Special Relativity

Relativity — theory of spacetime + connection with laws of physics

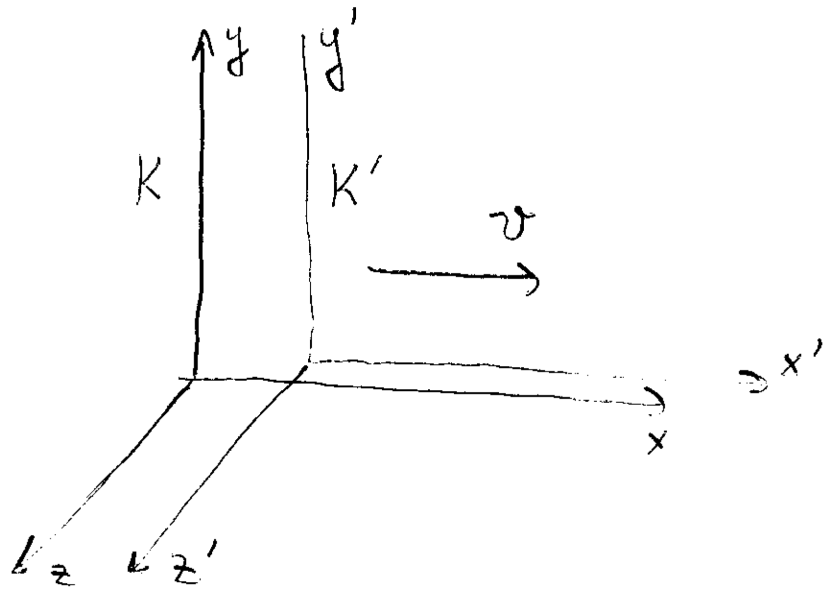
Event — localized occurence in space + time.

Spacetime — collection of all events

Spacetime coordinate system — 4 dimensional labels for events - continuous + differentiable.

Principle of Relativity

Laws of physics / results of experiments are independent of translational motion of system as a whole



Prior to 1905 - Galilean transformation

$$x' = x - vt \qquad \vec{v} \text{ rel. vel. of frames} = v\hat{x} \text{ constant}$$

$$y' = y \qquad z' = z \qquad t' = t$$

$\vec{u}$ = vel. of particle as measured in K

$\vec{u}'$ = vel of particle as measured in K'

$$= \frac{d\vec{x}'}{dt'} \Rightarrow \quad u'_x = u_x - v$$

$$u'_y = u_y \qquad u'_z = u_z$$

$x' = 0 \Rightarrow x = vt$

transformations are linear and invertible

velocity addition laws

Newton's laws $\qquad \vec{F} = m\vec{a}$

$$\vec{a} = \frac{d\vec{u}}{dt}$$

$$\vec{a}' = \frac{d\vec{u}'}{dt'} = \frac{d\vec{u}}{dt} \qquad \vec{v} = \text{const} \qquad t' = t$$

$+ \quad \vec{F}' = \vec{F} \quad$ so $\quad \vec{F}' = m\vec{a}'$

Maxwell's equations in vacuum

$$\vec{\nabla} \times \vec{B} = -\frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = -\nabla^2 \vec{B} + \cancel{\vec{\nabla}(\vec{\nabla}\cdot\vec{B})}^{\,0}$$

$$= -\frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla} \times E$$

$$= -\frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} \qquad \boxed{\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{B} = 0}$$

Now $$\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} + \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} = \frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}$$

so under Galilean transformations

$$\left[\frac{\partial^2}{\partial x'^2} - \frac{1}{c^2}\left(\frac{\partial^2}{\partial t'^2} - 2v\frac{\partial^2}{\partial t'\partial x'} + v^2\frac{\partial^2}{\partial x'^2}\right)\right]\vec{B} = 0$$

No transformation of B can put this in original form.

