# Motion in $\vec{E}$ & $\vec{B}$ fields

## ① Motion in constant, uniform $\vec{B}$ $(\vec{E}=0)$

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right) \qquad \frac{dU}{dt} = q\vec{v}\cdot\vec{E}$$

here $\vec{E}=0$, $U = \text{const} \Rightarrow \gamma = \text{const}$

Recall $\vec{p} = \gamma m \vec{v}$

so
$$\frac{d\vec{v}}{dt} = \vec{v}\times\frac{q}{\gamma mc}\vec{B} \equiv \vec{v}\times\vec{\omega}_B$$

$$\omega_B \equiv \frac{qB}{\gamma mc}$$ gyration or precession frequency

if $\vec{B}$ is along $z$ direction

$$\frac{dv_x}{dt} = v_y\omega_B \qquad \frac{dv_y}{dt} = -v_x\omega_B \qquad \frac{dv_z}{dt} = 0$$

circular motion in $xy$ plane (about origin by choice of coordinates), uniform motion in $z$

$$\vec{v}(t) = v_\perp\left(\sin\omega_B t\,\hat{x} + \cos\omega_B t\,\hat{y}\right) + v_\parallel\hat{z}$$

$$\vec{x}(t) = \frac{v_\perp}{\omega_B}\left(-\cos\omega_B t\,\hat{x} + \sin\omega_B t\,\hat{y}\right) + v_\parallel t\,\hat{z}$$

Will be convenient to have coordinate free form

$$\vec{v}(t) = v_{\parallel}\hat{\epsilon}_3 + \omega_B a\left(\hat{\epsilon}_1 - i\hat{\epsilon}_2\right)e^{-i\omega_B t}$$

$\hat{\epsilon}_3$ along $\vec{B}$ $\hat{\epsilon}_1, \hat{\epsilon}_2$ orthogonal unit vectors to $\vec{B}$

we really mean $\vec{v}(t) = \text{Re}\left\{\text{above expression}\right\}$

$$\vec{v}(t) = v_{\parallel}\hat{\epsilon}_3 + \omega_B a\left(\cos\omega_B t\,\hat{\epsilon}_1 - \sin\omega_B t\,\hat{\epsilon}_2\right)$$

for RH system $\hat{\epsilon}_1 \times \hat{\epsilon}_2 = \hat{\epsilon}_3$

$$\vec{x}(t) = \vec{X}_0 + v_{\parallel}t\hat{\epsilon}_3 + ia\left(\hat{\epsilon}_1 - i\hat{\epsilon}_2\right)e^{-i\omega_B t}$$

ie. $\vec{x}(t) = \vec{X} + v_{\parallel}t\,\hat{\epsilon}_3 + a\sin\omega_B t\,\hat{\epsilon}_1 + a\cos\omega_B t\,\hat{\epsilon}_2$

differs from earlier
expression by 90° phase
in other words, let $\hat{\epsilon}_1 = \hat{y}$ and $\hat{\epsilon}_2 = -\hat{x}$
to get right-handed system.

$$a = \text{gyration radius} = \frac{v_\perp}{\omega_B} = \frac{\gamma m v_\perp c}{qB} = \frac{c p_\perp}{qB}$$

Numerically $\quad \dfrac{p_\perp}{\text{MeV}/c} = \dfrac{3.00 \times 10^{-4} B}{\text{Gauss}} \dfrac{a}{\text{cm}} \quad$ for $|q| = e$

in galaxy $\quad B \simeq 3 \times 10^{-6}$ Gauss $\qquad$ in cluster $10^{-6}$ Gauss

between clusters - unknown

cosmic rays $\quad p_\perp \simeq 10^{4} - 10^{15} \; \dfrac{\text{MeV}}{c}$

$$a \simeq 10^{13} - 10^{24} \text{ cm}$$

$$\simeq 10^{-5} \text{ pc} - 1 \text{ Mpc}$$

② Combined $\vec{E}$ and $\vec{B}$ - assume static + uniform

we'll do special case when they are perpendicular

Recall $\quad \vec{E}' = \gamma(\vec{E} + \vec{\beta} \times \vec{B}) - \dfrac{\gamma^2}{\gamma+1} \vec{\beta}(\vec{\beta} \cdot \vec{E})$

$$\vec{B}' = \gamma(\vec{B} - \vec{\beta} \times \vec{E}) - \frac{\gamma^2}{\gamma+1} \vec{\beta}(\vec{\beta} \cdot \vec{B})$$

