$\int \nabla^2 \frac{1}{r}\, d^3r = \int \left(\vec{\nabla} \cdot \vec{\nabla} \frac{1}{r}\right) d^3r$$

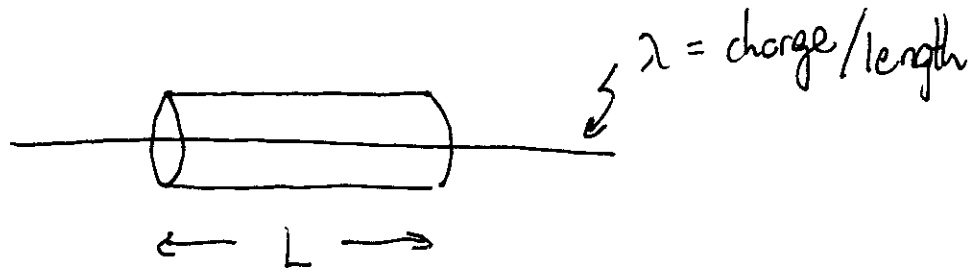
$$= \oint \vec{\nabla} \frac{1}{r} \cdot \hat{n}\, da = \oint \left(-\frac{1}{R^2}\right) R^2\, d\Omega$$

$$= -4\pi$$

Gauss's law

$$\int_V \vec{\nabla} \cdot \vec{E}\, d^3x = \oint_S \vec{E} \cdot \hat{n}\, da = \int_V d^3x\, 4\pi\rho$$

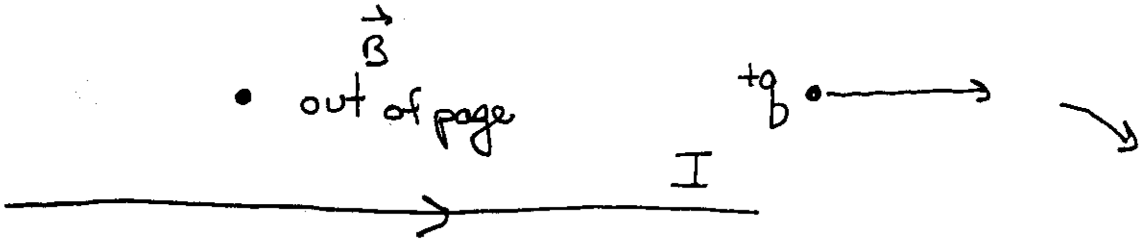
e.g. infinite charged wire



$$\vec{E} = E_s \hat{s}$$

$$2\pi s E_s \cancel{L} = 4\pi \lambda \cancel{L} \qquad E_s = \frac{2\lambda}{s}$$

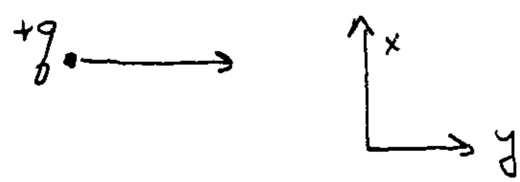
Magnetic force law $\vec{F} = q \frac{\vec{v}}{c} \times \vec{B}$

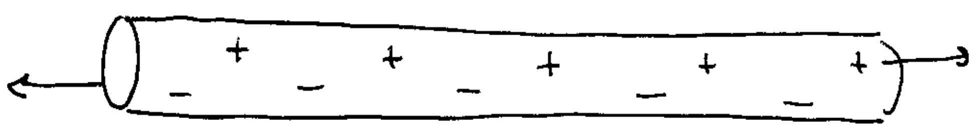


× $\vec{B}$ out into page

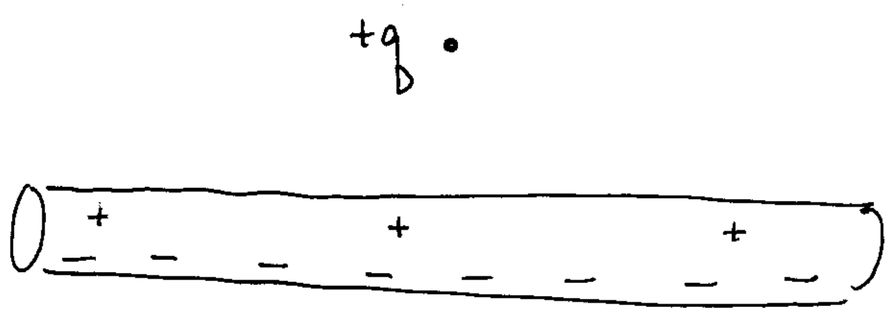
positive charge moving with current deflected toward current – like currents attract!

