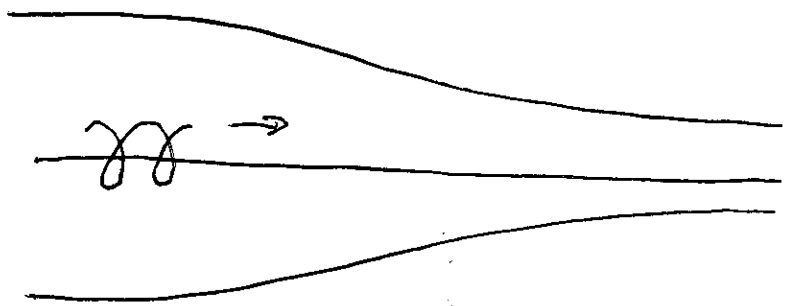
Particle spirals along field which is getting stronger or weaker



If field is static + no $\vec{E}$, then energy is conserved

$$\frac{dU}{dt} = q\vec{E}\cdot\vec{v} = 0 \Rightarrow U = \text{const}$$

$$\Rightarrow \gamma = \text{const} \Rightarrow \boxed{v^2 = v_\parallel^2 + v_\perp^2 = \text{const}}$$

Also, if change is slow (many cycles over time for field to change) then there is an adiabatic invariant

$$= \begin{cases} Ba^2 & \text{flux threading particle orbit} \\ \omega_B a^2 & \omega_B = \frac{qB}{\gamma mc} \\ v_\perp^2/\omega_B \text{ or } v_\perp^2/B & \end{cases}$$

i.e. $$\frac{v_\perp^2}{B(z)} = \frac{v_{\perp 0}^2}{B_0}$$

and we have $$v_0^2 = v_{\parallel}^2 + v_{\perp 0}^2 \frac{B(z)}{B_0}$$

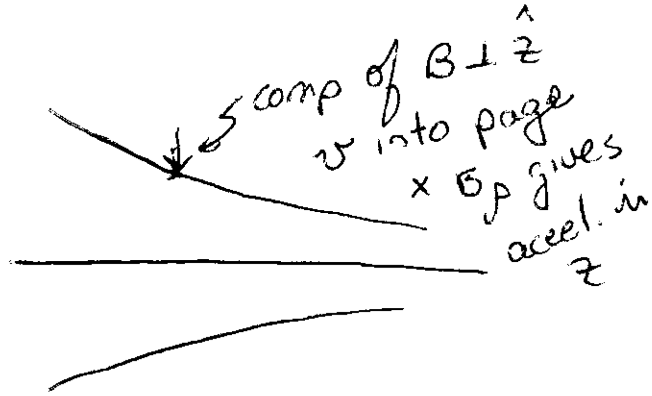
v_0^2 defined as $v^2 = \text{const}$

constants defined so that $v_{\parallel} = 0$ for $z = z_0$, $B(z_0) = B_0$

Problem reduces to that of motion of single particle in 1D potential $\propto B(z)$

z_0 is a turning point

In terms of Lorentz force



Lets use cylindrical coordinates centered on central line of $\vec{B}$

$$\vec{B} = \vec{B}(\rho, z) \quad + \text{ in } \rho - z \text{ plane}$$

Along z-axis $\quad \vec{B}(\rho, z) = B(z)\,\hat{z}$

Near z-axis $\quad \vec{\nabla}\cdot\vec{B} = 0$ implies $\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\rho B_\rho(\rho, z) = -\dfrac{dB}{dz}$

so $\quad B_\rho = -\dfrac{1}{2}\rho\dfrac{dB}{dz}$

$\vec{B}$ static so γ is constant

$$\ddot{\rho} - \rho\dot{\phi}^2 = \frac{q}{\gamma m c}\left(\vec{v}\times\vec{B}\right)_\rho$$

$$\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi} = \frac{q}{\gamma m c}\left(\vec{v}\times\vec{B}\right)_\phi$$

$$\ddot{z} = \frac{q}{\gamma m c}\left(\vec{v}\times\vec{B}\right)_z$$

0th order $\rho = a$ + first equation gives

$$-\rho\dot{\phi}^2 = \frac{q}{\gamma m c}\rho\dot{\phi}B_z \quad \text{or} \quad \dot{\phi} = -\frac{qB}{\gamma m c} \checkmark$$

as well, second gives $\rho^2\dot{\phi} = \text{const.}$

positive ω_B implies neg. $\dot{\phi}$

$$= -a^2\omega_{B,0} = -\frac{v_{\perp 0}^2}{\omega_{B,0}}$$

$$\omega_{B,0} = \frac{qB}{\gamma m c}$$

At 0th or $\ddot{z} = 0$ but small B_ρ cross v_ϕ

implies z force.

