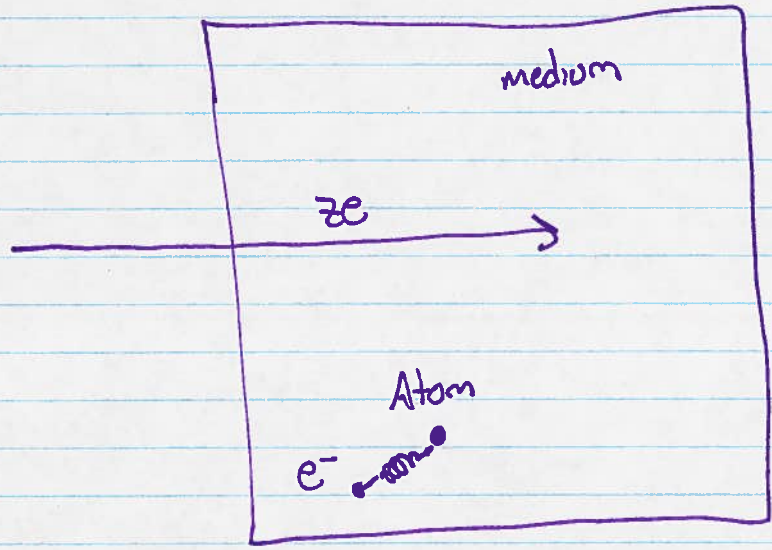


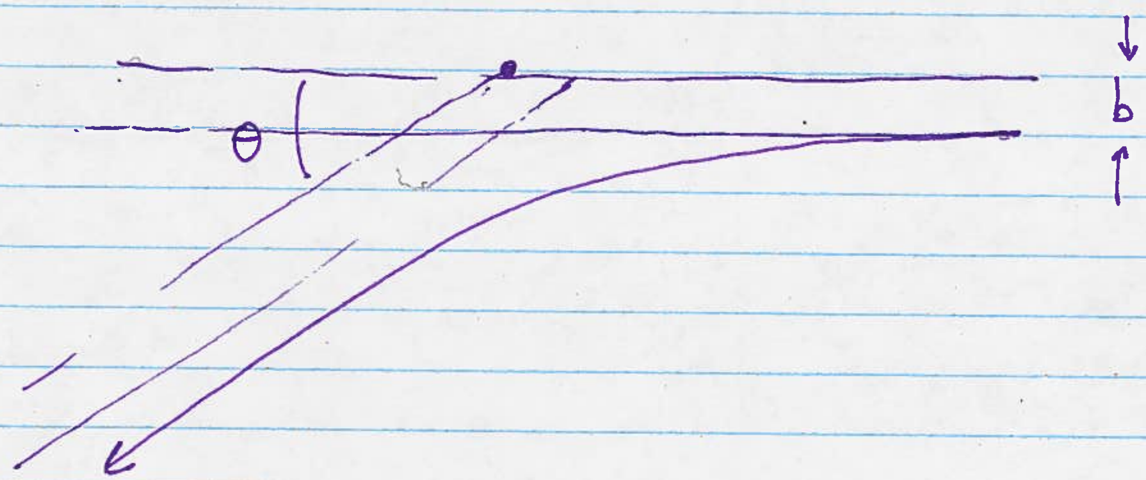

Here, interested in the interaction between the incoming particle + either the electrons or atomic nuclei:

