Relativistic Electrodynamics

Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \Rightarrow \quad \vec{\nabla} \times \vec{A} = \vec{B}$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla} \times \vec{A} \quad \Rightarrow \quad \vec{\nabla} \times \left(\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right) = 0$$

$$\Rightarrow \quad \vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\Phi \qquad \vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

Gauge freedom $\quad \vec{A} \rightarrow \vec{A}' = \vec{A} + \vec{\nabla}\lambda$ leaves $\vec{B}$ unchanged

But we also want $\vec{E}$ to be unchanged

$$\Phi \rightarrow \Phi' = \Phi - \frac{1}{c}\frac{\partial \lambda}{\partial t}$$

Now write equations in terms of potentials

$$\vec{\nabla}\cdot\left(-\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) = -\nabla^2\Phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = 4\pi\rho$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) - \frac{1}{c}\frac{\partial}{\partial t}\left(-\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right)$$

$$= -\nabla^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) + \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\Phi + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t} = \frac{4\pi}{c}\vec{J}$$

Looks messy!

But suppose we use our gauge freedom to set

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0$$

Then we have

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\left\{\begin{matrix}\Phi\\ \vec{A}\end{matrix}\right\} = 0$$

Can we do this?

suppose $$\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = f(\vec{x},t)$$

in a different gauge

$$\vec{\nabla}\cdot\vec{A}' + \nabla^2\lambda + \frac{1}{c}\frac{\partial\Phi'}{\partial t} - \frac{1}{c^2}\frac{\partial\lambda}{\partial t^2} = f(\vec{x},t)$$

now if we find λ so that $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\lambda = f$

then $\vec{\nabla}\cdot\vec{A}' + \frac{1}{c}\frac{\partial\Phi'}{\partial t} = 0$ ✓

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0 \qquad \text{Lorentz gauge}$$

suggests $A^\alpha = (\Phi, \vec{A})$ $\quad \partial_\alpha A^\alpha = 0$ Lorentz gauge is Lorentz covariant

+ we have

$$\Box A^\alpha = \frac{4\pi}{c}J^\alpha \qquad J^\alpha = (c\rho, \vec{J})$$

Back to fields

$$E_x = -\frac{\partial \Phi}{\partial x} - \frac{1}{c}\frac{\partial A_x}{\partial t}$$

$$\partial^\alpha = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z} \right) = \frac{\partial}{\partial x_\alpha}$$

so $$E_x = \partial^1 A^0 - \partial^0 A^1$$

$$B_x = (\vec{\nabla} \times \vec{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

$$= \partial^3 A^2 - \partial^2 A^3$$

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$$ anti-symmetric 2-index tensor
6-comp $\vec{E}, \vec{B}$

$$= \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

field equations

$$\partial_\alpha F^{\alpha\beta} = \partial_\alpha \partial^\alpha A^\beta - \partial^\beta \cancel{\partial_\alpha A^\alpha} \nearrow 0 \text{ in L.G.}$$

$$= \Box A^\beta = \frac{4\pi}{c} J^\beta$$

takes care of 2 "source" equations

Note $\partial_\beta \partial_\alpha F^{\alpha\beta} = 0 = \frac{4\pi}{c} \partial_\beta J^\beta$ ✓ Maxwell term! Charge cons.

$$F_{\alpha\beta} = g_{\alpha\gamma} g_{\beta\delta} F^{\gamma\delta}$$

$\vec{E} \to -\vec{E}$
$\vec{B} \to \vec{B}$

$$= \begin{pmatrix} 0 & E_x & E_y & E_z \\ & 0 & -B_z & B_y \\ & & 0 & -B_x \\ & & & 0 \end{pmatrix}$$

for homogeneous equations, define

$$\epsilon^{\alpha\beta\gamma\delta} = \begin{cases} +1 & \alpha=0,\ \beta=1,\ \gamma=2,\ \delta=3 \text{ + even perm.} \\ -1 & \text{for odd permutations} \\ 0 & \text{if any two are same} \end{cases}$$

$$\mathcal{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$$

$$= \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ & 0 & E_z & -E_y \\ & & 0 & E_x \\ & & & 0 \end{pmatrix}$$

$$\partial_\alpha \mathcal{F}^{\alpha\beta} = 0$$

$$\beta = 0 \Rightarrow \vec{\nabla}\cdot\vec{B} = 0$$

$$\beta = 1,2,3 \Rightarrow \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$$

