Wave equation

linearity allows for superposition

$$\partial_\alpha F^{\alpha\beta} = \partial_\alpha(\partial^\alpha A^\beta - \partial^\beta A^\alpha) \underset{\substack{\text{Lorentz}\\\text{gauge}}}{=} \partial_\alpha \partial^\alpha A^\beta = \frac{4\pi}{c} J^\beta$$

Solve $\quad \partial_\alpha \partial^\alpha_x D(x,x') = \delta^4(x-x')$

x - spacetime coordinate of observer

x' - coordinate of source

D describes response at x to idealized source at x'

$$\delta^4(x-x') = \delta(ct-ct')\,\delta(x-x')\,\delta(y-y')\,\delta(z-z')$$

Armed with $D(x,x')$ we can write

$$\vec{A}^\alpha(x) = \frac{4\pi}{c}\int d^4x'\, D(x,x')\, J^\alpha(x')$$

$$\Box_x A^\alpha(x) = \frac{4\pi}{c}\int d^4x'\, \Box_x D(x,x')\, J^\alpha(x')$$

$$= \frac{4\pi}{c} \int d^4x' \, \delta^4(x-x') \, J^\alpha(x')$$

$$= \frac{4\pi}{c} J^\alpha(x)$$

Now we could imagine BC's - here we'll assume none (apart from assumption that fields are finite at infinity)

$\Rightarrow$ $D(x, x')$ depends only on $x - x'$

$$z^\alpha = x^\alpha - x^{\alpha'} \qquad \Box_x = \Box_z$$

$$\Box_z D(z) = \delta^4(z)$$

Fourier analysis

$$D(z) = \left(\frac{1}{2\pi}\right)^4 \int d^4k\, e^{-ik\cdot z}\, \tilde{D}(k)$$

$$\int d^4z\, e^{ik'\cdot z}\, D(z) = \left(\frac{1}{2\pi}\right)^4 \int d^4k \int d^4z\, e^{i(k'-k)\cdot z}\, \tilde{D}(k)$$

$$= \int d^4k\, \tilde{D}(k)\, \delta^4(k-k') = \tilde{D}(k')$$

$$\Box_z D_z = \left(\frac{1}{2\pi}\right)^4 \int d^4k\, \tilde{D}(k)\, e^{-ik\cdot z} (-k^2)$$

$$= \delta^4(z) = \left(\frac{1}{2\pi}\right)^4 \int d^4k\, e^{-ik\cdot z}$$

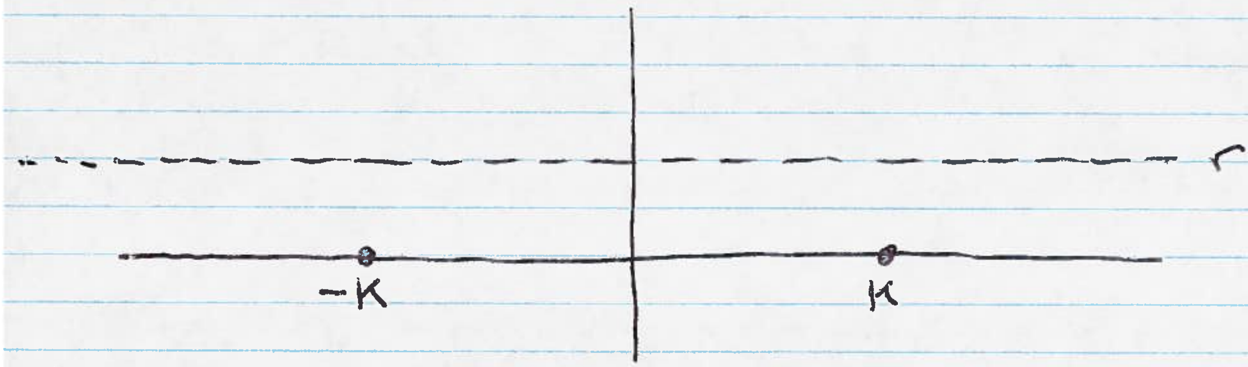
$$\Rightarrow \quad \tilde{D}(k) = -\frac{1}{k^2}$$

