Thomson scattering

Incident wave causes particle to accelerate

Accelerating particle radiates.

Can be expressed as a scattering problem

Energy in incident wave scattered to other angles

⇒ Primer on plane waves + polarization

Wave equation in vacuum for $\vec{E}$

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \vec{E} = 0$$

$$\vec{E}(\vec{x}, t) = \vec{A} \cos(\vec{k} \cdot \vec{x} - \omega t) + \vec{B} \sin(\vec{k} \cdot \vec{x} - \omega t)$$

$$\vec{k} \cdot \vec{A} = \vec{k} \cdot \vec{B} = 0 \quad (\text{from } \vec{\nabla} \cdot \vec{E} = 0)$$

Pick two polarization vectors

$$\hat{\epsilon}_1 \,,\, \hat{\epsilon}_2 \quad \text{with} \quad \hat{\epsilon}_1 \cdot \vec{k} = \hat{\epsilon}_2 \cdot \vec{k} = 0$$

$$\hat{\epsilon}_1 \times \hat{\epsilon}_2 = \hat{k} \quad \text{and cyclic perm.} \quad \text{e.g. } \vec{k} = k\hat{z} \quad \hat{\epsilon}_1 = \hat{x} \quad \hat{\epsilon}_2 = \hat{y}$$

$$\vec{E}(\vec{x},t) = \left(A_1\hat{\epsilon}_1 + A_2\hat{\epsilon}_2\right)\cos\phi + \left(B_1\hat{\epsilon}_1 + B_2\hat{\epsilon}_2\right)\sin\phi$$

$$\phi = \vec{k}\cdot\vec{x} - \omega t$$

write using complex notation

$$\vec{E}(\vec{x},t) = Re\left(e^{i\phi}\left[(A_1 - iB_1)\hat{\epsilon}_1 + (A_2 - iB_2)\hat{\epsilon}_2\right]\right)$$

$$= Re\left(\vec{\mathcal{E}}\right) \qquad \vec{\mathcal{E}} \text{ complex electric field}$$

$$\vec{\mathcal{E}} = e^{i(\vec{k}\cdot\vec{x} - \omega t)} \left(E_1 \hat{\epsilon}_1 + E_2 \hat{\epsilon}_2 \right)$$

$$E_1 = A_1 - iB_1 = a_1 e^{i\delta_1}$$

$$E_2 = A_2 - iB_2 = a_2 e^{i\delta_2}$$

$\hat{\epsilon}_1, \hat{\epsilon}_2$ convenient basis to describe linear polarization

- case when $\delta_1 = \delta_2$
- then polarization angle is $\cos\varphi = a_1/a_2$

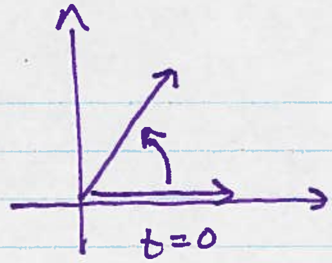
Alternate basis $\hat{\epsilon}_\pm = \dfrac{\hat{\epsilon}_1 \pm i\hat{\epsilon}_2}{\sqrt{2}}$ i.e. $\delta_1 = 0$, $\delta_2 = \pm\frac{\pi}{2}$

$$\vec{\mathcal{E}} = e^{i(\vec{k}\cdot\vec{x} - \omega t)} \left(E_+ \hat{\epsilon}_+ + E_- \hat{\epsilon}_- \right)$$

for $\hat{\epsilon}_1 = \hat{x}$, $\hat{\epsilon}_2 = \hat{y}$ $\quad \vec{\mathcal{E}} = e^{i(\vec{k}\cdot\hat{x} - \omega t)} E_0 \hat{\epsilon}_+$

$$\Rightarrow \vec{E} = \frac{E_0}{\sqrt{2}} \left(\hat{x}\cos(kz - \omega t) - \hat{y}\sin(kz - \omega t) \right)$$

at fixed $z = 0$



E-field rotates ccw as we look down $\vec{k}$
(toward approaching wave)

left circular polarized / positive helicity

$\hat{\epsilon}_-$ - right circular polarized - negative helicity

To pick out $\hat{\epsilon}$ polarization, compute $\hat{\epsilon}^* \cdot \vec{\mathcal{E}}$