Suppose $|\vec{E}| < |\vec{B}|$. find frame in which $\vec{E} = 0$

a) choose $\vec{u}$ = vel. of K' rel to K $\quad \vec{\beta} = \vec{u}/c$

b) $\vec{u}, \vec{E}, \vec{B}$ mutually perpendicular

c) $\vec{\beta}\cdot\vec{E} = 0 \qquad$ want $\quad \vec{E} + \vec{\beta}\times\vec{B} = 0$

let $\quad \vec{\beta} = \dfrac{\vec{E}\times\vec{B}}{B^2}$

$$\vec{\beta}\times\vec{B} = \frac{(\vec{E}\times\vec{B})\times\vec{B}}{B^2} = -\vec{E} \qquad (\vec{E}\cdot\vec{B}=0)$$

so $\quad \vec{E}' = 0$

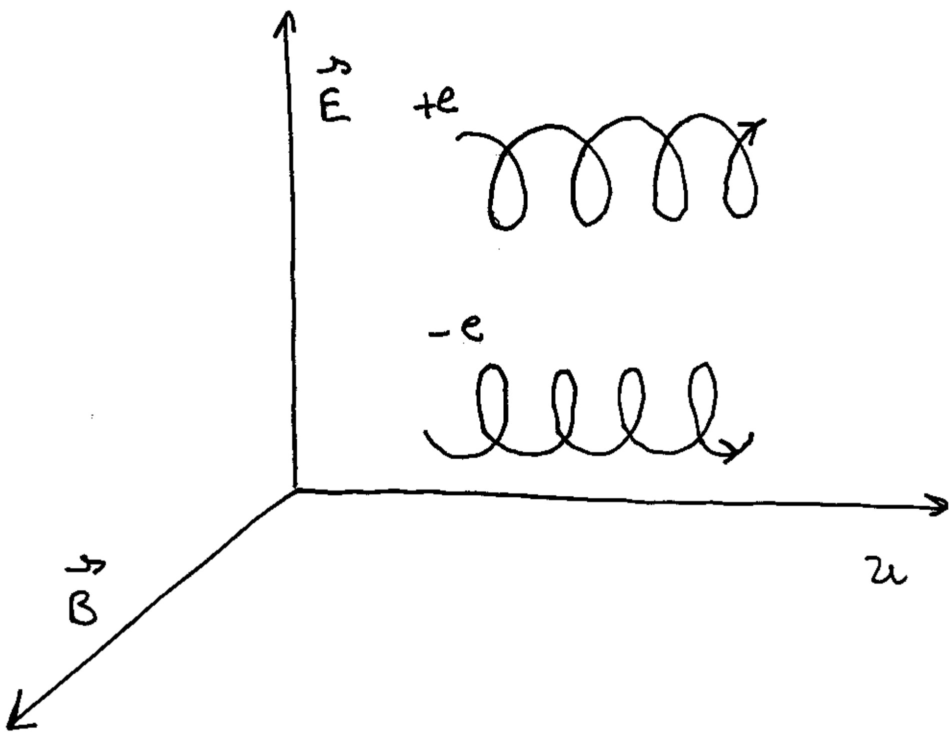
$$\vec{\beta}\times\vec{E} = \frac{\vec{E}\times\vec{B}}{B^2}\times\vec{E} = \frac{E^2}{B^2}\vec{B}$$

so $\quad \vec{B}' = \gamma\left(\vec{B} - \dfrac{E^2}{B^2}\vec{B}\right) = \gamma\left(1 - \dfrac{E^2}{B^2}\right)\vec{B}$

but $\quad \gamma = \dfrac{1}{(1-\beta^2)^{1/2}} = \dfrac{1}{\left(1-\frac{E^2}{B^2}\right)^{1/2}} = \dfrac{B}{(B^2-E^2)^{1/2}}$

so $$\vec{B}' = \frac{1}{\gamma}\vec{B} = \left(1 - \frac{E^2}{B^2}\right)^{1/2}\vec{B}$$

$$\vec{\beta} = \frac{E}{B}\hat{\beta} \qquad \gamma = \left(1 - \frac{E^2}{B^2}\right)^{-1/2}$$




Both +e + -e drift in same direction. + in plane containing $\vec{u}$ and $\vec{E}$

Case $|\vec{E}| > |\vec{B}|$ - transform to frame in which $\vec{B} = 0$ - then have hyperbolic motion (12.3)

is this the unique frame for $\vec{E} = 0$?