$$D(z) = \left(\frac{1}{2\pi}\right)^4 \int d^4k\, e^{-ik\cdot z}\, \frac{1}{k\cdot k}$$

$$= \left(\frac{1}{2\pi}\right)^4 \int d^3k \, e^{i\vec{k}\cdot\vec{z}} \int_{-\infty}^{\infty} dk_0 \, \frac{e^{-ik_0 z_0}}{k_0^2 - K^2}$$

$$K = |\vec{k}|$$

Treat k_0 as complex



$$\int_{-\infty}^{\infty} dk_0 \, \frac{e^{-ik_0 z_0}}{k_0^2 - K^2} = \oint dk_0 \, \frac{e^{-ik_0 z_0}}{k_0^2 - K^2}$$

If we can close contour where integrand vanishes

$$= -2\pi i \, \mathrm{Res}\left(\frac{e^{-ik_0 z_0}}{k_0^2 - K^2}\right)$$

for $z_0 < 0$ want $\mathrm{Im}(k_0) > 0 \Rightarrow -ik_0 z_0 \longrightarrow -\infty$
in semi-circle

$\Rightarrow$ close in UHP $\Rightarrow$ Res( ) = 0 $\Rightarrow$ D = 0

for $z_0 > 0$ close in LHP

$$\oint dk_0 \frac{e^{-ik_0 z_0}}{k_0^2 - K^2} = -2\pi i \left(\frac{e^{iKz_0}}{-2K} + \frac{e^{-iKz_0}}{2K} \right)$$

$$= -\frac{2\pi}{K} \left(\frac{e^{iKz_0} - e^{-iKz_0}}{2i} \right)$$

$$= -\frac{2\pi}{K} \sin K z_0$$

$$D_r(z) = \frac{\theta(z_0)}{(2\pi)^3} \int d^3k \, e^{i\vec{k}\cdot\vec{z}} \frac{\sin K z_0}{K}$$

$d^3k = 2\pi K^2 dK \, d\cos\theta \qquad \vec{k}\cdot\vec{z} = KR\cos\theta \quad R = |\vec{z}|$
$= |\vec{x} - \vec{x}'|$

θ-integral $\displaystyle\int_{-1}^{1} d\cos\theta \, e^{ikR\cos\theta} = \frac{1}{ikR}\left(e^{ikR} - e^{-ikR}\right)$

$$D_r = \frac{\theta(z_0)}{(2\pi)^2}\frac{1}{R}\int_0^\infty dk\left(\frac{e^{ikR} - e^{-ikR}}{i}\right)\left(\frac{e^{ikz_0} - e^{-ikz_0}}{2i}\right)$$

4 terms $e^{\pm ik(R \pm z_0)}$

combine in pairs + integrate from $-\infty$ to $+\infty$

$$= \frac{\theta(z_0)}{8\pi^2 R}\int_{-\infty}^{\infty} dk\left(e^{ik(z_0 - R)} - e^{ik(z+R)}\right)$$

$$= \frac{\theta(z_0)}{4\pi R}\left(\delta(z_0 - R) + \underbrace{\cancel{\delta(z_0 + R)}}_{0}\right)$$

0 from Heaviside fn.

$$= \frac{\theta(x_0 - x_0')}{4\pi R}\,\delta(x_0 - x_0' - R)$$

Recall $$\delta(f(x)) = \sum_i \frac{\delta(x_i)}{|(df/dx)|_i}$$ sum over zeros of f

$$\delta(x^2 - a^2) = \delta((x-a)(x+a)) = \frac{\delta(x-a) + \delta(x+a)}{2a}$$

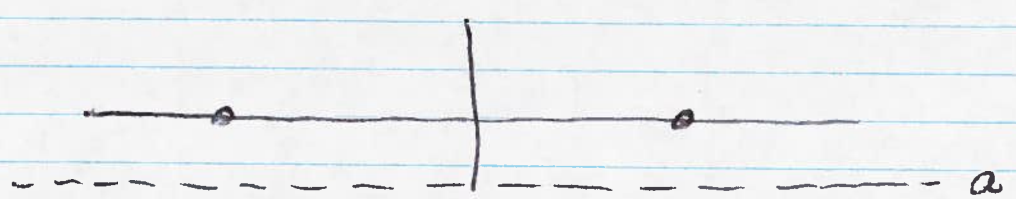
$$\delta[(x-x')^2] = \delta[(x_0 - x_0')^2 - |\vec{x} - \vec{x}'|^2]$$

$$= \delta[(x_0 - x_0')^2 - R^2]$$

$$= \frac{1}{2R}\left(\delta(x_0 - x_0' - R) + \cancelto{0 \text{ with } \Theta\text{-fn}}{\delta(x_0 - x_0' + R)}\right)$$

