Euclidean geometry

follows from postulates

or consequence of assumption that space is described by metric

$$ds^2 = dx^2 + dy^2 + dz^2$$

distance between (x, y, z) + $(x+dx, y+dy, z+dz)$ is ds.

<u>Vectors</u> directed

vector + vector = vector

vector × scalar = vector

vector · vector = scalar

prototype $d\vec{x} = (dx, dy, dz)$

I'll use $dx^i = (dx, dy, dz)$ $\quad i = 1-3$

$$= (dx^1, dx^2, dx^3)$$

dot product

$$ds^2 = d\vec{x} \cdot d\vec{x}$$

$$= g_{ij}\, dx^i dx^j \qquad g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

implied sum on i & j

Note position of indices!

metric tensor

Call dx^i, A^i, ... covariant vector

Define $dx_i = g_{ij}\, dx^j$ contravariant vector or dual vector

$$ds^2 = dx_j\, dx^j \qquad \text{implied sum}$$

ds^2 is invariant under coordinate transformations

$x = x(x', y', z')$ etc

e.g. $x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta + \frac{dx}{d\phi}d\phi$$

$$= \sin\theta\cos\phi\, dr + r\cos\theta\cos\phi\, d\theta + r\sin\theta\sin\phi\, d\phi$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$$

$$g_{ij} = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{pmatrix}$$

Euclidean space in spherical-polar coordinates

In general

$$dx^i = \frac{\partial x^i}{\partial x'^j} dx'^j$$

$$A^i = \frac{\partial x^i}{\partial x'^j} A'^j$$

transformation for <u>any</u> vector works same way prototypical dual vector

$$\nabla_i = \frac{\partial}{\partial x^i} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\frac{\partial}{\partial x^i} = \frac{\partial x^{j'}}{\partial x^i} \frac{\partial}{\partial x^{j'}}$$

$$A_i = \frac{\partial x^{j'}}{\partial x^i} A_j'$$

$$\vec{A} \cdot \vec{B} = A_i B^i = \frac{\partial x^{j'}}{\partial x^i} A_j' \frac{\partial x^i}{\partial x^{k'}} B^{k'}$$

$$= \delta^j_k A_j' B^{k'} = A_j' B^{j'}$$

special coordinate transformations leave form of metric unchanged - these are rotations + translations. Focus on rotations for rotation about z-axis

$$x = \cos\theta\, x' - \sin\theta\, y' \qquad y = \sin\theta\, x' + \cos\theta\, y'$$

$$\frac{\partial x^j}{\partial x^{i'}} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$ds^2 = dx^2 + dy^2 + dz^2 = dx'^2 + dy'^2 + dz'^2$$

form of equations vector = vector are unchanged "covariant" "invariant in form"

$$\vec{F} = m\vec{a} \Rightarrow \vec{F}' = m\vec{a}'$$

NB we can consider more general coordinate transformations – form of equations changes

We can also have 2 (or more) index tensors

$$A_{ij} \ , \ T^{i}{}_{kj} \ , \ \ldots$$

prototype g_{ij} metric tensor

$$g_{ij} = \frac{\partial x^{k'}}{\partial x^{i}} \frac{\partial x^{l'}}{\partial x^{j}} g'_{kl}$$

$$\delta^{l}_{n}$$

$$ds^2 = g_{ij}\, dx^i dx^j = \frac{\partial x^{k'}}{\partial x^{i}} \frac{\partial x^{l'}}{\partial x^{j}} g'_{kl} \frac{\partial x^{i}}{\partial x^{m'}} \frac{\partial x^{j}}{\partial x^{n'}} dx^{m'} dx^{n'}$$

$$\delta^{k}_{m}$$

$$= g'_{mn}\, dx^{m'} dx^{n'} \qquad \checkmark$$

Special / General relativity

spacetime described by a <u>metric</u>

In special relativity, interval ds defined by

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

$$= (dx^0)^2 - d\vec{x} \cdot d\vec{x}$$

$$= \eta_{\alpha\beta}\, dx^\alpha dx^\beta \qquad \eta_{\alpha\beta} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

interval between events t, x, y, z + $t+dt, x+dx, y+dy, z+dz$

If events are timelike separated (e.g. two events along worldline of particle) then

$$ds^2 = c^2 d\tau^2 \qquad d\tau = \text{proper time between events}$$

time between events as measured in frame in which they occur at same position

If two events are spacelike separated then

$$ds^2 = -dl^2$$

dl = proper distance = distance between events measured in frame in which they occur at same time