Consider Schrödinger equation in K

$$\left(-\frac{\hbar^2}{2m}\nabla^2 - i\hbar\frac{\partial}{\partial t}\right)\psi = V\psi$$

$$\left(-\frac{\hbar^2}{2m}\nabla'^2 - i\hbar\frac{\partial}{\partial t'} + i\hbar v\frac{\partial}{\partial x'}\right)\psi = V\psi$$

$$\psi = \psi' e^{i\frac{m}{\hbar}vx' + i\frac{mv^2}{2\hbar}t'}$$

gives $$\left(-\frac{\hbar^2}{2m}\nabla'^2 - i\hbar\frac{\partial}{\partial t'}\right)\psi' = V'\psi'$$

with $V = V'$

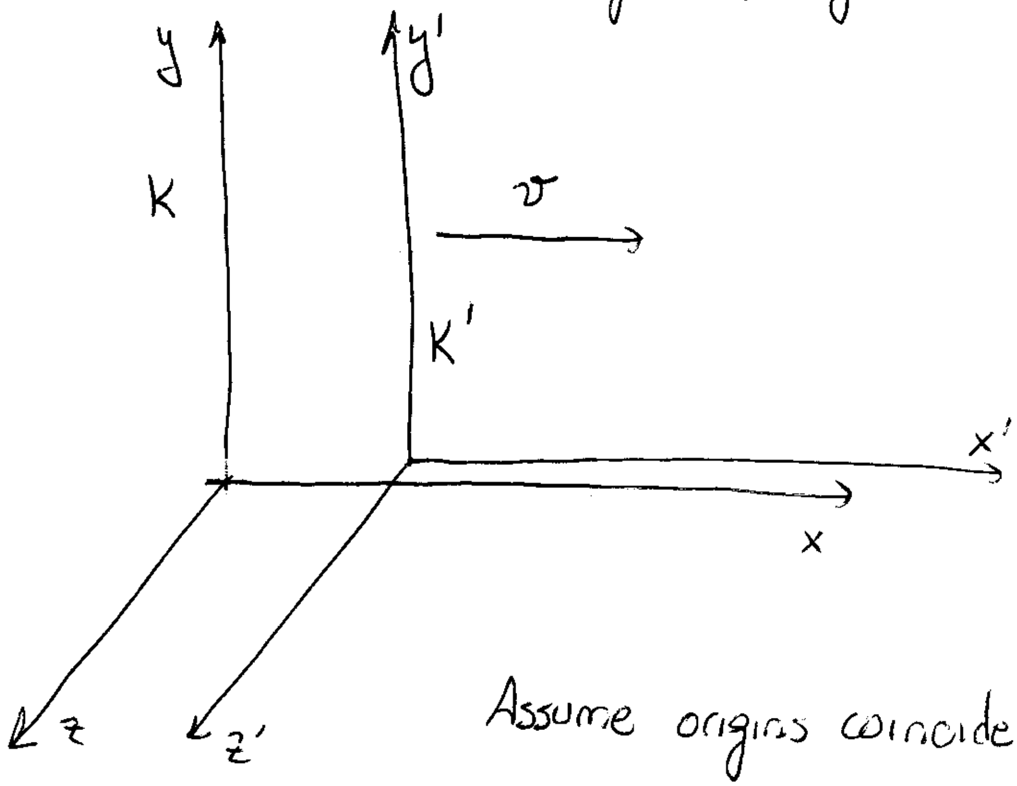
Back to Maxwell's equations

One passiblity is that light propogates in a medium - the ether. (Imagine we were writing equations for sound waves.)

Einstein's 2nd postulate

Speed of light is the same in all frames of reference independent of the motion of the source.

— Galilean velocity addition laws no longer apply.