Lorentz equation

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$$

we'll take $\vec{p}$ to be space part of 4-momentum

$$\frac{d\vec{p}}{d\tau} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)\frac{dt}{d\tau}$$

$$\frac{dt}{d\tau} = \gamma = \frac{u^0}{c} \qquad \frac{dt}{d\tau}\vec{v} = \gamma\vec{v} = \vec{u}$$

so
$$\frac{d\vec{p}}{d\tau} = \frac{q}{c}\left(u^0 E^x + u^y B^z - u^z B^y\right)$$

$$= \frac{q}{c}\left(F^{10}u_0 + F^{12}u_2 + F^{13}u_3\right) + \cancel{F^{11}u_1}^{\,0}$$

$$\boxed{\frac{dp^\alpha}{d\tau} = \frac{q}{c}F^{\alpha\beta}u_\beta}$$

we get an extra equation

$$\frac{dp^0}{d\tau} = \frac{q}{c}F^{0\beta}u_\beta = \frac{q\gamma}{c}\vec{E}\cdot\vec{v}$$

$$\Rightarrow \quad \frac{dU}{dt} = q\,\vec{E}\cdot\vec{v} \qquad \text{work energy!}$$

$U = cp_0$ particle energy

Transformation of fields

Suppose we have $\vec{E}, \vec{B}$ in frame K + we want fields $\vec{E}', \vec{B}'$ in K'

$$F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\delta}} \frac{\partial x'^{\beta}}{\partial x^{\gamma}} F^{\delta\gamma}$$

$$\frac{\partial x'^{\alpha}}{\partial x^{\delta}} = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = A$$

for K' moves at velocity $\vec{v} = v\hat{x}$ relative to K

$$F' = A F \tilde{A}$$

$$E_1' = E_1 \qquad E_2' = \gamma(E_2 - \beta B_3) \qquad E_3' = \gamma(E_3 + \beta B_2)$$

$$B_1' = B_1 \qquad B_2' = \gamma(E_2 + \beta B_3) \qquad B_3' = \gamma(E_3 - \beta B_2)$$

or $\quad E_{\parallel}' = E_{\parallel} \qquad \vec{E}_{\perp}' = \gamma(\vec{E}_{\perp} + \vec{\beta} \times \vec{B})$

Now $\vec{E}_\parallel = \vec{\beta}\,\frac{\vec{\beta}\cdot\vec{E}}{\beta^2}$ $\quad \vec{E} = \vec{E}_\parallel + \vec{E}_\perp$

so $\vec{E}' = \vec{\beta}\,\frac{\vec{\beta}\cdot\vec{E}}{\beta^2} + \gamma\left(\vec{E}_\perp + \vec{\beta}\times\vec{B}\right)$

$$\vec{E}_\perp = \vec{E} - \vec{E}_\parallel = \vec{E} - \frac{\vec{\beta}}{\beta^2}\vec{\beta}\cdot\vec{E}$$

$$= \gamma\left(\vec{E} + \vec{\beta}\times\vec{B}\right) - (\gamma - 1)\frac{\vec{\beta}}{\beta^2}\left(\vec{\beta}\cdot\vec{E}\right)$$

& $\beta^2 = 1 - \frac{1}{\gamma^2}$ so $\frac{1}{\beta^2} = \frac{\gamma^2}{\gamma^2 - 1}$