No - Boost along the direction of $\vec{B} + \vec{E}$ is still 0!

Note $\vec{E}\cdot\vec{B}$ and $E^2 - B^2$ are Lorentz invariants

So if $\vec{E}\cdot\vec{B} = 0$ $(\vec{E}\perp\vec{B})$ + $|\vec{E}| < |\vec{B}|$

we can find a frame in which $\vec{E}$ vanishes

If $\vec{E}$ + $\vec{B}$ not $\perp$, this wont be possible.

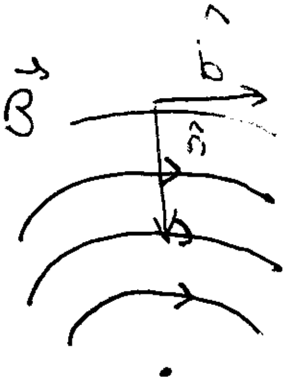
(3) Non-uniform Static fields

Assume distance over which fields change is large compared to gyration radius of particle.

Then we have spiralling along field lines
Gyration radius is set by local field strength
Center of gyration drifts slowly
Slow variations in gyration rate

Consider case where $\vec{\nabla} B \perp \vec{B}$

e.g. field of wire




particle <u>not</u> moving along direction of $\vec{B}$!

$\vec{B} = B(r)\hat{\phi}$ so gradient is in $\hat{\rho}$ direction for $\hat{\phi}$ field.

Use Taylor expansion to describe fields.

( first guess what happens to particle <u>not</u> moving along field lines )

$$\vec{B}(\vec{x}) = \vec{B}(\vec{x}_0) + (\vec{x} - \vec{x}_0) \cdot \vec{\nabla} B(\vec{x}_0)$$

set $\vec{x}_0 = 0$ , define $\vec{B}(\vec{x}_0) = \vec{B}_0$

$\vec{B} = B(\xi)\hat{b}$ $\qquad$ $\xi$ coordinate along $\hat{n}$ - direction of gradient

$\qquad\qquad\qquad\qquad$ $\vec{x} \cdot \hat{n} = \xi$

$$\vec{B}(\vec{x}) = \vec{B}_0 + (\vec{x} \cdot \hat{n})\left(\frac{\partial B}{\partial \xi}\right)_0 \hat{b} = \vec{B}_0\left(1 + (\hat{n} \cdot \vec{x})\frac{1}{B_0}\left(\frac{\partial B}{\partial \xi}\right)_0\right)$$

$$\vec{\omega}_B(\vec{x}) = \frac{e}{\gamma mc}\vec{B}(\vec{x}) = \vec{\omega}_0\left(1 + \frac{1}{B_0}\left(\frac{\partial B}{\partial \xi}\right)_0 \hat{n}\cdot\vec{x}\right)$$

Now $\quad \dfrac{d\vec{v}_\perp}{dt} = \vec{v}_\perp \times \vec{\omega}_B \quad$ assume no motion along field lines

write $\quad \vec{v}_\perp = \vec{v}_0 + \vec{v}_1$

$$\frac{d}{dt}(\vec{v}_0 + \vec{v}_1) = (\vec{v}_0 + \vec{v}_1) \times \vec{\omega}_0\left(1 + \frac{1}{B_0}\left(\frac{\partial B}{\partial \xi}\right)_0 \hat{n}\cdot\vec{x}\right)$$

so $\quad \dfrac{d\vec{v}_0}{dt} = \vec{v}_0 \times \vec{\omega}_0 \quad$ 0th order

$$\frac{d\vec{v}_1}{dt} = \vec{v}_1 \times \vec{\omega}_0 + \vec{v}_0 \times \left[\frac{1}{B_0}\left(\frac{\partial B}{\partial \xi}\right)_0 \hat{n}\cdot\vec{x}_0\right]\vec{\omega}_0$$