So $$D_r(x-x') = \frac{1}{2\pi}\Theta(x_0 - x_0')\,\delta[(x-x')^2]$$

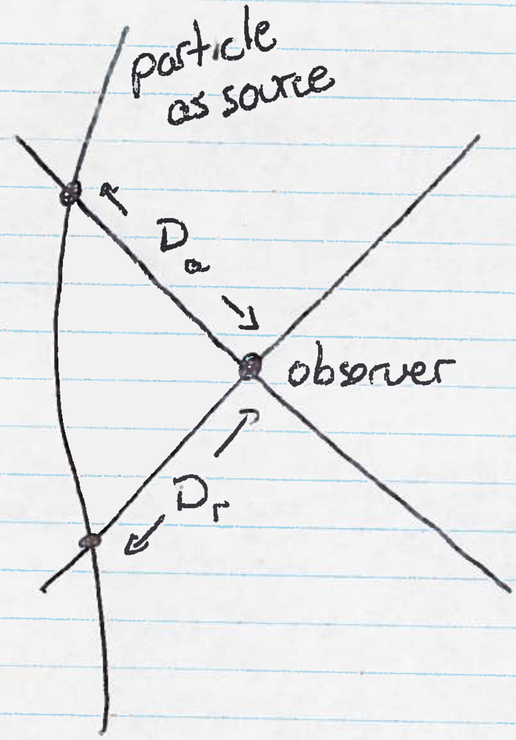
$$D_a(x-x') = \frac{1}{2\pi}\Theta(x_0' - x_0)\,\delta[(x-x')^2]$$



Lorentz invariant $\delta((x-x')^2)$

- source must be on light cone of observer

θ - function - picks either forward or backward light cone



$A_{in}(x)$ solution to source free equation $\Box A_{in} = 0$

$$A^\alpha(x) = A^\alpha_{in}(x) + \frac{4\pi}{c}\int d^4x' \, D_r(x-x') J^\alpha(x')$$

$$= A^\alpha_{out} + \frac{4\pi}{c}\int d^4x' \, D_a(x-x') J^\alpha(x')$$

Single point charge

$$\rho(\vec{x},t) = e\delta^3(\vec{x}-\vec{r}(t))$$

$$\vec{J}(\vec{x},t) = e\vec{v}(t)\delta^3(\vec{x}-\vec{r}(t))$$

$\vec{r}(t)$ path of particle in "lab" $\quad \vec{v} = d\vec{r}/dt$

$$J^\alpha = (c\rho, \vec{J}) = e(c, \vec{v}(t))\delta^3(\vec{x}-\vec{r}(t))$$

Not obviously covariant (4-vector)

$$\rho = e\int dt'\,\delta(t-t')\,\delta^3(\vec{x}-\vec{r}(t'))$$

$$r^\alpha = (ct', \vec{r}) \qquad \delta(t-t') = c\,\delta(x^0-r^0)$$

$$\rho = ec\int dt'\,\delta^4(x-r)$$

$$= ec\int \frac{dt'}{d\tau}\,d\tau\,\delta^4(x-r)$$

$$J^0 = c\rho = ec\int d\tau\, u^0\,\delta^4(x-r(\tau)) \qquad u^\alpha = \left(c\frac{dt}{d\tau}, \frac{d\vec{r}}{d\tau}\right)$$

