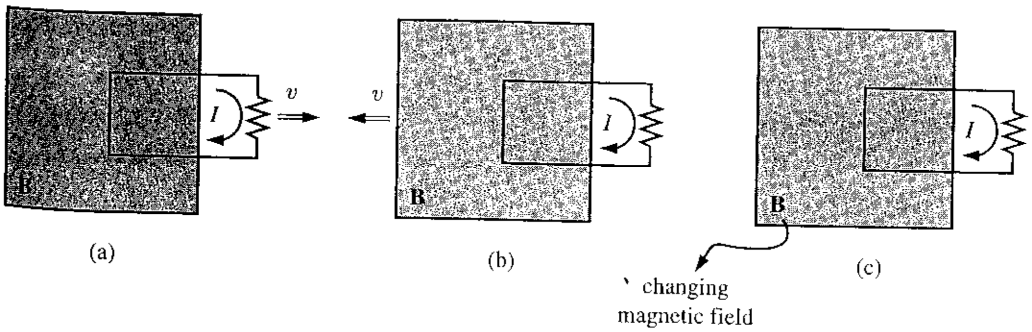
Experiment 1. He pulled a loop of wire to the right through a magnetic field (Fig. 7.20a). A current flowed in the loop.

Experiment 2. He moved the *magnet* to the *left*, holding the loop still (Fig. 7.20b). Again, a current flowed in the loop.

Experiment 3. With both the loop and the magnet at rest (Fig. 7.20c), he changed the *strength* of the field (he used an electromagnet, and varied the current in the coil). Once again, current flowed in the loop.

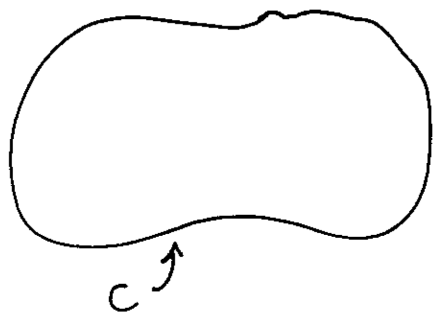


(a) (b) (c)

Electrodynamics

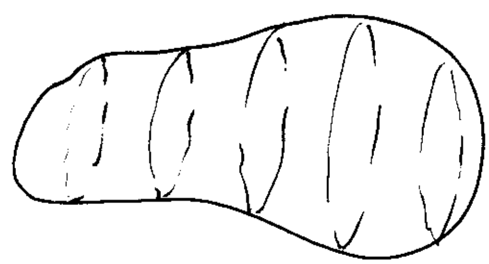
1831 Faraday observes that current in loop is induced if

a) adjacent current turned off
b) adjacent circuit is moved
c) permanent magnet is thrust through circuit



F = magnetic flux linked by circuit

$$= \int_S \vec{B} \cdot \hat{n} \, da$$



independent of S

Consider two surfaces S_1, S_2

integral over S_1 − integral over S_2 = integral over closed surface

$$= \oint \vec{B} \cdot \hat{n} \, da = \int \vec{\nabla} \cdot \vec{B} \, d^3x = 0$$

$$\mathcal{E} = \oint_C \vec{E}' \cdot d\vec{l}$$ integral of $\vec{E}'$ along $d\vec{l}$

$\vec{E}'$ is field in rest frame of $d\vec{l}$

Faraday finds $\quad \mathcal{E} = -k \frac{dF}{dt}$

sign: Lenz's law - induced mag. field opposes changing flux.

k - set by units.

$$\oint_C \vec{E}' \cdot d\vec{l} = -k \frac{d}{dt} \int_S \vec{B} \cdot \hat{n} \, da$$

NB. C need not correspond to physical wire!

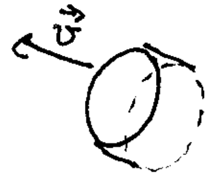
total time derivative

F changes if C changes or if $\vec{B}$ itself changes

A statement about fields!