Assume origins coincide at $t = t' = 0$

$y' = y \qquad z' = z$

(transformation involves x, t)

want $\quad x' = x'(x, t) \qquad t' = t'(x, t)$

Consider a spherical light signal at $t = 0$

Wave front will be at t, x, y, z so that

$$c^2 t^2 = x^2 + y^2 + z^2$$ Einstein's 2nd postulate

Could use K' coordinates

$$c^2 t'^2 = x'^2 + y'^2 + z'^2$$

$$\Rightarrow \quad c^2 t^2 - x^2 - y^2 - z^2 = c^2 t'^2 - x'^2 - y'^2 - z'^2$$

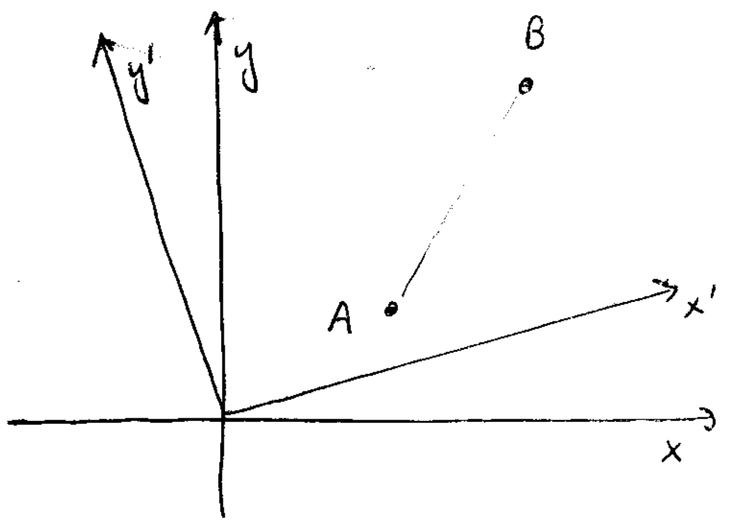
More generally true

Let $t_A, x_A, y_A, z_A \qquad t_B, x_B, y_B, z_B$

be two events, not necessarily connected by a light signal.

$$c^2 (t_A - t_B)^2 - (\vec{x}_A - \vec{x}_B)^2 = c^2 (t_A' - t_B')^2 - (x_A' - x_B')^2$$

Reminiscent of rotations in plane



$$s^2 = (x_A - x_B)^2 + (y_A - y_B)^2$$

$$= (x'_A - x'_B)^2 + (y'_A - y'_B)^2$$

where

$$x' = \cos\theta\, x + \sin\theta\, y$$
$$y' = -\sin\theta\, x + \cos\theta\, y$$

for events, the "interval" $c^2(t_A - t_B)^2 - (\vec{x}_A - \vec{x}_B)^2$

is a spacetime invariant

$$ct' = \cosh\psi\, ct - \sinh\psi\, x$$

$$x' = -\sinh\psi\, ct + \cosh\psi\, x$$

+ use $\cosh^2\psi - \sinh^2\psi = 1$ identity!

Note $x' = 0 \Rightarrow x = vt$

$$x' = 0 \quad \Rightarrow \quad x = \frac{\sinh \psi}{\cosh \psi}\, ct \qquad \beta = \frac{v}{c} = \tanh \psi$$

$$\cosh \psi = \frac{1}{(1 - \tanh^2 \psi)^{1/2}} = \frac{1}{(1 - \beta^2)^{1/2}} = \gamma$$

$$ct' = \gamma\left(ct - \frac{v}{c} x\right) \qquad x' = \gamma\left(x - \frac{v}{c} ct\right)$$