- Rutherford scattering
- ionization / atomic transitions
- density effect

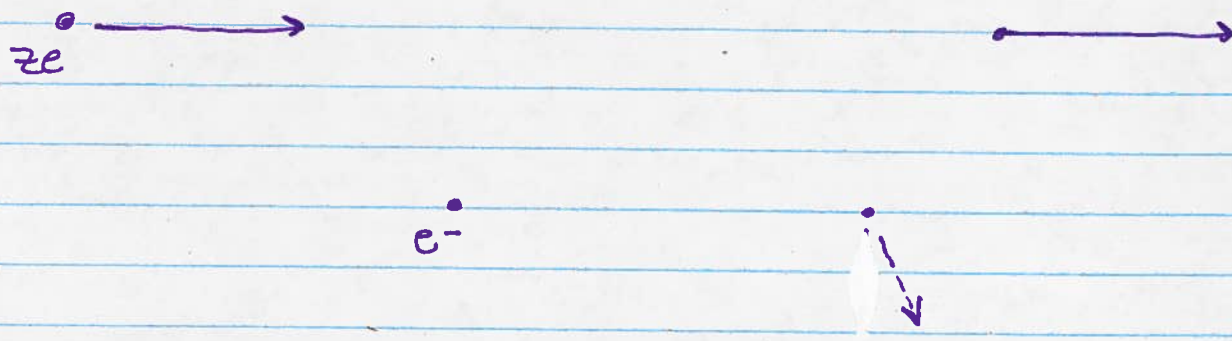
Bohr, Bethe, Fermi

if incoming particle is much heavier than $e^-$ (i.e. not an electron,

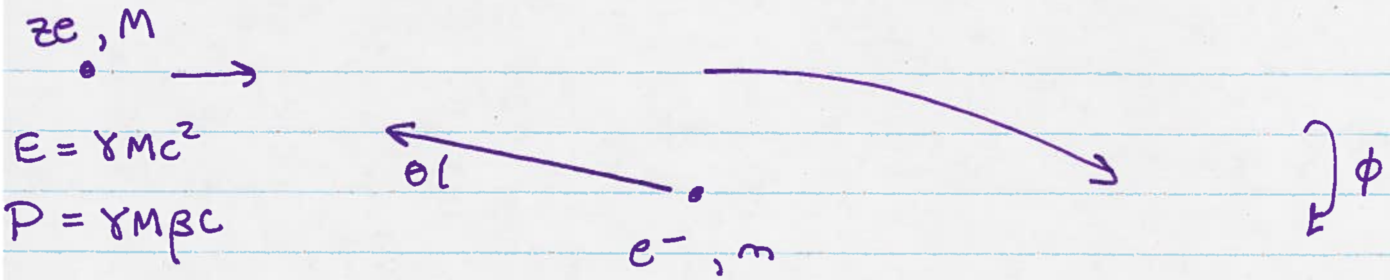
consider interaction in rest frame of incoming particle




lab




electron picks
up energy




View as elastic Rutherford scattering in center-of-mass frame = rest frame of heavy incipent particle

we are neglecting binding energy of atom!

NR Rutherford formula

$$\frac{d^2\vec{r}}{dt} = -\frac{ze^2}{mr^2}\hat{r}$$

or
$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 + \frac{ze^2}{mr^2} = 0$$

$$\frac{d^2\theta}{dt^2} + \frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt} = 0 \quad \Rightarrow \quad L = r^2\frac{d\theta}{dt} = \text{const}$$

so $$\frac{d^2r}{dt^2} - \frac{L^2}{r^3} + \frac{ze^2}{mr^2} = 0$$

$$\frac{dr}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta}\left(\frac{1}{u}\right) = -\frac{d\theta}{dt}\frac{1}{u^2}\frac{du}{d\theta} \qquad u = 1/r$$