Lets suppose C moves with velocity $\vec{v}$ (assume constant)

$\vec{B}$ may have explicit time-dependence



or $\vec{B}$ may change with position, + hence $\vec{B}$ at surface changes with time.

$$\frac{d\vec{B}}{dt} = \frac{\partial \vec{B}}{\partial t} + \left(\frac{d\vec{x}}{dt}\cdot\vec{\nabla}\right)\vec{B}$$

$$= \frac{\partial \vec{B}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{B}$$

$$= \frac{\partial \vec{B}}{\partial t} + \vec{\nabla}\times(\vec{v}\times\vec{B}) + \vec{v}\cancelto{0}{(\vec{\nabla}\cdot\vec{B})}$$

$$\frac{d}{dt}\int_S (\vec{B}\cdot\hat{n})\,da = \int_S \frac{\partial \vec{B}}{\partial t}\cdot\hat{n}\,da + \int \vec{\nabla}\times(\vec{v}\times\vec{B})\cdot\hat{n}\,da$$

$$= \int_S \frac{\partial \vec{B}}{\partial t}\cdot\hat{n}\,da + \oint_C (\vec{v}\times\vec{B})\cdot d\vec{l}$$

so $$\oint \left(\vec{E}' - k\,\vec{v}\times\vec{B}\right)\cdot d\vec{l} = -k\int \frac{\partial \vec{B}}{\partial t}\cdot \hat{n}\, da$$

Two interpretations:

1) $\vec{E}'$ is field in rest frame of circuit

$\mathcal{E} = \int \vec{E}'\cdot d\vec{l}$ has two contributions

one from $\partial \vec{B}/\partial t$ + one from motion thru spatially varying $\vec{B}$

2) different circuit at rest in lab, coincides instantaneously with moving circuit

$\mathcal{E} = \int \vec{E}\cdot d\vec{l}$ $\qquad \vec{E} = \vec{E}' - k\,\vec{v}\times\vec{B}$

only contribution is from $\partial\vec{B}/\partial t$

$\vec{E}$ + $\vec{B}$ lab fields

$\vec{E}' = \vec{E} + k\,\vec{v}\times\vec{B}$ field transformation (Galilean)

Consider a charges particle at rest in moving circuit

As viewed in rest frame of moving circuit, it experiences an electric force $\vec{F} = q\vec{E}'$

As viewed in lab, it experiences both electric & magnetic forces $\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$

Hence, since $\vec{F}$'s should be equal (in Galilean relativity)

$$\boxed{k = \frac{1}{c}} \qquad \vec{E}' = \vec{E} + \frac{\vec{v}}{c} \times \vec{B}$$

for fixed circuit

$$\oint_C \vec{E} \cdot d\vec{l} = \int_S (\vec{\nabla} \times \vec{E}) \cdot \hat{n}\, da$$

$$= -\frac{1}{c} \int_S \frac{\partial \vec{B}}{\partial t} \cdot \hat{n}\, da$$

$$\Rightarrow \quad \boxed{\vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0}$$

Notes: Faraday's law is correct in SR

Transformation law is only valid in $\frac{v}{c} \ll 1$ limit

(i.e., correct to $O(v^2/c^2)$)

Maxwell's displacement current

In general $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$

$$\frac{\partial}{\partial t}\left(\vec{\nabla} \cdot \vec{E} = 4\pi\rho \right) \Rightarrow \frac{\partial \rho}{\partial t} = \frac{1}{4\pi} \vec{\nabla} \cdot \frac{\partial \vec{E}}{\partial t}$$

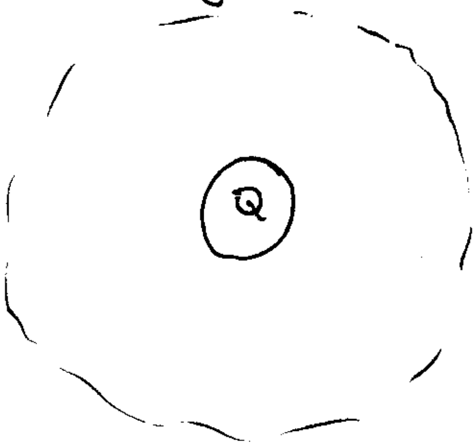
$$= - \vec{\nabla} \cdot \vec{J}$$

$$\Rightarrow \quad \vec{\nabla} \cdot \left(\vec{J} + \frac{1}{4\pi} \frac{\partial \vec{E}}{\partial t} \right) = 0$$

Since $\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{B} \right) = 0$ we have

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

Consider a spherical charge distribution leaking out spherically.



$$\vec{E} = \frac{Q}{R^2}\hat{r}$$

$$\vec{J} = -\frac{1}{4\pi R^2}\frac{dQ}{dt}\hat{r}$$

$$\frac{\partial \vec{E}}{\partial t} = \frac{1}{R^2}\frac{dQ}{dt}\hat{r}$$

so $\quad \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = 0 \quad$ as it must!