and so we have

$$J^\alpha(x) = ec\int d\tau\, u^\alpha(\tau)\,\delta^4(x - r(\tau))$$

manifestly covariant

$\delta^4(x)$ is Lorentz invariant

$\int d^4x\, \delta^4(x) = \begin{cases} 1 \\ 0 \end{cases}$

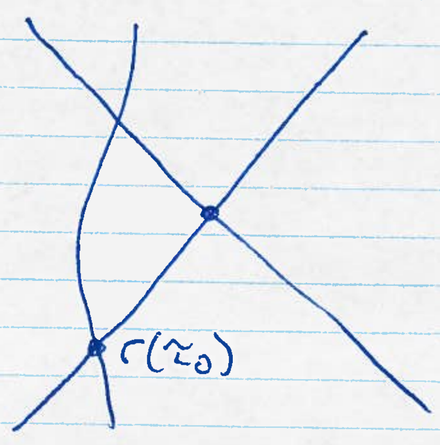
$$A^\alpha(x) = \frac{\cancel{4\pi}^{\,2}}{\cancel{c}}\int d^4x' \,\frac{1}{\cancel{2\pi}}\,\Theta(x_0 - x_0')\,\delta[(x-x')^2]$$

$$\times\, e\cancel{c}\int d\tau\, u^\alpha(\tau)\,\delta^4(x' - r(\tau))$$

Do x' integral (trivially)

$$= 2e\int d\tau\, u^\alpha(\tau)\,\Theta(x_0 - r_0(\tau))\,\delta[(x - r(\tau))^2]$$

light cone condition $(x - r(\tau_0))^2 = 0$



imposed by δ-function

defines τ_0 - retarded time

$$\delta\left((x-r(\tau))^2\right) = \frac{\delta(\tau-\tau_0)}{\left| df/d\tau \right|_{\tau=\tau_0}}$$

$$f = (x - r(\tau))^2$$

$$\frac{df}{d\tau} = -2(x-r(\tau))^\alpha \frac{dr_\alpha}{d\tau}$$

so $$A^\alpha = e \left.\frac{u^\alpha(\tau)}{(x-r)\cdot u}\right|_{\tau=\tau_0}$$

Manifestly a 4-vector!

term in denominator is a scalar dot-product.

Liénard-Wiechert potentials

$$u \cdot (x - r(\tau_0)) = u_0 (x_0 - r_0(\tau_0)) - \vec{u} \cdot (\vec{x} - \vec{r})$$

$$= \gamma c R - \gamma \vec{v} \cdot \hat{n} R$$

where $\vec{x} - \vec{r} = R \hat{n}$ and $x_0 - r_0(\tau_0) = R$

NB. $(x - r(\tau))^2 = (x_0 - r_0)^2 - |\vec{x} - \vec{r}|^2 = R^2 - R^2 = 0$ light cone

so $u \cdot (x - r) = \gamma c R (1 - \vec{\beta} \cdot \hat{n})$

$$\Phi(\vec{x}, t) = \left. \frac{e}{(1 - \vec{\beta} \cdot \hat{n}) R} \right|_{ret}$$

for particle at rest we recover Coulomb potential

$$\vec{A}(\vec{x}, t) = \left. \frac{e \vec{\beta}}{(1 - \vec{\beta} \cdot \hat{n}) R} \right|_{ret}$$

Now find $\vec{E}$ and $\vec{B}$ or $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$

$$\partial^\alpha A^\beta = 2e\, \partial^\alpha \left\{ \int d\tau\, u^\beta(\tau)\, \Theta[x_0 - r_0(\tau)]\, \delta[(x - r(\tau))^2] \right\}$$

derivative acts on x_0 and x

$$\partial^0\, \Theta(x_0 - r_0(\tau)) = \delta(x_0 - r_0(\tau))$$

But $\delta(x_0 - r_0(\tau))\, \delta[(x - r(\tau))^2]$ corresponds to

$R = 0$ case – rather pathological (particle sits on top of observer) – ignore

so $$\partial^\alpha A^\beta = 2e \int d\tau\, u^\beta(\tau)\, \Theta(x_0 - r_0(\tau))\, \partial^\alpha \delta[(x - r(\tau))^2]$$