$$\ddot{z} = \frac{q}{2\gamma m c}\rho^2\dot{\phi}\frac{dB}{dz}$$

$$\simeq -\frac{v_{\perp 0}^2}{2}\frac{q}{\gamma m c}\frac{1}{\omega_{B,0}}\frac{dB}{dz} = -\frac{v_{\perp 0}^2}{2B_0}\frac{dB}{dz}$$

so $$\frac{1}{2}\dot{z}^2 + \frac{v_{\perp 0}^2}{2}\frac{B(z)}{B_0} = \frac{v_0^2}{2} \quad \text{integration const.}$$

Lagrangian mechanics

non relativistic formalism

$$L = L(q_i, \dot{q}_i, t)$$

$$A = \int_{t_1}^{t_2} L\, dt$$

A an extremum $\quad \delta A = 0$

$$\delta A = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt$$

$$\frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right) - \delta q_i \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i}$$

so $$\delta A = \int_{t_1}^{t_2} \left(\left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i + \cancelto{0}{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right)} \right) dt$$

$$\Rightarrow \quad \frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0$$

NR case $L = K - V = \frac{1}{2}mv^2 - V(\vec{x})$

gives $-\vec{\nabla}V - \frac{d}{dt}m\vec{v} = 0$ or $\vec{F} = m\vec{a}$

Now A should be a Lorentz invariant scalar

$$A = \int_{t_1}^{t_2} L\,dt = \int_{\tau_1}^{\tau_2} L\frac{dt}{d\tau}\,d\tau \qquad \tau \text{ proper time}$$

$$= \int_{\tau_1}^{\tau_2} \gamma L\,d\tau$$

so γL must be a Lorentz invariant scalar

for a free particle, only 4-vector is u^α, only scalar is $u^\alpha u_\alpha = -c^2$

$$L_{free} = -\frac{mc^2}{\gamma} = -mc^2\left(1 - \frac{v^2}{c^2}\right)^{1/2}$$

$$\simeq -mc^2 + \frac{1}{2}mv^2 \quad \text{NR limit}$$

$$\frac{\partial L}{\partial q_i} = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{d}{dt}\left(\frac{mv_i}{(1 - v^2/c^2)^{1/2}}\right) = \frac{d}{dt}\gamma m v_i = 0$$

i.e. $p_i = \text{const.}$

Now we include interactions

NR Lagrangian $-$ $K - V$ $\qquad V = e\Phi$

i.e. $L_{int} = -e\Phi$

Less obvious what to do with Lorentz force.

Use relativity + covariance. For particle at rest $u^\alpha = (c, 0)$; $A^\alpha = (\Phi, \vec{A})$

so $L_{int} = -\dfrac{e}{\gamma c} u^\alpha A_\alpha$ then γL_{int} is Lorentz invariant Scalar + $L_{int} \simeq -e\Phi$ for particle initially at rest.

$$= -e\Phi + \frac{e}{c}\vec{v}\cdot\vec{A}$$

so the Lagrangian for the Lorentz force is $\dfrac{e}{c}\vec{v}\cdot\vec{A}$

$$L = -mc^2\left(1 - v^2/c^2\right)^{1/2} + \frac{e}{c}\vec{v}\cdot\vec{A} - e\Phi$$

variation gives $\dfrac{d}{dt}\gamma m\vec{v} = e\left(\vec{E} + \dfrac{\vec{v}}{c}\times\vec{B}\right)$

where $\vec{E} = -\vec{\nabla}\Phi - \dfrac{\partial\vec{A}}{\partial t}$ $\quad \vec{B} = \vec{\nabla}\times\vec{A}$ and $\dfrac{d}{dt} = \dfrac{\partial}{\partial t} + \dfrac{d\vec{x}}{dt}\dfrac{\partial}{\partial\vec{x}}$

$$= \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}$$

Canonical momentum $\vec{P}$ conjugate to position $\vec{x}$

$$P_i = \frac{\partial L}{\partial v_i} = \gamma m v_i + \frac{e}{c} A_i = p_i + \frac{e}{c} A_i$$