1st order

$\vec{x}_0$ is zeroth order motion

Lets eliminate $\vec{v}_0$ in favor of $\vec{x}_0$

Recall $\quad \vec{v}_0 = \omega_0 a(\hat{\epsilon}_1 - i\hat{\epsilon}_2)e^{-i\omega_0 t}$

$$\vec{x}_0 = v_{\parallel} t\hat{\epsilon}_3 + ia(\hat{\epsilon}_1 - i\hat{\epsilon}_2)e^{-i\omega_0 t} \qquad \underline{X} = 0$$

center of gyration radius

$$\vec{\omega}_0 \times \hat{\epsilon}_3 = 0$$

$$\vec{\omega}_0 \times \hat{\epsilon}_2 = -\omega_0 \hat{\epsilon}_1$$

$$\vec{\omega}_0 \times \hat{\epsilon}_1 = \omega_0 \hat{\epsilon}_2$$

gives $\quad \vec{v}_0 = -\vec{\omega}_0 \times \vec{x}_0$

$$\frac{d\vec{v}_1}{dt} = \left[\vec{v}_1 - \frac{1}{B_0}\left(\frac{\partial B}{\partial \xi}\right)_0 \vec{\omega}_0 \times \vec{x}_0(\hat{n}\cdot\vec{x}_0)\right] \times \vec{\omega}_0$$

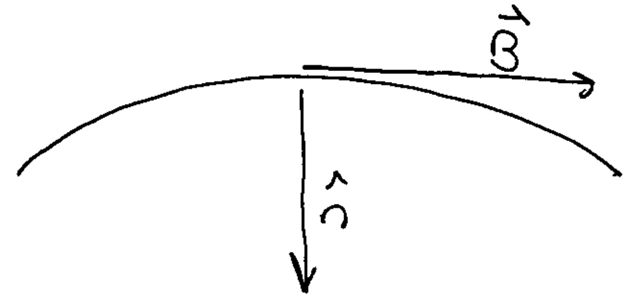
We care about non-oscillating term - net motion averaged over cycles.

$$\langle \vec{v}_1 \rangle \equiv \vec{v}_G = \frac{1}{B_0}\left(\frac{\partial B}{\partial \xi}\right)_0 \vec{\omega}_0 \times \langle \vec{x}_0(\hat{n}\times\vec{x}_0)\rangle$$

Key term is quadratic appearance of $\vec{x}_0$

in other words, compute $\langle \vec{x}_0(\hat{n}\cdot\vec{x}_0)\rangle$

Quick and dirty




Particle orbits in plane containing $\hat{n}$ + $\perp$ to $\vec{B}$

$\hat{\epsilon}_1$ out of page

$$\vec{x}_0 = a(\sin\omega_B t\,\hat{\epsilon}_1 + \cos\omega_B t\,\hat{n})$$

$$\langle \vec{x}_0(\hat{n}\cdot\vec{x}_0)\rangle = a^2\langle \sin\omega_B t\cos\omega_B t\,\hat{\epsilon}_1 + \cos^2\omega_B t\,\hat{n}\rangle$$

$$= \frac{a^2}{2}\hat{n}$$

Gradient drift velocity is $\vec{v}_G = \frac{a^2}{2}\frac{1}{B_0}\left(\frac{\partial B}{\partial \xi}\right)_0 \vec{\omega}_B \times \hat{n}$

$$\vec{\omega}_B = \omega_B \frac{\vec{B}}{B}$$

$$\vec{\nabla}_\perp B = \hat{n}\frac{\partial B}{\partial \xi}$$

$$\boxed{\frac{\vec{v}_G}{a\omega_B} = \frac{a}{2B^2}\vec{B}\times\vec{\nabla}_\perp B}$$

positively + negatively charged particles drift in opposite directions

More generally, particle spirals along field lines

Consider simple set-up

$\vec{B} = B(\rho)\hat{\phi}$ in cylindrical $\rho, \phi, z$ coordinates
e.g field of infinite straight wire