ds^2 invariant under general coordinate transformations

e.g. transform to cylindrical coordinates

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - dz^2$$

then transform to rotating frame $\phi = \phi' - \omega t'$
$t = t'$

$$ds^2 = c^2 dt^2 - dr^2 - r^2 (d\phi - \omega dt)^2 - dz^2$$

I've dropped primes

<u>special</u> set of transformations leave form of metric unchanged

3 rotations + 3 Lorentz transformations (+ translations)

$$ds^2 = \eta_{\alpha\beta}\, dx^\alpha dx^\beta$$

dx^α is vector $\qquad \eta_{\alpha\beta}\, dx^\alpha = dx_\beta$ dual vector

$ds^2 = dx_\beta\, dx^\beta$ dot product of two vectors - a scalar

Coordinate transformations

$$dx^\alpha = \frac{\partial x^\alpha}{\partial x^{\beta'}}\, dx^{\beta'}$$

e.g.

$$x^0 = \gamma(x^{0'} + \beta x^{1'}) = \cosh\psi\, x^{0'} + \sinh\psi\, x^{1'}$$

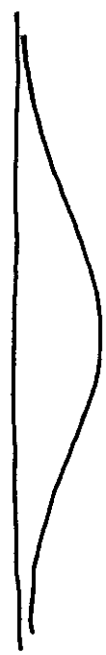
$$x^1 = \gamma(x^{1'} + \beta x^{0'}) = \sinh\psi\, x^{0'} + \cosh\psi\, x^{1'}$$

$$\frac{\partial x^\alpha}{\partial x^{\beta'}} = \begin{pmatrix} \cosh\psi & \sinh\psi & & \\ \sinh\psi & \cosh\psi & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$ds^2 = +(dx^{0'})^2 - (dx^{1'})^2 - (dx^{2'})^2 - (dx^{3'})^2$$

form is unchanged under special transformations.

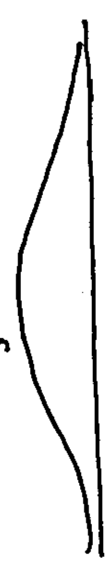
Twin paradox



accelerating twin

$$\Delta\tau = \int dt\,(1-\beta^2)^{1/2} < \Delta\tau_{\text{non-accel.}}$$

If we work in rest frame of accelerating twin then we need to transform metric



non accel. twin

But this transformation will put metric in different form.

So calculation of

$$d\tau^2 = \frac{ds^2}{c^2} = \frac{1}{c^2} g_{\alpha\beta}\, dx^\alpha dx^\beta$$

will be more complicated

Vectors + dual vectors

$$A^\alpha = (A^0, \vec{A})$$

$$A_\alpha = g_{\alpha\beta} A^\beta = (A^0, -\vec{A})$$

$$\underset{\sim}{A} \cdot \underset{\sim}{B} = \eta_{\alpha\beta} A^\alpha B^\beta$$

$$= A_\alpha B^\alpha = A^0 B^0 - \vec{A} \cdot \vec{B}$$

$$\partial_\alpha = \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla} \right) = \left(\frac{1}{c} \frac{\partial}{\partial t}, \vec{\nabla} \right)$$

$$\partial^\alpha = \frac{\partial}{\partial x_\alpha} = \left(\frac{\partial}{\partial x^0}, -\vec{\nabla} \right) = \left(\frac{1}{c} \frac{\partial}{\partial t}, -\vec{\nabla} \right)$$

$$\partial^\alpha A_\alpha = \frac{\partial A^0}{\partial x^0} + \vec{\nabla} \cdot \vec{A} = \partial_\alpha A^\alpha \qquad \text{scalar!}$$

Recall $\quad \dfrac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \quad$ continuity equation

Define 4-vector $J^{\alpha} = (c\rho, \vec{J})$

$$\partial_\alpha J^\alpha = \frac{1}{c}\frac{\partial}{\partial t} c\rho + \vec{\nabla}\cdot\vec{J} = 0 \quad \checkmark$$

Also, let $A^\alpha = \partial^\alpha f$ $\quad \partial_\alpha A^\alpha = \partial_\alpha \partial^\alpha f = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) f$ wave operator