$\Rightarrow \quad \frac{\gamma-1}{\beta^2} = \frac{\gamma^2(\gamma-1)}{(\gamma-1)(\gamma+1)} = \frac{\gamma^2}{\gamma+1}$

$$\vec{E}' = \gamma\left(\vec{E} + \vec{\beta}\times\vec{B}\right) - \frac{\gamma^2}{\gamma+1}\vec{\beta}\left(\vec{\beta}\cdot\vec{E}\right)$$

$$\vec{B}' = \gamma\left(\vec{B} - \vec{\beta}\times\vec{E}\right) - \frac{\gamma^2}{\gamma+1}\vec{\beta}\left(\vec{\beta}\cdot\vec{B}\right)$$

Field of a charge in constant motion

K' rest frame of charge $\quad$ K rest frame of observer

Observer at P $\quad x_1 = x_3 = 0 \quad x_2 = b$

or $\quad x_1' = -vt', \; x_3' = 0, \; x_2' = b$

find fields in K' - simple Coulomb fields

$$E_1' = \frac{q x_1'}{r'^3} \qquad E_2' = \frac{q x_2'}{r'^3} \qquad E_3' = \frac{q x_3'}{r'^3}$$

$$r' = \left(x_1'^2 + x_2'^2 + x_3'^2\right)^{1/2} = \left((vt')^2 + b^2\right)^{1/2}$$

so $$E_1' = -\frac{q v t'}{r'^3} \qquad E_2' = \frac{q b}{r'^3} \qquad E_3' = 0$$

$$\vec{B}' = 0$$

Now for observer P in K

2 steps - transform fields / write in terms of K quantities

$$E_1 = E_1' \qquad E_2 = \gamma E_2'$$

so $$E_1 = -\frac{\cancel{q} v t'}{r'^3} \qquad E_2' = \frac{\gamma \cancel{q} b}{r'^3}$$

need t' in terms of $t, \vec{x}$

$$t' = \gamma(t - v x_1/c^2) \quad \text{or} \quad t = \gamma(t' + v x_1'/c^2)$$

but $x_1' = -vt'$ so $t = \gamma t'(1 - v^2/c^2) = t'/\gamma$

$$r' = (b^2 + \gamma^2 v^2 t^2)^{1/2}$$

so $$E_1 = -\frac{\cancel{q}\gamma v t}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \qquad E_2 = \frac{\gamma \cancel{q} b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}$$

so $$\vec{E} = \frac{q\gamma}{(b^2+\gamma^2v^2t^2)^{3/2}}\left(-vt\,,\,b\,,\,0\right)$$

$$\vec{r} = \left(-vt, b, 0\right)$$ vector from particle to observer

$$\gamma^2v^2t^2 + b^2 = \gamma^2r^2 + (1-\gamma^2)b^2$$

$$= \gamma^2r^2\left(1 - \frac{\gamma^2-1}{\gamma^2}\frac{b^2}{r^2}\right) \qquad \frac{1}{\gamma^2} = 1-\beta^2$$

$$= \gamma^2r^2\left(1 - \beta^2\sin^2\psi\right) \qquad \sin\psi = \frac{b}{r}$$

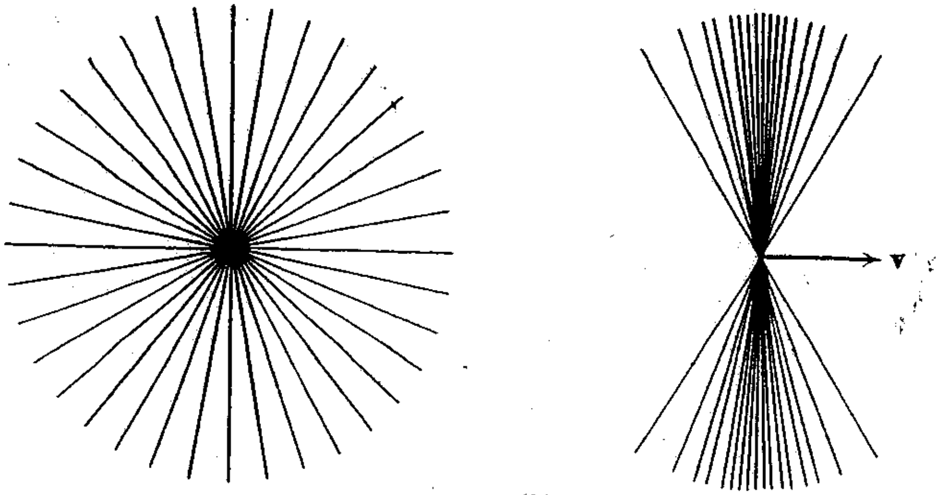
so $$\vec{E} = \frac{q\vec{r}}{\gamma^2r^3\left(1-\beta^2\sin^2\psi\right)^{3/2}}$$ points back to particle!