Poynting's theorem

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$ Lorentz force for pt charge

$$\vec{F} \cdot d\vec{l} = dW$$ work done on charge over distance $d\vec{l} = \vec{v}\,dt$

$$= q\vec{E} \cdot \vec{v}\,dt$$ $\vec{v} \cdot (\vec{v} \times \vec{B}) = 0$

$$\rightarrow \int_{V} \rho \vec{v} \cdot \vec{E}\,dt\,d^3x$$

so $$\frac{dU}{dt} = \int \vec{J} \cdot \vec{E}\,d^3x$$ $\vec{J} = \frac{c}{4\pi}\vec{\nabla} \times \vec{B} - \frac{1}{4\pi}\frac{\partial \vec{E}}{\partial t}$

$$= \frac{1}{4\pi}\int\left[c\left(\vec{\nabla} \times \vec{B}\right) \cdot \vec{E} - \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}\right] d^3x$$

$$\vec{E}\cdot(\vec{\nabla}\times\vec{B}) = -\vec{\nabla}\cdot(\vec{E}\times\vec{B}) + \vec{B}\cdot(\vec{\nabla}\times\vec{E})$$

$$= -\vec{\nabla}\cdot(\vec{E}\times\vec{B}) - \frac{1}{c}\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}$$

also $$\int d^3x\, \vec{E}\cdot\frac{\partial\vec{E}}{\partial t} = \frac{1}{2}\frac{d}{dt}\int d^3x\, E^2$$

$$\frac{dW}{dt} = -\frac{1}{8\pi}\int d^3x\,(E^2+B^2) - \frac{c}{4\pi}\oint da\,\hat{n}\cdot(\vec{E}\times\vec{B})$$

$$\text{or } -\frac{c}{4\pi}\int d^3x\,\vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

$$W = \int d^3x\, u_{mech}$$ u_{mech} = mechanical energy density

$$u_{EM} = \frac{1}{8\pi}(E^2+B^2)$$ energy density in field

$$\vec{S} = \frac{c}{4\pi}(\vec{E}\times\vec{B})$$ energy flux

$$\frac{\partial}{\partial t}(u_{mech}+u_{EM}) = -\vec{\nabla}\cdot\vec{S}$$ cons. of energy

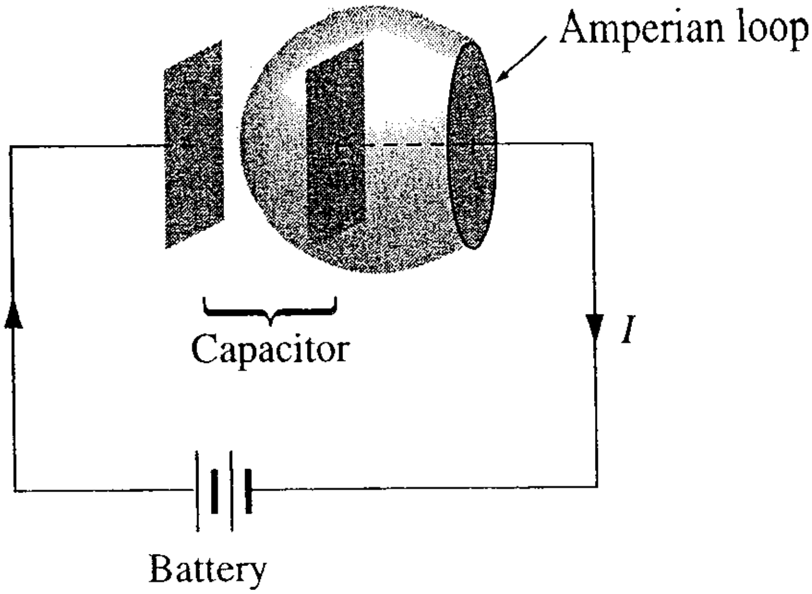


Figure 7.42

Surface 1 $\quad \vec{E} = 0 \qquad \vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J}$

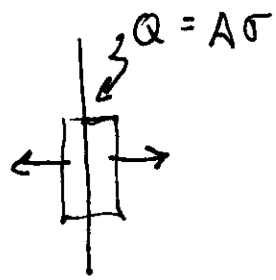
Get B_ϕ from Amperian loop

$$\int da\, \hat{n} \cdot \vec{\nabla} \times \vec{B} = \oint d\vec{l} \cdot \vec{B} = 2\pi s B_\phi$$

$$= \frac{4\pi}{c} \int da\, \hat{n} \cdot \vec{J} = \frac{4\pi}{c} I \qquad B_\phi = \frac{2I}{cs}$$