Need $\partial^\alpha \delta(f)$ where $f = (x - r)^2$

$$\partial^\alpha \delta(f) = \partial^\alpha f \frac{d}{df} \delta(f) = \partial^\alpha f \frac{d\tau}{df} \frac{d}{d\tau} \delta(f)$$

$$\partial^\alpha f = 2(x-r)^\alpha \qquad \frac{df}{d\tau} = -2u\cdot(x-r)$$

so $$\partial^\alpha A^\beta = -2e\int d\tau\, \theta(x_0 - r_0)\, \frac{u^\beta (x-r)^\alpha}{u\cdot(x-r)}\, \frac{d}{d\tau}\, \delta\left[(x-r)^2\right]$$

integrate by parts

$$= 2e\int d\tau\, \theta(x_0 - r_0)\, \delta\left[(x-r)^2\right] \frac{d}{d\tau}\left[\frac{(x-r)^\alpha u^\beta}{u\cdot(x-r)}\right]$$

again use trick that

$$\delta\left[(x-r)^2\right] = \frac{\delta(\tau-\tau_0)}{2u\cdot(x-r)}$$

$$\partial^\alpha A^\beta = \frac{e}{u\cdot(x-r)}\, \frac{d}{d\tau}\left[\frac{(x-r)^\alpha u^\beta}{u\cdot(x-r)}\right]\Bigg|_{\tau=\tau_0}$$

$\tau=\tau_0$ retarded time

$$F^{\alpha\beta} = \frac{e}{u\cdot(x-r)}\frac{d}{d\tau}\left[\frac{(x-r)^\alpha u^\beta - (x-r)^\beta u^\alpha}{u\cdot(x-r)}\right]_{\tau=\tau_0}$$

manifestly, an anti-symmetric 2 index tensor!

For actual problems, we'd like to write in terms of particle's ordinary velocity + acceleration + position (say in lab frame)

$$(x-r)^\alpha = \left(x_0 - r_0(\tau), \vec{x}-\vec{r}\right)$$

$$= \left(R, R\hat{n}\right)$$ defines unit vector $\hat{n}$ + R

$$u^\alpha = \left(\gamma c, \gamma c\vec{\beta}\right)$$ 4-velocity

$$\frac{du^\alpha}{d\tau} = \left(c\frac{d\gamma}{d\tau}, c\left(\frac{d\gamma}{d\tau}\vec{\beta} + \gamma\frac{d\vec{\beta}}{d\tau}\right)\right)$$

$$\frac{d\gamma}{d\tau} = \frac{dt}{d\tau}\frac{d\gamma}{dt} = \gamma \frac{d}{dt}\frac{1}{(1-\beta^2)^{1/2}} = \gamma \frac{\vec{\beta}\cdot\dot{\vec{\beta}}}{(1-\beta^2)^{3/2}}$$

$$= \gamma^4 \vec{\beta}\cdot\dot{\vec{\beta}}$$

$$\frac{d}{d\tau}\Big(u\cdot(x-r)\Big) = (x-r)^\alpha \frac{du_\alpha}{d\tau} - \frac{dr^\alpha}{d\tau}u_\alpha$$

$$= (x-r)\cdot\frac{du}{d\tau} - c^2$$

When the dust settles

$$\vec{E}(\vec{x},t) = e\left[\frac{\hat{n}-\vec{\beta}}{\gamma^2(1-\vec{\beta}\cdot\hat{n})^3R^2}\right]_{ret}$$

$$+ \frac{e}{c}\left[\frac{\hat{n}\times\{(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\}}{(1-\vec{\beta}\cdot\hat{n})^3R}\right]_{ret}$$

1st term involves $\vec{\beta}$, not $\dot{\vec{\beta}}$ + goes as $1/R^2$ Coulomb

2nd term involves $\dot{\vec{\beta}}$ (is zero if $\dot{\vec{\beta}}=0$) goes as $1/R$

Radiation!