for $\psi = \pi/2$ (particle crosses $x=0$) $\vec{E} = \frac{q}{b^2}\hat{y} \times \gamma$

increased by γ

$\psi = 0$ observer on x-axis $\vec{E} = \frac{q}{(vt)^2}\hat{x}\ \frac{1}{\gamma^2}$

decreased by $\frac{1}{\gamma^2}$



(*b*)

Fig. 11.9 Fields of a uniformly moving charged particle. (*a*) Fields at the observation point *P* in Fig. 11.8 as a function of time. (*b*) Lines of electric force for a particle at rest and in motion ($\gamma = 3$).

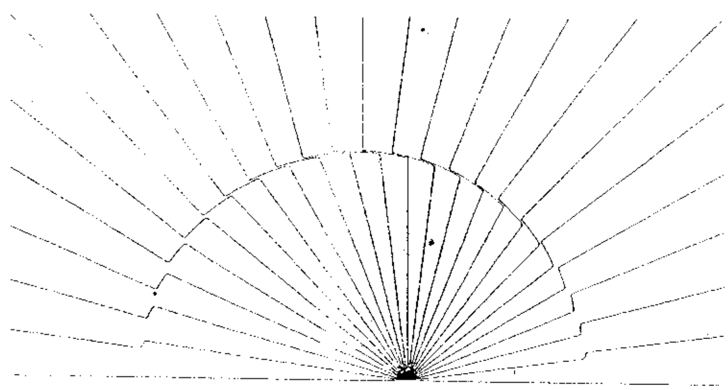


FIG. 11. Electric field lines of a charge accelerated from rest by a uniform force to a final velocity $\beta = 0.20$, $t_0 = 32\Delta t$.

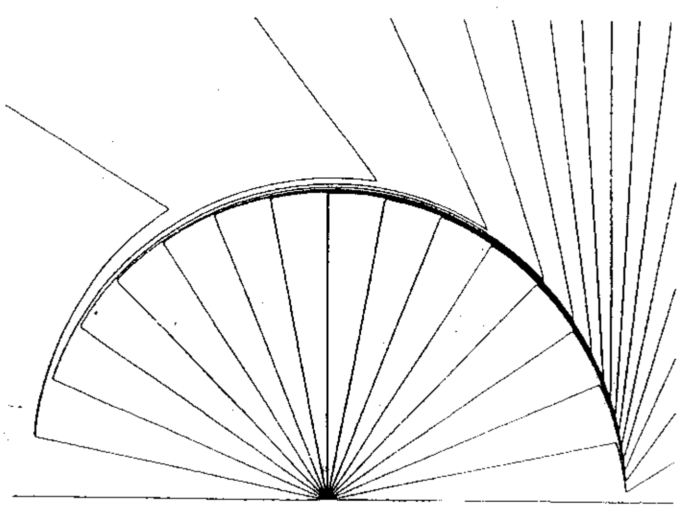


FIG. 12. Electric field lines of a charge decelerated from an initial velocity $\beta = 0.95$ to rest by a uniform force, $t_0 = 16\Delta t$.