Surface 2
Same loop! $\quad \vec{J} = 0 \quad$ but $\quad \frac{\partial \vec{E}}{\partial t} = 0$

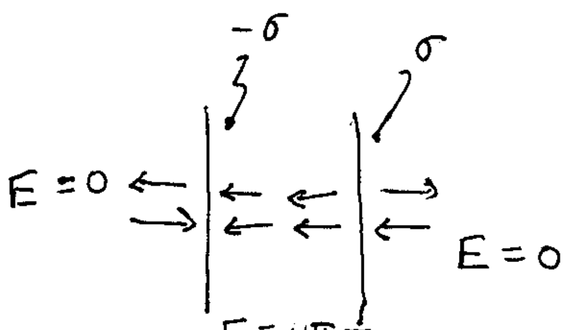
single plate



$2E \times A = 4\pi\sigma A \qquad E = 2\pi\sigma$

$$= \frac{2\pi Q}{A}$$

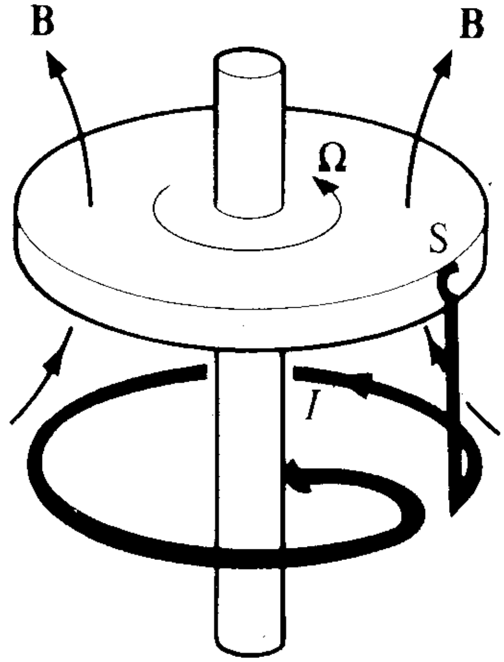
for parallel plate



so $\quad \frac{1}{c} \int \frac{\partial \vec{E}}{\partial t} \cdot \hat{n}\, da = \frac{4\pi}{c}\frac{dQ}{dt} = \frac{4\pi}{c} I$

Disk is kept at fixed Ω

Begin with a small current which gives rise to B-flux thru disk. What happens?



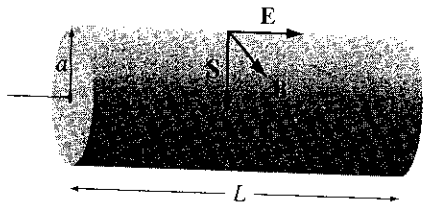


Figure 8.1

Imagine a very long solenoid with radius R, n turns per unit length, and current I. Coaxial with the solenoid are two long cylindrical shells of length l—one, *inside* the solenoid at radius a, carries a charge $+Q$, uniformly distributed over its surface; the other, *outside* the solenoid at radius b, carries charge $-Q$ (see Fig. 8.7; l is supposed to be much greater than b). When the current in the solenoid is gradually reduced, the cylinders begin to rotate, as we found in Ex. 7.8. *Question*: Where does the angular momentum come from?[4]

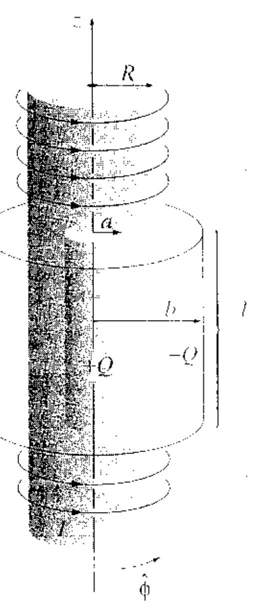


Figure 8.7

Voltage drop V over length L $\quad V = E \cdot L$

for magnetic field $\quad \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J}$

or along Amperian loop at surface of wire

$$\int da\, \hat{n} \cdot (\vec{\nabla} \times \vec{B}) = \oint d\vec{l} \cdot \vec{B} = 2\pi a B_\phi$$

$$= \frac{4\pi}{c} \int \vec{J} \cdot \hat{n}\, da = \frac{4\pi}{c} I_{encl}$$

so $\quad B_\phi = \frac{2 I_{encl}}{ca}$

at Surface $\quad \vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B} = -\frac{\cancel{c}}{4\pi} \frac{V}{L} \frac{2 I_{encl}}{\cancel{c} a} \hat{s}$