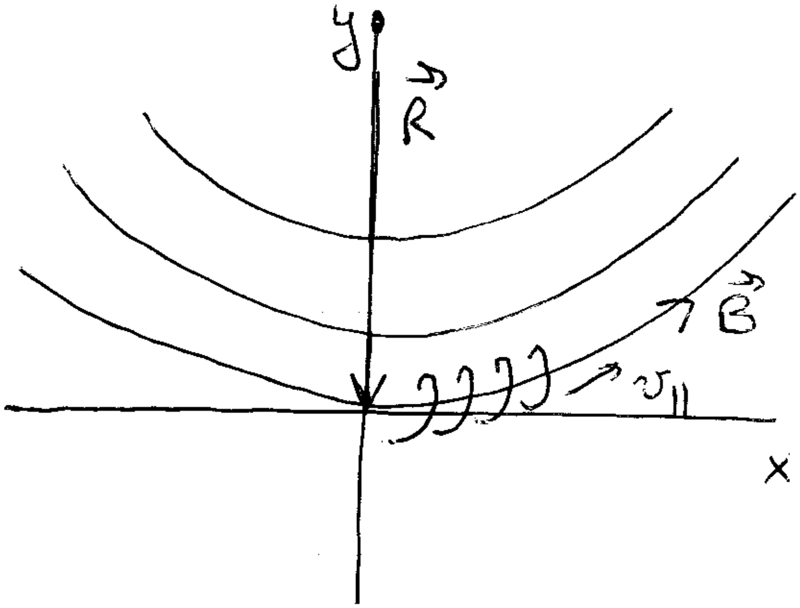
$\vec{\nabla} \cdot \vec{B} = 0$ ✓

Assume no currents (for simplicity) + static

$$\vec{\nabla} \times \vec{B} = \hat{z}\,\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho B_\phi = 0 \quad \Rightarrow \quad B_\phi \propto \frac{1}{\rho}$$

write $\vec{B} = B_0 \frac{R}{\rho}\hat{\phi}$ $\quad \vec{\omega}_B = \omega_0 \frac{R}{\rho}\hat{\phi}$

Lorentz force law $\vec{a} = \vec{v} \times \vec{\omega}_B$




z out of page

$$\vec{B} = B(\rho)\,\hat{\phi}$$

$$= B_0 \frac{R}{\rho}\,\hat{\phi}$$

$$\omega_0 = \frac{qB_0}{\gamma mc} \quad \text{const}$$

$\vec{\nabla}\cdot\vec{B} = 0$ required

$\vec{\nabla}\times\vec{B} = 0$ implies $\vec{J} = 0$

$v_{\parallel}$ in motion in $\hat{\phi}$ direction

$v_{\perp}$ in $\rho, z$ plane   z out of page

Gradient drift is in $+z$ direction (imagine motion in yz plane)

$$\vec{v}_G = \frac{a^2}{2B_0}\left(\frac{\partial B}{\partial \xi}\right)_0 \vec{\omega}_0 \times \hat{n} \qquad \hat{n}\frac{\partial B}{\partial \xi} = \vec{\nabla}_{\perp} B$$

$$\frac{\vec{v}_G}{a\omega_0} = \frac{a}{2B^2}\,\vec{B}\times\vec{\nabla}_{\perp} B$$

$$\vec{r} = \rho\hat{\rho} + z\hat{z}$$

$$\vec{v} = \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\hat{z}$$

$$\vec{v} \times \vec{\omega}_B = \omega_0 \frac{R}{\rho}\dot{\rho}\hat{z} - \omega_0 \frac{R}{\rho}\dot{z}\hat{\rho}$$

$$\vec{a} = (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\rho} + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\hat{\phi} + \ddot{z}\hat{z}$$