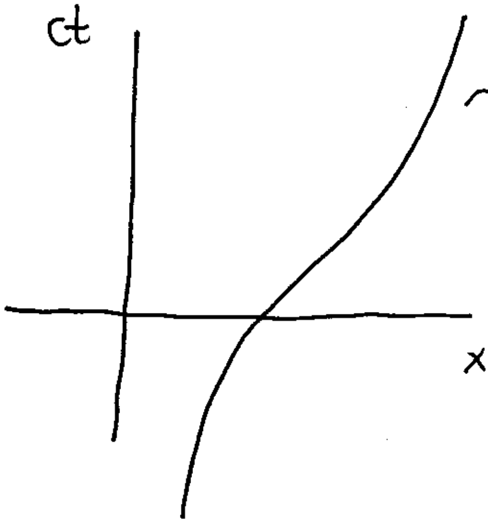
D'alembertian

$\Box f$

---

4 - velocity

Let dx^α by element along worldline of particle



$\rightarrow x = x(t)$

or

$x^\alpha = x^\alpha(\tau)$

τ is proper time for particle

worldline $\quad t = t(\tau)$

$\quad \vec{x} = \vec{x}(\tau)$

Now $\quad d\tau^2 = \frac{ds^2}{c^2} = dt^2 - d\vec{x}\cdot d\vec{x}/c^2$

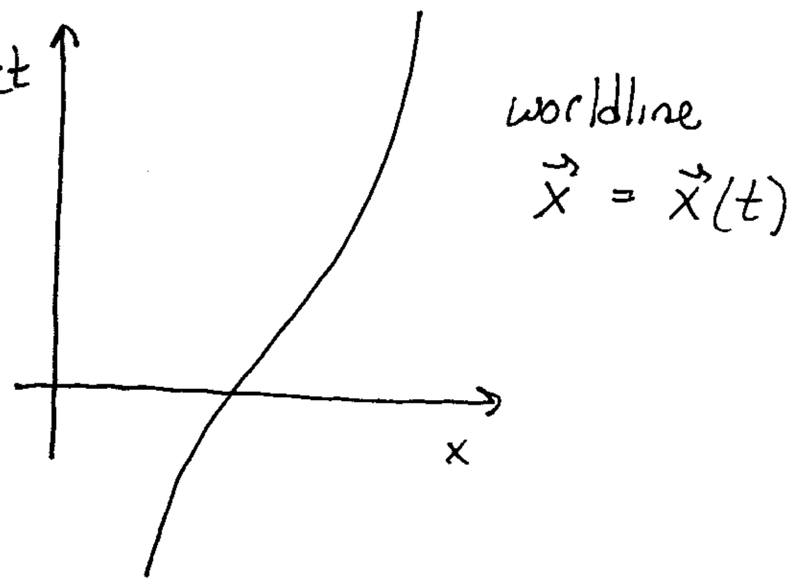
or

$$1 = \left(\frac{dt}{d\tau}\right)^2 - \frac{1}{c^2}\left(\frac{d\vec{x}}{d\tau}\right)^2$$

Define 4-vector $J^\alpha = (c\rho, \vec{J})$

$$\partial_\alpha J^\alpha = \frac{1}{c}\frac{\partial}{\partial t} c\rho + \vec{\nabla}\cdot\vec{J} = 0$$

4-velocity



parameterize by proper time $x^\alpha = x^\alpha(\tau)$

i.e. $t = t(\tau)$ $\quad \vec{x} = \vec{x}(\tau)$

$$u^\alpha = \frac{dx^\alpha}{d\tau} = \left(\frac{dx^0}{d\tau}, \frac{d\vec{x}}{d\tau}\right)$$

Recall $\frac{dt}{d\tau} = \gamma$ $\quad x^0 = ct$ so

$$u^\alpha = (\gamma c, \gamma\vec{v})$$

$$d\tau^2 = \frac{1}{c^2} ds^2 = \frac{1}{c^2} g_{\alpha\beta}\, dx^\alpha dx^\beta$$

or $$1 = \frac{1}{c^2} g_{\alpha\beta} \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = \frac{1}{c^2} u^\alpha u_\alpha$$

so components of u^α are not all independent

0th component : γc

Consider 4-momentum $p^\alpha = m u^\alpha$

$$p^0 = \gamma m c$$

in non-relativistic limit $$p^0 \simeq mc\left(1 + \frac{v^2}{2c^2}\right)$$

$$= \frac{1}{c}\left\{ mc^2 + \frac{1}{2} m v^2 \right\}$$

$$= \frac{1}{c}\left\{ \begin{array}{c}\text{rest mass}\\ \text{energy}\end{array} + \text{kinetic} \right\}$$