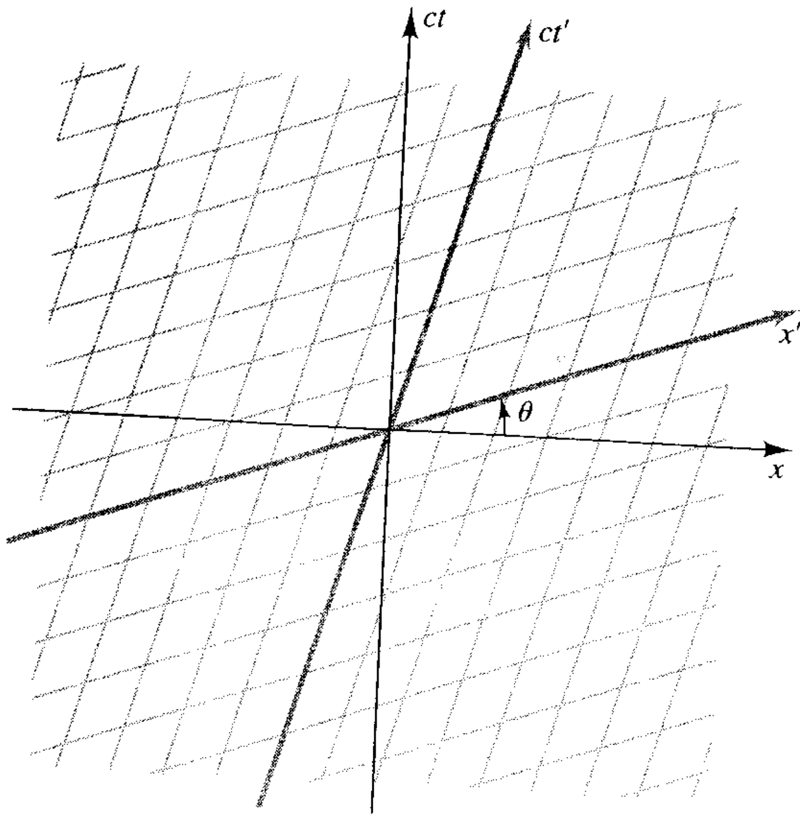
4-vector notation

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$$

$$x^{0'} = \gamma(x^0 - \beta x^1)$$

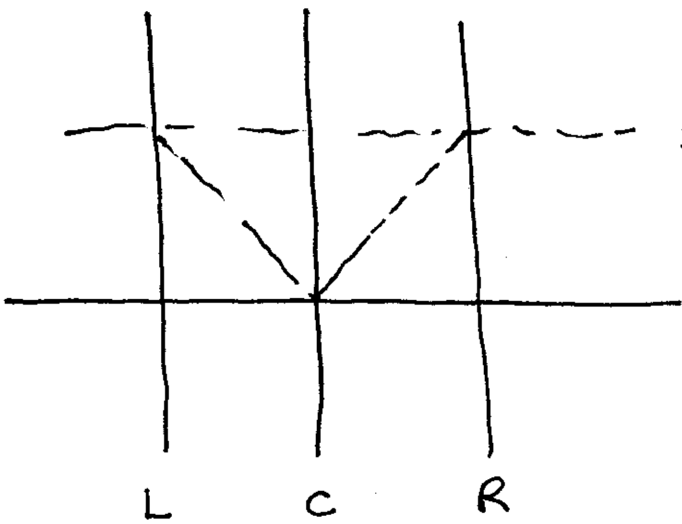
$$x^{1'} = \gamma(x^1 - \beta x^0)$$

$$x^{2'} = x^2 \qquad x^{3'} = x^3$$

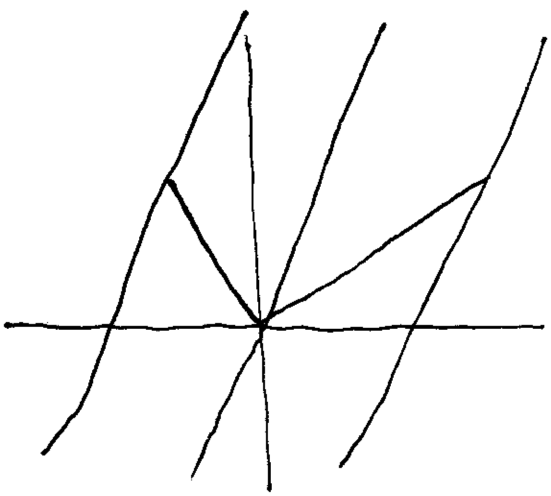


ct
ct'
x'
θ
x

Einstein's train paradox

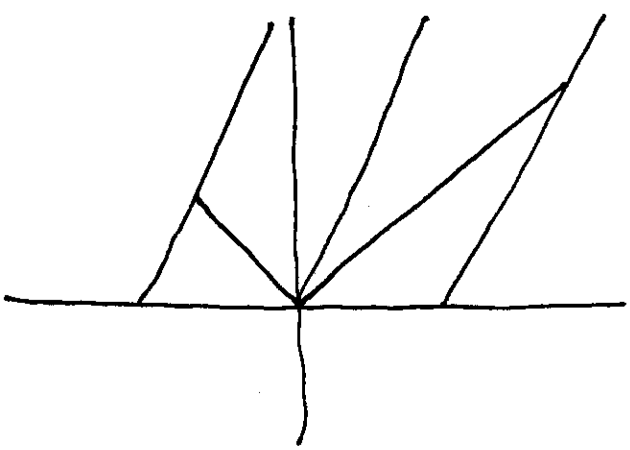


signal received at same time in rest frame of train



Galilean