Since $\hat{\epsilon}_+^* \cdot \hat{\epsilon}_+ = 1 \qquad \hat{\epsilon}_+^* \cdot \hat{\epsilon}_- = 0$ etc

Stokes parameters

In general, one can use either linear basis or circular polarization basis

$$\vec{E} = \left(\hat{\epsilon}_1 E_1 + \hat{\epsilon}_2 E_2\right) e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

$$= \left(\hat{\epsilon}_+ E_+ + \hat{\epsilon}_- E_-\right) e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

$$E_1 = a_1 e^{i\delta_1} \qquad E_2 = a_2 e^{i\delta_2}$$

$$E_+ = a_+ e^{i\delta_+} \qquad E_- = a_- e^{i\delta_-}$$

$$S_0 = |\hat{\epsilon}_1 \cdot \vec{E}|^2 + |\hat{\epsilon}_2 \cdot \vec{E}|^2 = a_1^2 + a_2^2$$

$$\hat{\epsilon}_1 = \frac{\hat{\epsilon}_+ + \hat{\epsilon}_-}{\sqrt{2}} \qquad \hat{\epsilon}_2 = \frac{\hat{\epsilon}_+ - \hat{\epsilon}_-}{\sqrt{2}\, i}$$

$$S_0 = |\hat{\epsilon}_+^* \cdot \vec{E}|^2 + |\hat{\epsilon}_2^* \cdot \vec{E}|^2 = a_+^2 + a_-^2$$

overall amplitude of wave

$$S_1 = |\hat{\epsilon}_1 \cdot \vec{E}|^2 - |\hat{\epsilon}_2 \cdot \vec{E}|^2$$ preponderance of x over y

$$= a_1^2 - a_2^2$$

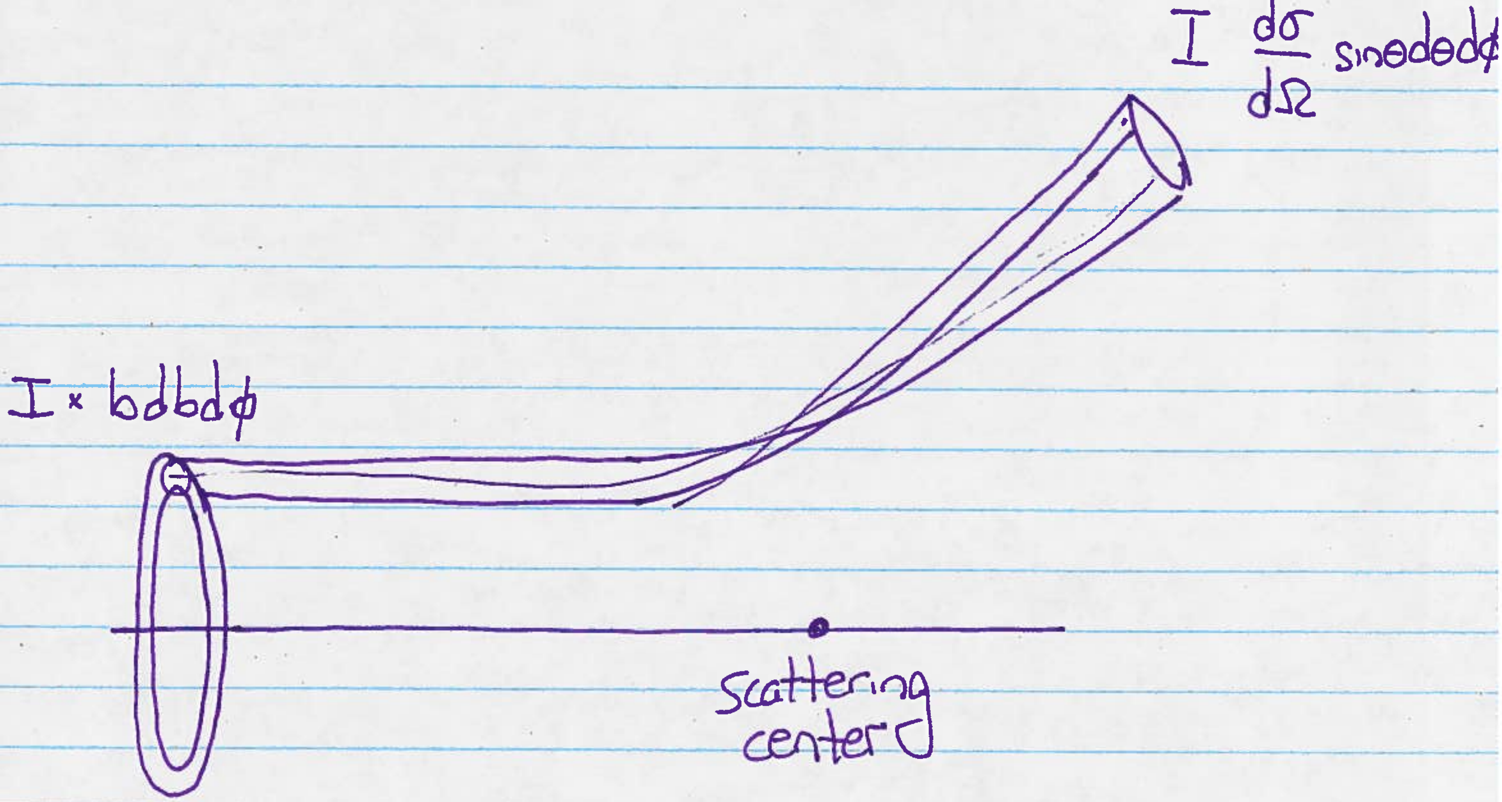
$$= 2 a_+ a_- \cos(\delta_+ - \delta_-)$$

