Larmor's formula + Relativistic generalization

Consider particle which is accelerating but in a frame where $|\vec{\beta}| \ll 1$

$$\vec{E}_a = \frac{e}{c} \left. \frac{\hat{n} \times (\hat{n} \times \dot{\vec{\beta}})}{R} \right)_{ret}$$

$\vec{E}_a$ is "acceleration field"

term that goes like $1/R$ (ignore Coulomb term which goes like $1/R^2$)

$\vec{S}$ = energy flux energy/time/area

$$= \frac{c}{4\pi}\left(\vec{E} \times \vec{B}\right) = \frac{c}{4\pi}\left(\vec{E}_a \times (\hat{n} \times \vec{E}_a)\right)$$

$$= \frac{c}{4\pi}\left(|\vec{E}_a|^2 \hat{n} - \underbrace{\cancel{(\vec{E}_a \cdot \hat{n})}}_{0} \vec{E}_a\right)$$

$$= \frac{c}{4\pi} |\vec{E}_a|^2 \hat{n}$$

Now if we want flux in some polarization direction $\vec{\hat{p}}$

$$\vec{S}_{\hat{p}} = \frac{c}{4\pi} \left| \vec{E}_a \cdot \hat{p} \right|^2 \hat{n}$$

Power radiated into area element $\hat{n}_{obs}\, dA$ is

$$dP = \vec{S} \cdot \hat{n}_{obs}\, dA$$

standard practice to let $\hat{n}_{obs} = \hat{n} R^2 d\Omega$

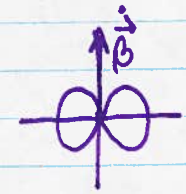
$$d\Omega = \sin\theta\, d\theta\, d\phi = d\cos\theta\, d\phi \qquad \text{solid angle element}$$

$$\frac{dP}{d\Omega} = \frac{c}{4\pi} \left| R \vec{E}_a \right|^2 \hat{n} = \frac{e^2}{4\pi c} \left| \hat{n} \times (\hat{n} \times \dot{\vec{\beta}}) \right|^2$$

$$\left| \hat{n} \times (\hat{n} \times \dot{\vec{\beta}}) \right|^2 = \left| \dot{\vec{\beta}} - (\hat{n} \cdot \dot{\vec{\beta}})\hat{n} \right|^2 = \dot{\beta}^2 - 2(\hat{n} \cdot \dot{\vec{\beta}})^2 + (\hat{n} \cdot \dot{\vec{\beta}})^2$$

$$= \dot{\beta}^2 \left(1 - (\hat{n} \cdot \hat{\dot{\beta}})^2\right) = \dot{\beta}^2 \sin^2 \Theta$$

Θ is angle between $\hat{n}$ and $\dot{\vec{\beta}}$



dipole pattern!

for total power, integrate over all angles

- take z-axis to be along $\dot{\vec{\beta}}$-direction

$$P = \int_{-1}^{1} d\cos\Theta \int_{0}^{2\pi} d\phi \; \frac{e^2}{4\pi c} \dot{\beta}^2 \sin^2\Theta$$

$$\int_{-1}^{1} du \, (1-u^2) = 2 - \frac{2}{3} = \frac{4}{3}$$

$$P = \frac{2}{3} \frac{e^2}{c^3} |\dot{\vec{v}}|^2$$

Larmor formula

Relativistic generalization

Key $\quad P$ is a Lorentz invariant

Conservation law $\qquad T^{\mu\nu} = \left(\begin{array}{c|c} u & \vec{S}/c \\ \hline \vec{S}/c & T_{ij} \end{array}\right)$

$$u = \frac{1}{8\pi}\left(E^2 + B^2\right)$$

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}$$

$\partial_\mu T^{\mu\nu} = 0 \qquad$ conservation of energy and momentum

$$\partial_\mu T^{\mu 0} = \frac{1}{c}\left(\frac{\partial}{\partial t}u + \vec{\nabla}\cdot\vec{S}\right) = 0$$

Integrate over finite volume

$$\frac{d}{dt}\int d^3x\, u = -\int d^3x\, \vec{\nabla}\cdot\vec{S} = -\oint da\, \hat{n}\cdot\vec{S}$$

$$= -P$$

so we have $\quad P\,dt = -d\left\{\int d^3x\, u\right\}$

or $\quad \int P\,dt = \int_{init} d^3x\, u - \int_{final} d^3x\, u$

$u = T^{00}$ transforms as t-t part of 2 index tensor

What about $u\,d^3x$?

u transforms same as $A^0 dt$ $\qquad$ dt is t part of 4-vector

$u\,d^3x \sim A^0\,dt\,d^3x \sim A^0 d^4x$ $\qquad$ transforms as time part of 4-vector

But $\quad d^3x\, u = P\,dt \quad$ so P must be Lorentz scalar!