signals still received at same time. Light to left & right moves at different speeds



Special Relativity

light moves at speed c

time to receive is different

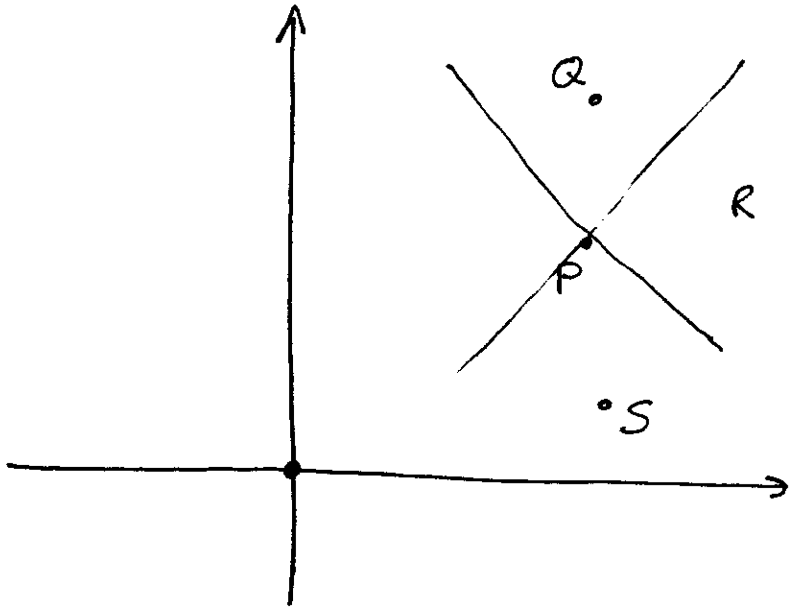
"lack of simultaneity"

General 4-vector

$$A^\mu = (A^0, A^1, A^2, A^3)$$
$$= (A^0, \vec{A})$$

$$A^{0\prime} = \gamma(A^0 - \beta A^1) \qquad A^{2\prime} = A^2$$
$$A^{1\prime} = \gamma(A^1 - \beta A^0) \qquad A^{3\prime} = A^3$$

$$(A^{0\prime})^2 - \vec{A}^\prime \cdot \vec{A}^\prime = (A^0)^2 - \vec{A} \cdot \vec{A}$$ Lorentz invariant