$$= -L\frac{du}{d\theta}$$

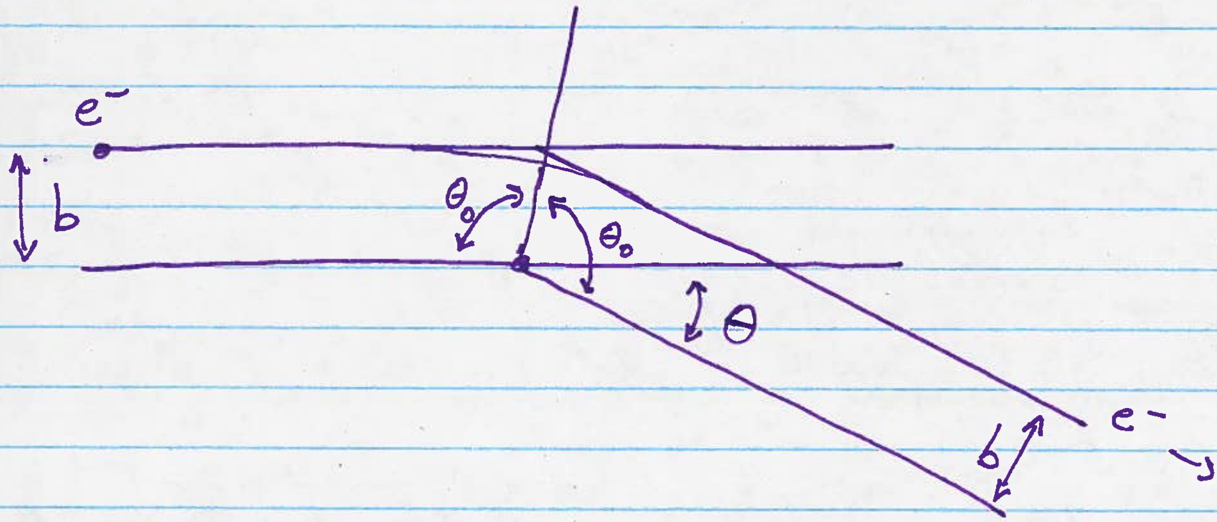
$$\frac{d^2r}{dt^2} = \frac{d\theta}{dt}\frac{d}{d\theta}\left(-L\frac{du}{d\theta}\right) = -\frac{L^2}{r^2}\frac{d^2u}{d\theta^2}$$

so we have $$\frac{d^2u}{d\theta^2} + u - \frac{ze^2}{mL^2} = 0$$

solution $$u = u_0\cos(\theta - \theta_0) + \frac{ze^2}{mv^2b} \qquad L = vb$$

also $$\frac{du}{d\theta} = -u_0\sin(\theta - \theta_0)$$

$\theta_0$ is angle at closest approach




scattering angle is $2\theta_0 - \pi = \theta$

find $\theta_0$ : $r \to \infty$, $u \to 0$, $\theta = 0$

$$u_0 \cos(\theta_0) = -\frac{ze^2}{mv^2 b^2}$$

$$u_0 \sin(\theta_0) = -\frac{v}{L}$$

eliminate $u_0$

$$\frac{1}{\tan\theta_0} = \frac{ze^2}{mv^2 b} = \frac{1}{\tan\left(\frac{\theta}{2} + \pi/2\right)}$$

$$= \tan\theta/2$$

$$\boxed{\tan\theta/2 = \frac{ze^2}{mv^2 b}}$$

or $\quad b = \dfrac{ze^2}{mv^2}\dfrac{1}{\tan\theta/2}$

for cross section

$$I\, b\, db\, d\phi = I \frac{d\sigma}{d\Omega} d\Omega$$

so $\quad \dfrac{d\sigma}{d\Omega} = b\left|\dfrac{db}{d\theta}\right|\dfrac{1}{\sin\theta}$

$$= \frac{ze^2}{mv^2}\frac{1}{\tan\theta/2}\frac{ze^2}{mv^2}\frac{1}{\tan^2\theta/2}\frac{1}{\cos^2\theta/2}\frac{1}{2}\frac{1}{2\sin\theta/2\cos\theta/2}$$

+ with $p = mv$

$$= \left(\frac{ze^2}{2pv}\right)^2 \frac{1}{\sin^4\theta/2}$$

correct with $p = \gamma m\beta c$ relativistic case!

$\theta$ is scattering angle in c/m = rest frame of incident particle.