$$S_2 = 2\,\mathrm{Re}\left[(\hat{\epsilon}_1 \cdot E)^* (\hat{\epsilon}_2 \cdot \vec{E})\right] = 2 a_1 a_2 \cos(\delta_2 - \delta_1)$$

$$S_3 = |\epsilon_+^* \cdot \vec{E}|^2 - |\epsilon_-^* \cdot \vec{E}|^2 = a_+^2 - a_-^2$$

in reality, 3 parameters $a_1, a_2, \delta_1 - \delta_2$

or $a_+, a_-, \delta_+ - \delta_-$



$I \times b\,db\,d\phi$
$I \dfrac{d\sigma}{d\Omega} \sin\theta\,d\theta\,d\phi$
scattering center

Recall $$\frac{dP}{d\Omega} = \frac{c}{4\pi}\left|R\vec{E}_a\right|^2$$

$$= \frac{e^2}{4\pi c^3}\left|\hat{n}\times(\hat{n}\times\dot{\vec{v}})\right|^2$$

Non-relativistic limit!

for radiation into polarization $\hat{\epsilon}$, we want

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}\left|\hat{\epsilon}^*\cdot\left(\hat{n}\times(\hat{n}\times\dot{\vec{v}})\right)\right|^2$$

$$= \frac{e^2}{4\pi c^3}\left|\hat{\epsilon}^*\cdot\left((\hat{n}\cdot\dot{\vec{v}})\hat{n}-\dot{\vec{v}}\right)\right|^2$$

$$= \frac{e^2}{4\pi c^3}\left|\hat{\epsilon}^*\cdot\dot{\vec{v}}\right|^2$$

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{e^2}{4\pi c^3}\left\langle\left|\hat{\epsilon}^*\cdot\dot{\vec{v}}\right|^2\right\rangle$$

Incident wave $\vec{E}(\vec{x},t) = \hat{\epsilon}_0 E_0 e^{i(\vec{k}_0\cdot\vec{x}-\omega t)}$

equation of motion $m\vec{a} = e\vec{E}$

so $\dot{\vec{v}} = \frac{e}{m}\hat{\epsilon}_0 E_0 e^{i(\vec{k}_0\cdot\vec{x}-\omega t)}$