for event P, spacetime is divided into
future, past, spacelike separated
Q S R

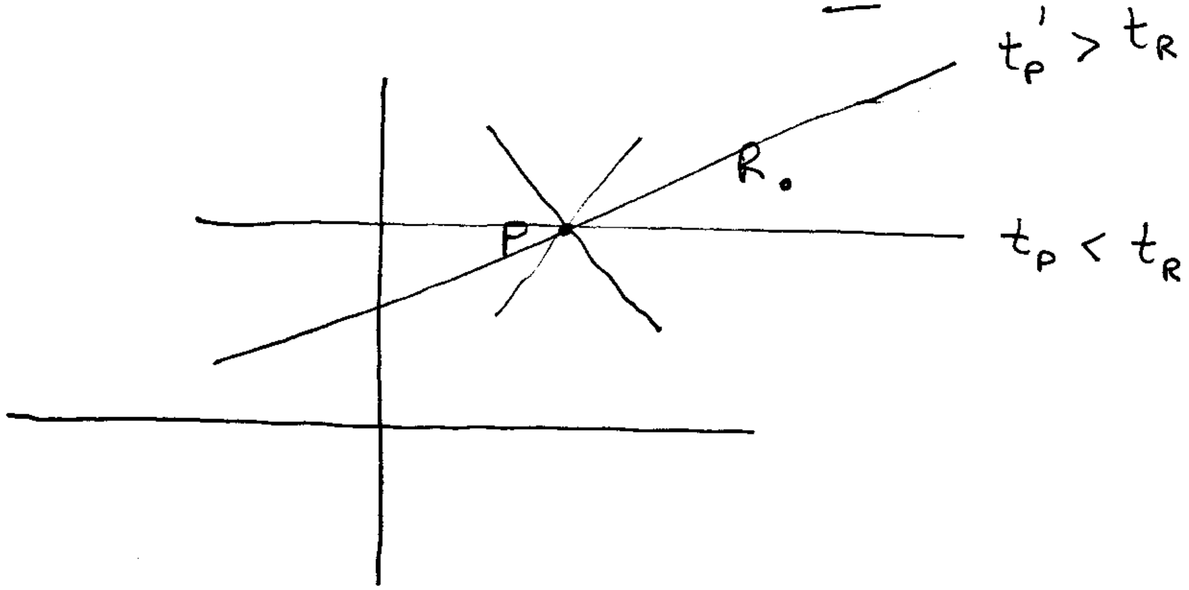
$$(\Delta s)^2 = (x_{P,0} - x_{Q,0})^2 - |\vec{x}_P - \vec{x}_Q|^2$$

spacetime invariant

> 0 if timelike separated (past or future)