$p^\alpha$ initial 4 momentum of $e^-$ in rest frame of heavy incoming particle

$$= (\gamma mc, \gamma m\beta c, 0, 0)$$

$p^{\alpha'}$ final 4-momentum

$$= (\gamma mc, \gamma mc\beta\cos\theta, \gamma mc\beta\sin\theta, 0)$$

Lorentz scalar $-(p^\alpha - p^{\alpha'})^2 = (\gamma mc\beta)^2\left((\cos\theta - 1)^2 + \sin^2\theta\right)$

$$= 2p^2(2 - 2\cos\theta)$$

$$= 4p^2\sin^2\theta/2$$

$$\equiv Q^2$$

$$\frac{d\sigma}{dQ^2} = \int_0^{2\pi} \frac{d\sigma}{d\Omega}\frac{\sin\theta\, d\theta}{dQ^2}\, d\phi = 2\pi \frac{d\sigma}{d\Omega}\frac{\sin\theta\, d\theta}{dQ^2}$$

$$\frac{dQ^2}{d\theta} = 4p^2\sin\theta/2\cos\theta/2 = 2p^2\sin\theta$$

i.e. $\dfrac{\sin\theta\, d\theta}{dQ^2} = \dfrac{1}{2p^2}$

so $$\frac{d\sigma}{dQ^2} = 2\pi \left(\frac{ze^2}{2pv}\right)^2 \frac{1}{\sin^4\theta/2} \frac{1}{2p^2}$$

denominator has $p^4 \sin^4 \theta/2 = Q^4/16$

so $$\frac{d\sigma}{dQ^2} = 4\pi \left(\frac{ze^2}{vQ^2}\right)^2$$

Consider $Q^2$ in initial rest frame of electron

$$Q^2 = -(p-p')^2 \qquad p^\alpha = (mc, \vec{0})$$

$$p'^\alpha = (\gamma mc, \gamma m\vec{v})$$

$$= -\left((\gamma-1)^2 - \gamma^2\beta^2\right) m^2c^2$$

$$= (2\gamma - 2) m^2c^2 = 2(\gamma-1)mc^2 \times m = 2Tm$$

$T = (\gamma-1)mc^2$ kinetic energy gained by electron

= k.e. lost by particle

$$\frac{d\sigma}{dT} = \frac{2\pi z^2 e^4}{mc^2\beta^2 T^2}$$

cross section as a function of energy loss

$N$ atoms/volume $\qquad Z$ electrons/atom

$$\frac{dE}{dx} = \text{energy lost by incoming particle per unit length}$$

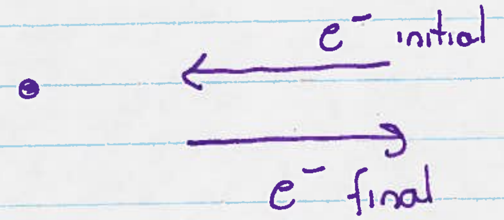
$$= NZ \int T \frac{d\sigma}{dT}\, dT$$

$T_{min}$ set by binding energy $\sim \hbar\langle\omega\rangle$

$T_{max}$ ? in frame of incoming particle

most energetic case is one in which electron back-scatters.

rest frame of incident particle




$\gamma mc^2$, $\gamma m\beta c$

boost to lab frame

$$E = \gamma(E' + \beta c p') = \gamma^2(mc^2 + mc^2\beta^2)$$

$$T_{max} = E - mc^2 = (\gamma^2 - 1 + \beta^2\gamma^2)mc^2$$

$$= 2\gamma^2\beta^2 mc^2$$

$$\boxed{\frac{dE}{dx}(T > \epsilon) = 2\pi N Z \frac{z^2 e^4}{mc^2\beta^2} \ln\left(\frac{2\gamma^2\beta^2 mc^2}{\epsilon}\right)}$$

Case of collisions at lower energies must be handled quantum mechanically. Form is similar

$$\frac{dE}{dx}(T<\epsilon) = 2\pi N Z \frac{z^2 e^4}{mc^2\beta^2} \ln\left[B_q^2\right]$$

$$B_q(\varepsilon) = \frac{\gamma v(2m\varepsilon)^{1/2}}{\hbar\langle\omega\rangle}$$

add two expressions - sum of logs = log of product

$$\frac{dE}{dx}(T<\varepsilon) = 4\pi N Z \frac{z^2 e^4}{mc^2\beta^2} \ln\left(\frac{2\gamma^2\beta^2 mc^2}{\hbar\langle\omega\rangle}\right)$$

also spin modifies expression

$$\ln(\;) \rightarrow \ln(\;) - \beta^2$$