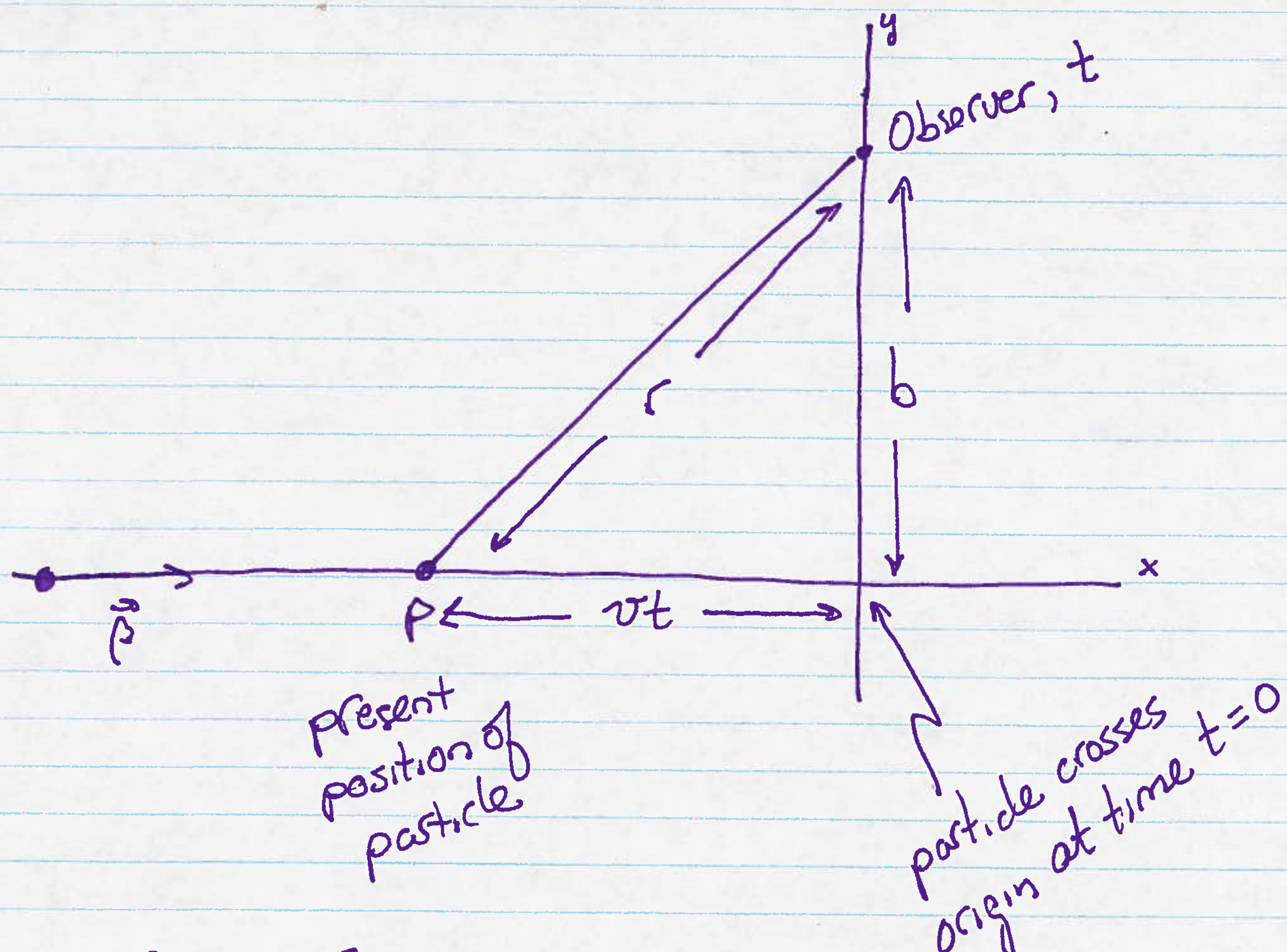
Find $\vec{B} = (\hat{n} \times \vec{E})_{ret}$