NR limit

$$P = \frac{2}{3}\frac{e^2}{m^2c^3}\frac{d\vec{p}}{dt}\cdot\frac{d\vec{p}}{dt}$$

$$\longrightarrow -\frac{2}{3}\frac{e^2}{m^2c^3}\frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau}$$

$$-\frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau} = \frac{d\vec{p}}{d\tau}\cdot\frac{d\vec{p}}{d\tau} - \frac{1}{c^2}\left(\frac{dU}{d\tau}\right)^2$$

$$U^2 = c^2p^2 + m^2c^4 \qquad U\,dU = c^2p\,dp \qquad dU = \frac{c^2p}{U}dp$$

$$= c\beta\,dp$$

$$P = \frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{d\vec{p}}{d\tau}\cdot\frac{d\vec{p}}{d\tau} - \beta^2\left(\frac{dp}{d\tau}\right)^2\right)$$

lots of algebra

$$P = \frac{2}{3}\frac{e^2}{c}\gamma^6\left(\dot{\beta}^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2\right)$$

Liénard result

Linac

$$\frac{d\vec{p}}{d\tau}\cdot\frac{d\vec{p}}{d\tau} = \left(\frac{dp}{d\tau}\right)^2$$

$$P = \frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{dp}{d\tau}\right)^2(1-\beta^2) \qquad 1-\beta^2 = \frac{1}{\gamma^2} = \left(\frac{d\tau}{dt}\right)^2$$

$$= \frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{dp}{dt}\right)^2$$

$$\frac{dp}{dt} = \frac{1}{c\beta}\frac{dU}{dt} = \frac{1}{c}\frac{dt}{dx}\frac{dU}{dt} = \frac{1}{c}\frac{dU}{dx}$$

change in energy
per unit distance
along Linac

$$\frac{P}{dU/dt} = \frac{\text{radiated power}}{\text{change in energy}}$$ must be small for acceleration

$$= \frac{2}{3}\frac{e^2}{m^2c^3}\frac{1}{v}\frac{dU}{dX}$$

take $v = c$

prefactor is $\frac{e^2}{m^2c^4} = \frac{e^2}{mc^2}\frac{1}{mc^2}$

$\frac{e^2}{mc^2}$ = classical radius of electron

$= 2.8 \times 10^{-13}$ cm

$= r_{cl}$

$$\frac{P}{dU/dt} = \frac{2}{3}\frac{d(E/mc^2)}{d(x/r_{cl})}$$

SLAC 3 km $\Delta E = 50$ GeV $= 100\, mc^2$

3 km $= 10^{18}\, r_{cl.}$ so $\frac{P}{dU/dt}$ is $\sim 10^{-16} \lll 1$

Circular accelerators $\quad |\vec{p}| \simeq \text{const}$

Write $\quad \vec{p} = |\vec{p}|(\cos\omega t\,\hat{x} \pm \sin\omega t\,\hat{y})$

$$\frac{d\vec{p}}{d\tau} = \gamma\omega|\vec{p}|(-\sin\omega t\,\hat{x} \pm \cos\omega t\,\hat{y})$$

so $\quad \dfrac{d\vec{p}}{d\tau}\cdot\dfrac{d\vec{p}}{d\tau} = \gamma^2\omega^2|\vec{p}|^2 \gg \left(\dfrac{dp}{d\tau}\right)^2$

and we have $\quad P = \dfrac{2}{3}\dfrac{e^2}{m^2c^3}\gamma^2\omega^2|\vec{p}|^2$

ω angular frequency $= v/R \qquad R$ radius of orbit

$|\vec{p}|^2 = \gamma^2 m^2 v^2$

$$P = \frac{2}{3}\frac{e^2}{\cancel{m^2}\cancel{c^3}}\gamma^2\frac{\cancel{c^2}\beta^2}{R^2}\gamma^2\cancel{m^2}c^{\cancel{2}}\beta^2 = \frac{2}{3}\frac{e^2c}{R^2}\beta^4\gamma^4$$

Relavent quantity $\delta E = P \times \delta T$ $\quad \delta T$ time/revolution

$$= \frac{2\pi R}{v}$$

$$\delta E = \frac{4\pi}{3}\frac{e^2}{R}\beta^3\gamma^4 \qquad \boxed{\gamma = \frac{E}{mc^2}}$$

goes as 4th power of E + −1 power of R !

$$\frac{\delta E}{\text{MeV}} = 8.85\times 10^{-2}\left(\frac{E}{\text{GeV}}\right)^4\left(\frac{m}{R}\right) \quad \text{for electrons}$$

FNAL $\quad E = \text{TeV} \quad R = 1\text{km} \quad \delta E \simeq 10^8\,\text{MeV}$ NOGO

NB 50,000 keV/s

$$\frac{M_e}{M_p} = \frac{1}{1836}$$

for protons $$\frac{\delta E}{\text{MeV}} = 7.79\times 10^{-15}\left(\frac{E}{\text{GeV}}\right)^4\left(\frac{m}{R}\right)$$

so at FNAL $\simeq 10^{-5}$

To go to 10 TeV, need a bigger ring