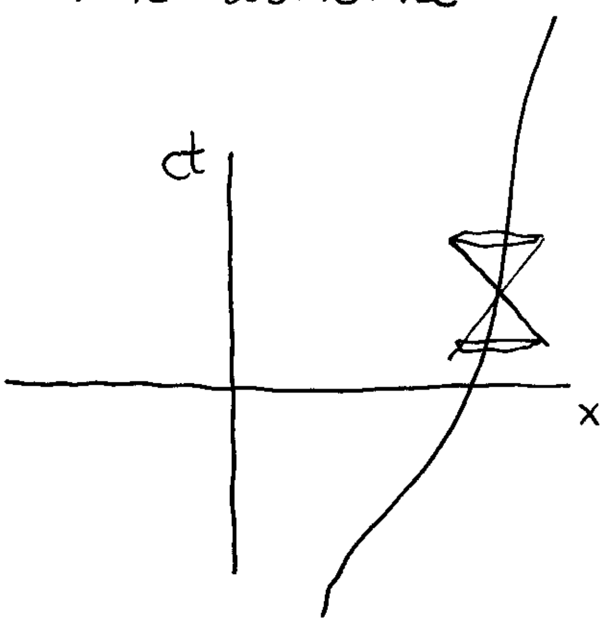
< 0 if spacelike separated

$= 0$ if on light cone

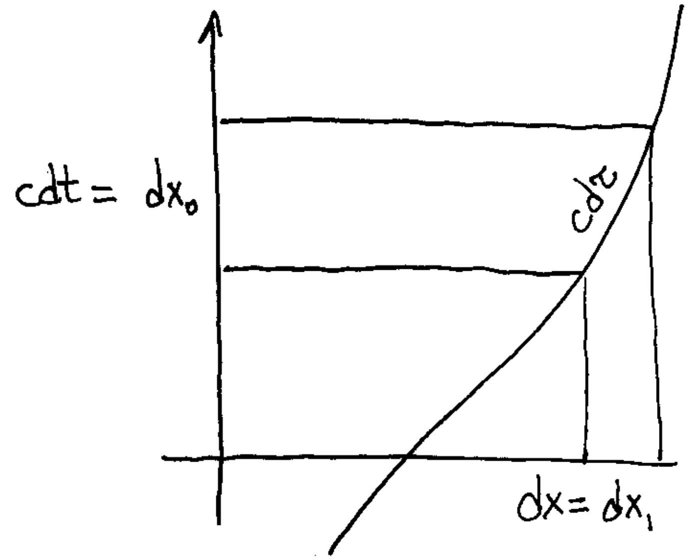


No answer to question "which happened first"

Particle worldline $\vec{x} = \vec{x}(t)$ $\vec{v} = \dfrac{d\vec{x}}{dt}$



threads lightcone



$$c^2 d\tau^2 = c^2 dt^2 - dx^2$$ Lorentz invariant

$$= c^2 dt^2 (1 - \beta^2)$$

$$d\tau = \frac{dt}{\gamma}$$ time dilation

$d\tau$ = proper time

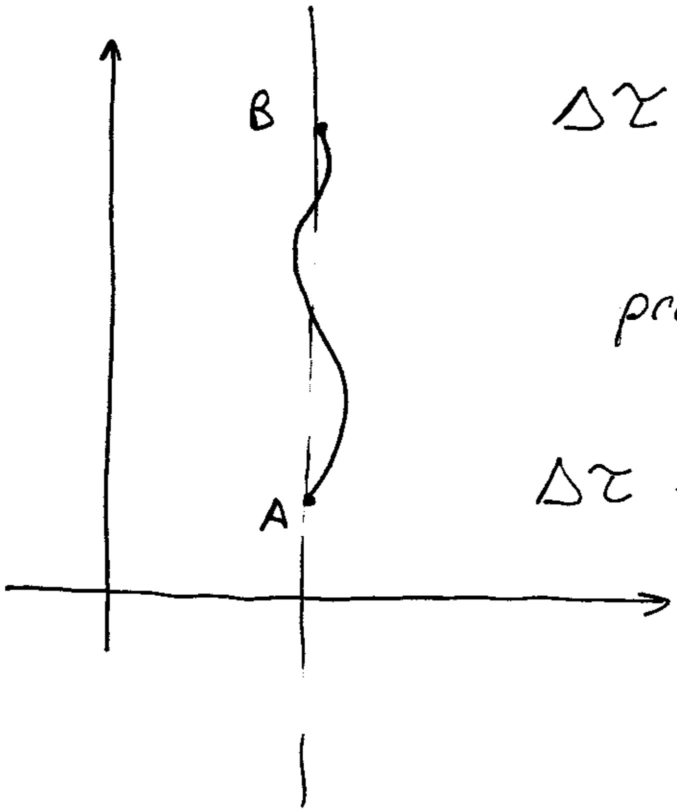
time measured by clock carried by particle

dt time interval measured by two clocks at different places in lab

$> d\tau$

Consider two timelike seperated events.

Find worldline of particle in constant motion
Let this particle define rest frame.



$$\Delta\tau = \int_A^B dt = t_B - t_A$$

proper time for this inertial observer

$\Delta\tau$ for another observer

$$= \int_A^B dt(1-\beta^2)$$

$$\leq \Delta\tau_{inertial}$$

Stay-at-home twin is older!

Can't we treat accelerating twin as twin at rest?

Yes, but requires more general treatment of spacetime.