① $\ddot{\rho} - \rho\dot{\phi}^2 = -\omega_0 \frac{R}{\rho}\dot{z}$

② $\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi} = 0$

③ $\ddot{z} = \omega_0 \frac{R}{\rho}\dot{\rho}$

② conservation of ang. momentum $\rho^2\dot{\phi} = L = \text{constant}$

$\equiv v_{\parallel} R$

motion along field lines

$$\dot{z} = R\omega_0 \ln \rho/R + v_0 \qquad v_0 \text{ integration constant}$$

$$\ddot{\rho} - \frac{L^2}{\rho^3} = -\omega_0^2 R^2 \frac{\ln \rho/R}{\rho} - \omega_0 v_0 \frac{R}{\rho}$$

has form $\ddot{\rho} = -\frac{dV_{eff}}{d\rho}$ $\quad V_{eff}$ eff. potential

Now we don't really care about details of motion in $\rho$

We find $\rho = R + x_{eq}$ + oscillations (gyrations)

$x_{eq}$ is solution to $V' = 0$ + small compared to $R$

$$V' = -\frac{L^2}{\rho^3} + \omega_0^2 R^2 \frac{\ln \rho/R}{\rho} + \omega_0 v_0 \frac{R}{\rho}$$

$$\simeq -\frac{L^2}{R^3} + 3\frac{L^2}{R^3}\frac{x}{R} + \omega_0^2 R \frac{x}{R} + \omega_0 v_0 - \omega_0 v_0 \frac{x}{R}$$

$$= 0 \quad \text{for} \quad \frac{x_{eq}}{R} = \frac{L^2/R^3 - \omega_0 v_0}{\omega_0^2 R + 3L^2/R^3 - \omega_0 v_0}$$

Now we assume $\omega_0 R = v_\perp$

$$\gg \begin{cases} v_0 \\ L/R = v_\parallel \end{cases}$$

then
$$\frac{X_{eq}}{R} = \frac{L^2}{\omega_0^2 R^4} - \frac{v_0}{\omega_0 R}$$

$$= \frac{v_\parallel^2}{\omega_0^2 R^2} - \frac{v_0}{\omega_0 R}$$

$$= \frac{\langle x \rangle}{R}$$

$$\langle \dot{z} \rangle = v_0 + \omega_0 \langle x \rangle = \frac{v_\parallel^2}{\omega_0 R}$$

$$= v_c \qquad \text{curvature drift}$$

$v_c$ is in $+z$ direction or $\vec{R} \times \vec{B}$ direction

so we write $\quad \vec{v}_c = \dfrac{v_{\parallel}^2}{\omega_0 R} \dfrac{\vec{R} \times \vec{B}}{RB}$

Same direction as gradient drift.

Recall $\quad \vec{v}_G = \dfrac{a^2 \omega_B}{2B^2} \vec{B} \times \vec{\nabla}_{\perp} B$

for our example $\quad \vec{B} = B_0 \dfrac{R}{\rho} \hat{\phi} \qquad \vec{\nabla}_{\perp} B = -\dfrac{\vec{R}}{R^2} B$

so $\quad \vec{v}_G = \dfrac{\omega_B a^2}{2R} \dfrac{\vec{R} \times \vec{B}}{RB} \qquad v_{\perp} = \omega_B a$

so $\quad \vec{v}_D = \vec{v}_c + \vec{v}_G = \dfrac{1}{\omega_0 R} \left( v_{\parallel}^2 + \frac{1}{2} v_{\perp}^2 \right) \dfrac{\vec{R} \times \vec{B}}{RB}$

$$\frac{v_D}{\mathrm{cm/s}} = \frac{172\,\frac{T}{K}}{\frac{R}{m}\;\frac{B}{\mathrm{gauss}}}$$

i.e. $v_{\parallel}^2 + \frac{1}{2} v_{\perp}^2 \simeq \frac{kT}{m}$ $\qquad \omega_B = \frac{eB}{\gamma m c}$

so $v_D \simeq \frac{kT}{\cancel{m}} \frac{\gamma \cancel{m} c}{eB} \frac{1}{R}$ $\qquad$ independent of mass – $\gamma \simeq 1$

e.g. $R = 1$ meter $\quad B = 10^3$ Gauss

$U = 1$ eV $\quad T = 10^4$ K $\qquad v_D \simeq 2 \times 10^3$ cm/s

$v_{\parallel,\perp} \simeq 100 - 1000$ m/s $\qquad \simeq 10$ m/s

i.e. 1–10% of thermal velocity