1) Particle at rest $\vec{E} = e\,\dfrac{\hat{n}}{R^2}$ $\vec{B} = 0$

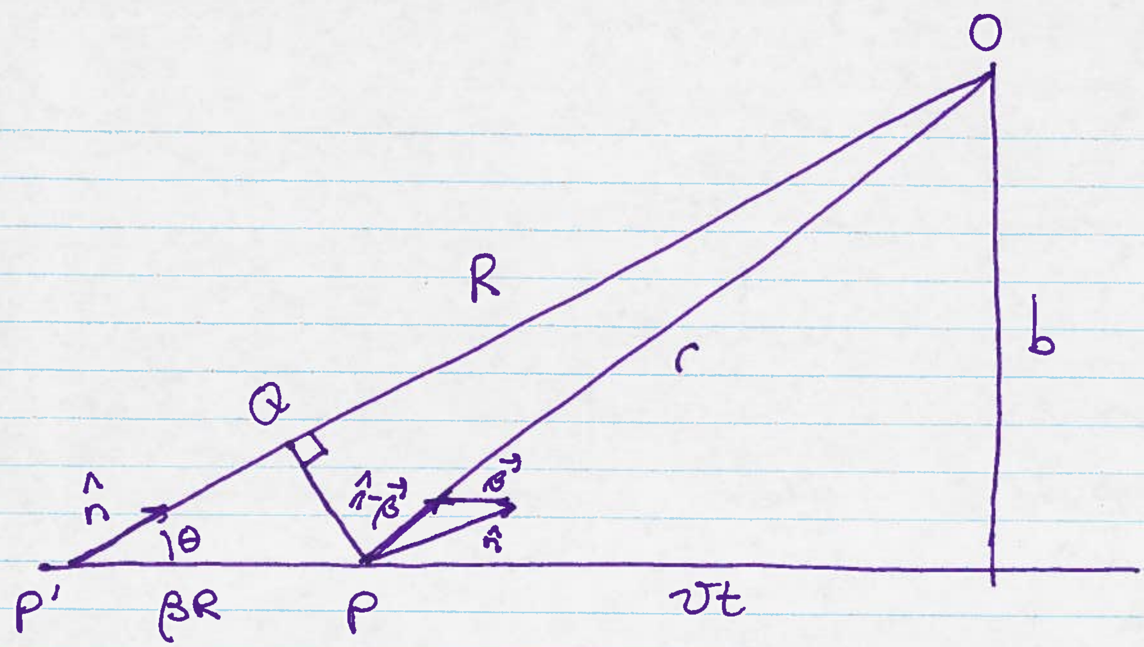
2) Uniform motion

$$\vec{E} = e\left(\frac{\hat{n} - \vec{\beta}}{\gamma^2(1-\vec{\beta}\cdot\hat{n})^3 R^2}\right)_{ret}$$

NB eva



$r^2 = b^2 + v^2t^2$ instantaneous position of particle gives distance r



R retarded distance $= c\Delta t$

Δt is light travel time from P' to observer

$$P'P = v\Delta t = \beta R$$

$$P'Q = \beta R\cos\theta = \vec{\beta}\cdot\hat{n}\,R$$

$$[QO]^2 = \left[\left(1-\vec{\beta}\cdot\hat{n}\right)R\right]^2$$ (denominator!)

$$= r^2 - (\beta R\sin\theta)^2$$

$\sin\theta = b/R$

$r^2 = v^2t^2 + b^2$

$$= b^2 + v^2t^2 - \beta^2 b^2$$

$$= (1-\beta^2)b^2 + v^2t^2 = \frac{1}{\gamma^2}\left(b^2 + \gamma^2 v^2 t^2\right)$$

so $$E_y = e \frac{\hat{n}\cdot\hat{y}}{\left(\gamma^2(1-\vec{\beta}\cdot\hat{n})^3 R^2\right)_{ret}} = \frac{\hat{n}\cdot\hat{y}\, R\gamma}{\left(\gamma^3(1-\vec{\beta}\cdot\hat{n})^3 R^3\right)_{ret}}$$

$$= \frac{e\gamma}{(b^2+\gamma^2 v^2 t^2)^{3/2}} \; R\sin\theta$$

$$= \frac{e\gamma b}{(b^2+\gamma^2 v^2 t^2)^{3/2}}$$

just as in previous section

$$E_x = e \frac{\hat{n}\cdot\hat{x} - \beta}{\gamma^2(1-\beta\cdot\hat{n})^3 R^2} = \frac{e\gamma}{R} \frac{1}{(b^2+\gamma^2 v^2 t^2)^{3/2}} (R\cos\theta - \beta R)$$

$$= \frac{e\gamma v t}{(b^2+\gamma^2 v^2 t^2)^{3/2}}$$

$$\vec{E} = \frac{e\gamma}{(b^2+\gamma^2 v^2 t^2)^{3/2}} \left(vt \, , \, b \, , \, 0 \right)$$

points to P not P′