$$= -\frac{1}{2\pi a L} V I_{encl} \hat{s}$$

so $\quad \oint \vec{S} \cdot \hat{n}\, da = V I$

$\hat{n}$ points inward to give flow of energy into wire

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B}$$

energy flux $\frac{\text{energy}}{\text{time area}}$

$[\vec{S}] = [\text{energy den.} \times c]$

$\vec{\mathcal{P}}$ = momentum density

$$= \frac{1}{4\pi c} \vec{E} \times \vec{B}$$

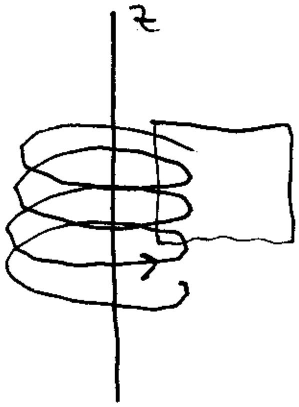
but $p = E/c$ for photons

Electric field: radial $\quad \vec{\nabla}\cdot\vec{E} = 4\pi\rho$

or $\quad 2\pi l s E_s = 4\pi Q \quad \Rightarrow \quad E_s = \dfrac{2Q}{ls}$

for $a < s < b$

$\vec{B}$ along symmetry (z) axis



B is zero outside

$$B_z \cdot l = \frac{4\pi}{c} I_{encl}$$

$$= \frac{4\pi}{c} n I l$$

so $\quad B_z = \begin{cases} \dfrac{4\pi}{c} n I & s < R \\ 0 & s \geq R \end{cases}$

in region $a < s < R$ we have $\quad \mathcal{P}_\phi = \dfrac{1}{4\pi c}(\vec{E}\times\vec{B})\cdot\hat{\phi}$

$$= -\frac{1}{4\pi}\frac{2Q}{ls}\frac{4\pi}{c^2} n I$$

$$= -\frac{2QnI}{lsc^2}$$

Turn off magnetic field and induced $\vec{E}$ in ϕ-direction causes cylinders to rotate

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\int da\, \hat{n} \cdot (\vec{\nabla} \times \vec{E}) = \oint \vec{E} \cdot d\vec{l}$$

$$= \begin{cases} E_\phi \cdot 2\pi a & \text{inner} \\ E_\phi \cdot 2\pi b & \text{outer} \end{cases}$$

$$= -\frac{1}{c}\int \frac{\partial \vec{B}}{\partial t} \cdot \hat{n}\, da$$

$$= -\frac{1}{c}\frac{d}{dt}\int \vec{B} \cdot \hat{n}\, da$$

$$= -\frac{4\pi}{c^2} n \frac{dI}{dt} \cdot \begin{cases} \pi a^2 & \text{inner} \\ \pi R^2 & \text{outer} \end{cases}$$

$$E_\phi = \begin{cases} -\dfrac{2\pi}{c^2} n a \dfrac{dI}{dt} & \text{inner} \\ -\dfrac{2\pi}{c^2} n \dfrac{R^2}{b} \dfrac{dI}{dt} & \text{outer} \end{cases}$$

Torque

$$N_{inner} = a \times Q \times E_\phi \hat{z} = -\frac{2\pi}{c^2} n Q a^2 \frac{dI}{dt} \hat{z}$$

$$N_{outer} = b \times (-Q) \times E_\phi \hat{z} = \frac{2\pi}{c^2} n Q R^2 \frac{dI}{dt} \hat{z}$$

$$\vec{L} = \int \left(\vec{N}_{inner} + \vec{N}_{outer} \right) dt \qquad \int \frac{dI}{dt} dt = -I$$

$$= \frac{2\pi}{c^2} n Q I \left(a^2 - R^2 \right) \hat{z}$$

$$\vec{\mathcal{P}}_\phi = -\frac{2QnI}{l s c^2}$$

$$\vec{L}_{EM} = \int \vec{r} \times \vec{\mathcal{P}}_{EM} \, d^3x = -\frac{2QnI\hat{z}}{lc^2} \int_0^l dz \int_a^R 2\pi s \, ds \times s \times \frac{1}{s}$$

$$= -\frac{2\pi}{c^2} n Q I \left(R^2 - a^2 \right) \hat{z} \quad \checkmark$$

What if we "turn off" electric field?

Connect cylinders by a wire

Then Lorentz force on radially flowing charge in B_z gives rise to torque on cylinder