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{e^2}{8\pi c^3}\frac{e^2}{m^2}|E_0|^2|\hat{\epsilon}^* \cdot \hat{\epsilon}_0|^2$$

factor of 1/2 when we time-averag

power/solid angle or energy/time/solid angle

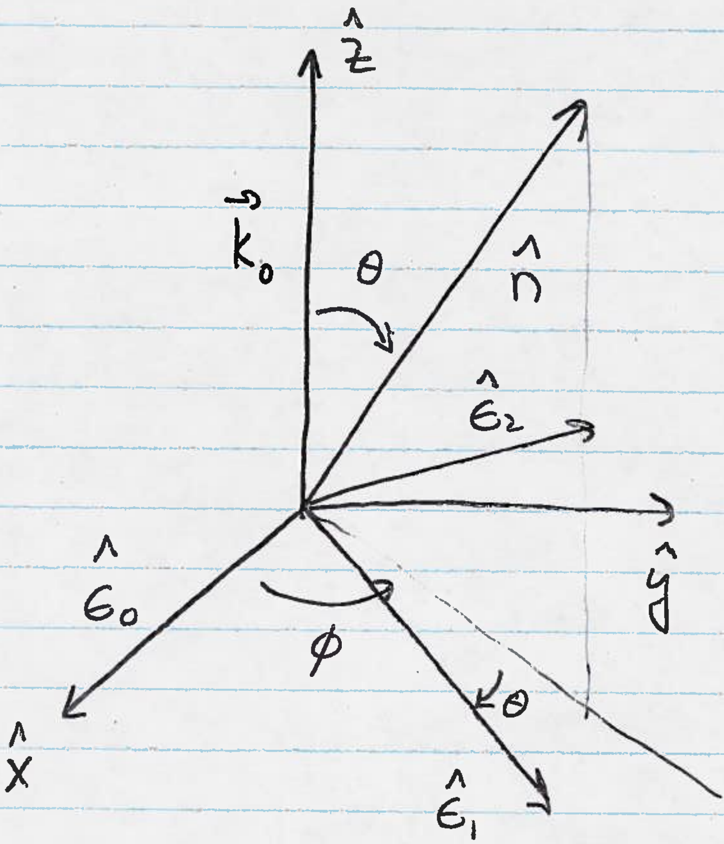
Incident wave $\langle\vec{S}\rangle = \frac{c}{4\pi}\langle\vec{E}\times\vec{B}\rangle$

$$= \frac{c}{8\pi}\vec{E}\cdot\vec{E}^*\,\hat{n} = \frac{c}{8\pi}|E_0|^2\,\hat{n}$$

$$\frac{\text{energy/time/solid angle}}{\text{energy/time/area}} = \frac{\text{area}}{\text{solid angle}} = \frac{d\sigma}{d\Omega}$$ differential cross section

$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2 \left|\hat{\epsilon}^* \cdot \hat{\epsilon}_0\right|^2}$$

$\frac{e^2}{mc^2}$ classical radius of the electron



$$\vec{k}_0 = k\hat{z}$$

$$\hat{\epsilon}_0 = \hat{x} \text{ or } \hat{y}$$ if linear

$$\hat{n} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$\hat{\epsilon}_1$ in plane containing $\vec{k}_0$ + $\hat{n}$

$$\hat{\epsilon}_1 \cdot \hat{n} = 0$$

$$= \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$

$$\hat{\epsilon}_2 = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

$\perp$ to both $\hat{\epsilon}_1$ + $\hat{n}$

for linear polarization with $\hat{\epsilon}_0 = \hat{x}$

$$\hat{\epsilon}_i^{*} \cdot \hat{\epsilon}_0 = \begin{cases} \cos\theta\cos\phi & i = 1 \\ -\sin\phi & i = 2 \end{cases}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2 \times \begin{cases} \cos^2\theta\cos^2\phi & \hat{\epsilon} = \hat{\epsilon}_1 \\ \sin^2\phi & \hat{\epsilon} = \hat{\epsilon}_2 \end{cases}$$

often interested in scattering cross section summed over final polarization states $\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2 \left(\cos^2\theta\cos^2\phi + \sin^2\phi\right)$

for $\hat{\epsilon}_0 = \hat{y}$ $\qquad \frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2 \left(\cos^2\theta\sin^2\phi + \cos^2\phi\right)$