conjugate momentum is the conserved quantity if L is indepent of the associated coordinate

ordinary (relativistic) momentum

Hamiltonian $\quad H = \vec{P} \cdot \vec{v} - L$

now $\quad \gamma m \vec{v} = \vec{P} - \frac{e}{c}\vec{A}$

$$\frac{v^2}{1 - v^2/c^2} = \left(\frac{\vec{P} - \frac{e}{c}\vec{A}}{m} \right)^2$$

solve for v^2 or γ

$$\frac{v^2}{c^2} = \frac{\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2}{\left(\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2 + m^2c^2\right)^{1/2}}$$

$$\vec{v} = \frac{c\vec{P} - e\vec{A}}{\left(\left(\vec{P} - \frac{e\vec{A}}{c}\right)^2 + m^2c^2\right)^{1/2}}$$

$$H = \left(\left(c\vec{P} - e\vec{A}\right)^2 + m^2c^4\right)^{1/2} + e\Phi$$

i.e. $\quad \vec{P} \cdot \vec{v} + mc\left(1 - v^2/c^2\right)^{1/2} - \frac{e}{c}\vec{v} \cdot \vec{A} + e\Phi$

H gives the total energy W of the particle
$U + e\Phi$ where U is kinetic energy
(conserved if fields are time independent)

$$\left(W - e\Phi\right)^2 = \left(c\vec{P} - e\vec{A}\right)^2 + \left(mc^2\right)^2$$

So p^α usual 4-momentum of particle

$$p_\alpha p^\alpha = (mc)^2$$

Lorentz invariant
$E^2 = (pc)^2 + (mc^2)^2$

$$P^\alpha = p^\alpha + \frac{e}{c} A^\alpha$$

$$= \left(\frac{E}{c} + \frac{e}{c}\Phi \, , \, \vec{p} + \frac{e}{c}\vec{A}\right)$$

canonical conjugate 4-momentum

i.e. $P^0 = \dfrac{E}{c} + \dfrac{e\Phi}{c}$ zeroth comp. of canonical momentum

$$H = \left((c\vec{P} - e\vec{A})^2 + m^2c^4 \right)^{1/2} + e\Phi$$

Hamiltonian.

Derive Lorentz force from Hamilton's equations

$$-\frac{\partial H}{\partial x_i} = \frac{dP^i}{dt} \qquad \frac{\partial H}{\partial P_i} = \frac{dx^i}{dt}$$

$$\downarrow$$

gives $\vec{v} = \dfrac{c\vec{P} - e\vec{A}}{(\qquad)^{1/2}}$ ✓

NR limit

$$H = mc^2 + \frac{1}{2m}\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2 + e\Phi$$

$$= mc^2 + \frac{(\vec{P})^2}{2m} - \frac{e}{mc}\vec{P}\cdot\vec{A} + \frac{e^2}{2mc^2}A^2 + e\Phi$$

Particle in uniform field $\vec{B}$

$$\vec{A} = -\frac{1}{2}\vec{r} \times \vec{B}$$

$$\vec{\nabla} \times \vec{A} = -\frac{1}{2}\vec{\nabla} \times (\vec{r} \times \vec{B})$$

$$= \frac{1}{2}\Big(\vec{B}\underbrace{(\vec{\nabla}\cdot\vec{r})}_{=3\vec{B}} - \underbrace{(\vec{B}\cdot\vec{\nabla})\vec{r}}_{=\vec{B}}\Big) = \vec{B} \checkmark$$

$$-\frac{e}{mc}\vec{P}\cdot\vec{A} = \frac{e}{2mc}\vec{P}\cdot(\vec{r}\times\vec{B}) = \frac{e}{2mc}\vec{B}\cdot(\vec{P}\times\vec{r})$$

$$= -\frac{e}{2mc}\vec{B}\cdot\vec{L}$$

Covariant treatment

Twin paradox - free particle follows path of maximal proper time

$$A \propto \int_{\tau_1}^{\tau_2} d\tau$$

$$c\,d\tau = \left(\eta_{\alpha\beta}\, dx^\alpha dx^\beta\right)^{1/2} = \left(\eta_{\alpha\beta} \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\right)^{1/2} ds$$

s is any monotonic parameter along path

$$A = -mc \int_{\tau_1}^{\tau_2} \left(\eta_{\alpha\beta} \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\right)^{1/2} ds$$

units of ang. momentum same as $\hbar$

$$L_{\text{free}} = -mc\left(\eta_{\alpha\beta} \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\right)^{1/2}$$