<u>first hint of relativity</u>





frame in which charge is moving
Lorentz force in −x direction



frame in which charge is at rest
electric force in −x direction

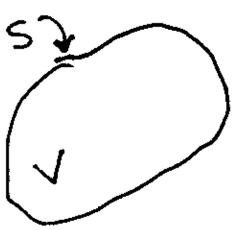
Magnetostatics $\vec{J}, \vec{B}$ time-independent

In general, charge is conserved

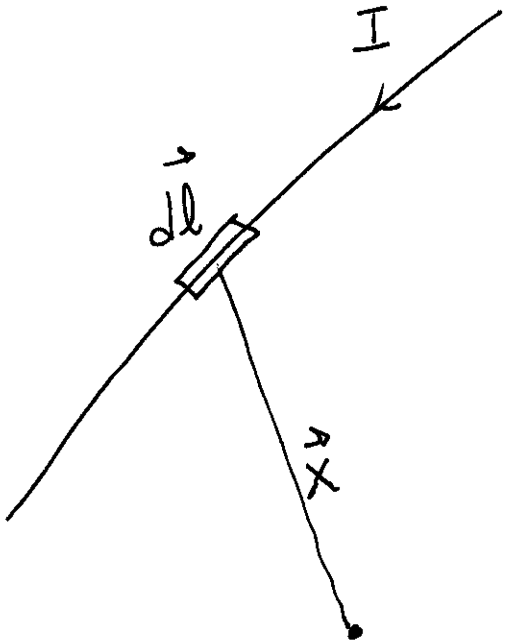
$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \qquad \text{continuity equation}$$

$$\int_V d^3x \left(\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} \right) = \underbrace{\frac{d}{dt} Q}_{\text{change of charge in } V} + \underbrace{\oint_S da\, \hat{n} \cdot \vec{J}}_{\text{flux out of } V \text{ thru } S} = 0$$

$$= 0$$



in magnetostatics, $\vec{\nabla} \cdot \vec{J} = 0$



$$d\vec{B} = \frac{I}{c}\,\frac{d\vec{l}\times\vec{x}}{|\vec{x}|^3}$$

but only makes sense for closed loop $(\vec{\nabla}\cdot\vec{J}=0)$

$[I]$ = statcoulombs/s = statamps

$[\vec{J}]$ = statcoulombs/s/cm^2 = statamps/cm^2

$$I\,d\vec{l} \longrightarrow J(\vec{x})\,d^3x$$

$J \sim I\,\delta^2(\vec{x}')$

δ-fn transverse to wire

$$\vec{B}(\vec{x}) = \frac{1}{c}\int \vec{J}(\vec{x}')\times\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\,d^3x'$$

$$= \frac{1}{c}\int\left(\vec{\nabla}_x\frac{1}{|\vec{x}-\vec{x}'|}\right)\times\vec{J}(\vec{x}')\,d^3x'$$

$$\vec{B}(\vec{x}) = \vec{\nabla} \times \left(\frac{1}{c} \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x' \right)$$

$$= \vec{\nabla} \times \vec{A} \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{B} = 0 \text{ automatically}$$

$$\vec{A}(\vec{x}) = \frac{1}{c} \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x' + \vec{\nabla}\psi$$

gradient of scalar
gauge freedom

Consider $\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A})$

$$= \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) = \frac{1}{c}\vec{\nabla}\left(\vec{\nabla}\cdot\int\frac{\vec{J}(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|}\right)$$

$$= \frac{1}{c}\vec{\nabla}\int d^3x'\,\vec{J}(\vec{x}')\cdot\vec{\nabla}\frac{1}{|\vec{x}-\vec{x}'|}$$

$$= -\frac{1}{c}\vec{\nabla}\int d^3x'\,\vec{J}(\vec{x}')\cdot\vec{\nabla}'\frac{1}{|\vec{x}-\vec{x}'|}$$

$$= \frac{1}{c}\vec{\nabla}\int d^3x'\left(\vec{\nabla}'\cdot\vec{J}(\vec{x}')\right)\frac{1}{|\vec{x}-\vec{x}'|} \quad + \text{bdy term}$$

$$= 0$$

second term: use $\nabla^2\frac{1}{|\vec{x}-\vec{x}'|} = -4\pi\,\delta(\vec{x}-\vec{x}')$

$$\vec{\nabla}\times\vec{B} = -\frac{1}{c}\int d^3x'\,\vec{J}(\vec{x}')\,\nabla^2\frac{1}{|\vec{x}-\vec{x}'|}$$

$$= \frac{4\pi}{c}\vec{J}$$