If initial beam is unpolarized, average over $\hat{\epsilon}_0 = \hat{x}, \hat{y}$

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2 \left(\frac{1 + \cos^2\theta}{2}\right)$$

Thomson scattering formula

Total cross section

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega$$

$$= 2\pi \left(\frac{e^2}{mc^2}\right)^2 \int_{-1}^{1} \frac{1+x^2}{2} dx = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2$$

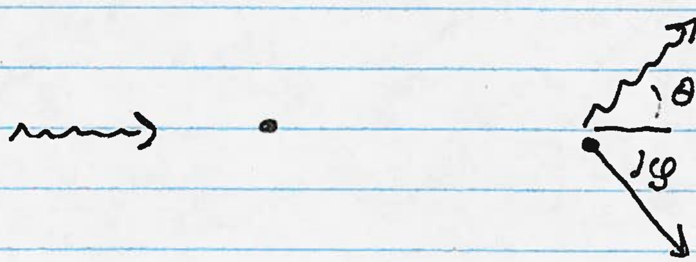
$= 0.665 \times 10^{-24}\ cm^2$

for electrons

Note: final ~~photon~~ EM wave has same wavelength as incident wave

Comes from our solution of eqn. of motion for electron.

What have we ignored?



Conservation of energy + momentum

$$m_e c^2 + E_\gamma = E_\gamma' + E_e'$$

$$p_\gamma = p_\gamma' \cos\theta + p_e' \cos\varphi$$

$$0 = p_\gamma' \sin\theta - p_e' \sin\varphi$$

eliminate quantities related to electron using

$$E_e'^2 - c^2 p_e'^2 (\cos^2\varphi + \sin^2\varphi) = m_e^2 c^4$$

$$(E_\gamma + m_e c^2 - E_\gamma')^2 - c^2 (p_\gamma - p_\gamma' \cos\theta)^2 - c^2 p_\gamma'^2 \sin^2\theta = m_e^2 c^4$$

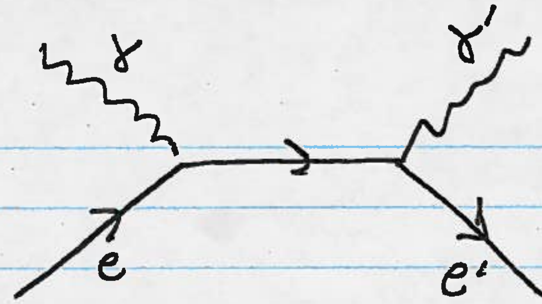
$$\cancel{E_\gamma^2} + \cancel{m_e^2 c^4} + \cancel{E_\gamma'^2} + \cancel{2}(E_\gamma - E_\gamma') m_e c^2 - \cancel{2} E_\gamma E_\gamma' - \cancel{c^2 p_\gamma^2} - \cancel{c^2 p_\gamma'^2} + \cancel{2} c^2 p_\gamma p_\gamma' \cos\theta = \cancel{m_e^2 c^4}$$

Now $E_\gamma = \hbar\omega = \hbar k c$

$$E_\gamma' (m_e c^2 + E_\gamma - E_\gamma \cos\theta) = E_\gamma m_e c^2$$

or $$\frac{E_\gamma'}{E_\gamma} = \frac{k'}{k} = \frac{1}{1 + \frac{\hbar\omega}{m_e c^2}(1 - \cos\theta)}$$ Compton formula

QM calculation



Ignore spin + find

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2 \left(\frac{k'}{k}\right)^2 \left|\hat{\epsilon}^* \cdot \hat{\epsilon}_0\right|^2$$

so cross section decreases relative to Thomson formula as we go to higher frequencies