Euler-Lagrange
$$\frac{d}{ds}\frac{\partial L}{\partial \frac{dx^\alpha}{ds}} = \frac{\partial L}{\partial x^\alpha}$$

for free particle $\frac{\partial L}{\partial x^\alpha} = 0$ and we have

$$\frac{\partial L}{\partial \frac{dx^\alpha}{ds}} = \text{const.} \qquad \text{i.e.} \qquad \frac{mc\,\frac{dx^\alpha}{ds}}{(\quad)^{1/2}} = 0$$

but $(\quad)^{1/2} ds = c\,d\tau$ so $\frac{dx^\alpha}{d\tau} = \text{constant.}$

for charged particle, must add interaction term.

At our disposal $\frac{dx^\alpha}{ds}$ and A^α

$$A = -\int_{s_1}^{s_2} \left[mc\left(\eta_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\right)^{1/2} + \frac{e}{c}\frac{dx^\alpha}{ds}A_\alpha(x)\right] ds$$

$$= \int_{s_1}^{s_2} L\,ds$$

$$\frac{\partial L}{\partial x^\alpha} = -\frac{e}{c}\frac{dx^\beta}{ds}\frac{\partial A_\beta}{\partial x^\alpha}$$

$$\frac{\partial L}{\partial \frac{dx^\alpha}{ds}} = -mc\frac{\frac{dx_\alpha}{ds}}{\left(\eta_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}\right)^{1/2}} - \frac{e}{c}A_\alpha(x)$$

$$= -m\frac{dx_\alpha}{d\tau} - \frac{e}{c}A_\alpha(x)$$

so $$\frac{d}{d\tau}\left(m\frac{dx^\alpha}{d\tau} + \frac{e}{c}A^\alpha(x)\right) = \frac{e}{c}\frac{dx^\beta}{d\tau}\frac{\partial A_\beta}{\partial x_\alpha}$$

and $$\frac{d}{d\tau} = \frac{dx^\beta}{d\tau}\frac{\partial}{\partial x^\beta}$$

so $$m\frac{d^2x^\alpha}{d\tau^2} + \frac{e}{c}\left(\partial^\beta A^\alpha - \partial^\alpha A^\beta\right)\frac{dx_\beta}{d\tau} = 0$$

$$m\frac{d^2x^\alpha}{d\tau^2} + \frac{e}{c}F^{\alpha\beta}\frac{dx_\beta}{d\tau} = 0 \quad \checkmark$$

Adiabatic invariants

q_i generalize coordinate

p_i conjugate momentum

$$\oint p_i \, dq_i \simeq \text{constant} = J$$

e.g. pendulum with changing length

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1-\cos\theta)$$

$$\simeq \frac{1}{2} m l^2 \dot{\theta}^2 - mgl\frac{\theta^2}{2} \qquad \theta = \theta_0 \cos\omega t \quad \omega = \left(\frac{g}{l}\right)^{1/2}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta}$$

$$J_\theta = \oint p_\theta \, dq_\theta = \int_0^T m l^2 \dot{\theta} \, d\theta \propto m l^2 \omega \theta_0^2 = m l^{3/2} g^{1/2} \theta_0^2$$

So $\quad l^{3/2}\theta_0^2 = \text{const}$

$$E = ml^2\dot{\theta}^2 = ml^2\theta_0^2\omega^2 \qquad \text{so } E/\omega \text{ is approx const.}$$

(though E, ω are not)

$$\vec{J} = \oint \vec{P}_\perp \cdot d\vec{l}$$

$$= \oint \gamma m \vec{v}_\perp \cdot d\vec{l} + \frac{e}{c}\oint \vec{A}\cdot d\vec{l}$$

$$d\vec{l} = \vec{v}_\perp dt = \frac{\vec{v}_\perp}{\omega_B} d\phi \qquad \frac{v_\perp^2}{\omega_B} = \omega_B a^2$$

$$\oint \vec{A}\cdot d\vec{l} = \int (\vec{\nabla}\times\vec{A})\cdot\hat{n}\, dA = \int \vec{B}\cdot\hat{n}\, dA = Ba^2$$

But $\quad B = \gamma \dfrac{mc\omega_B}{e}$

Also two terms have opposite sign.

$$J = \frac{e}{c}\pi B a^2 = \gamma m \omega_B a^2 \pi$$