Lagrangian for EM field

q_i are displacements of masses from equilibrium

i labels mass

$i \longrightarrow x^\alpha$ x^α label for coordinate

$\phi(x)$ the "coordinates" are now values of field at position x

$\longrightarrow \phi_k(x)$ k labels different types of fields

e.g $A^\alpha(x)$

$\dot{q}_i \longrightarrow \dfrac{\partial \phi_k}{\partial x^\alpha}$ we include both time + space derivatives

(toy model - gradient energy comes from spring)

$$A = \int L\, dt$$

$$\rightarrow \int \mathcal{L}\, d^3x\, dt \qquad \mathcal{L} \quad \text{Lagrangian density}$$

$$= \int \mathcal{L}\, d^4x \qquad d^4x \quad \text{is Lorentz invariant scalar}$$

$$= \frac{\epsilon_{\alpha\beta\gamma\delta}\, dx^\alpha dx^\beta dx^\gamma dx^\delta}{24}$$

$$\frac{\partial}{\partial x^\beta} \frac{\partial \mathcal{L}}{\partial(\partial_\beta \phi_k)} = \frac{\partial \mathcal{L}}{\partial \phi_k}$$

Electromagnetism

$$\mathcal{L} = -\frac{1}{16\pi} F^{\alpha\beta} F_{\alpha\beta} - \frac{1}{c} J_\alpha A^\alpha$$

$$= -\frac{1}{16\pi} \eta_{\mu\alpha} \eta_{\nu\beta} \left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) - \frac{1}{c} J_\alpha A^\alpha$$

all known physics

Schrodinger
Feynman
Einstein
Maxwell-Yang-Mills
Yukawa

$$\Psi = \int e^{\frac{i}{\hbar}} \int \left(\frac{R}{16\pi G} - F^2 + \bar{\psi} i D \psi - \lambda \varphi \bar{\psi} \psi + |D\varphi|^2 - V(\varphi) \right)$$

Planck
Newton
Dirac
Higgs

$$\mathcal{L}_{GWS} = \sum_f (\bar{\Psi}_f (i\gamma^\mu \partial\mu - m_f)\Psi_f - eQ_f \bar{\Psi}_f \gamma^\mu \Psi_f A_\mu) +$$

$$+ \frac{g}{\sqrt{2}} \sum_i (\bar{a}_L^i \gamma^\mu b_L^i W_\mu^+ + \bar{b}_L^i \gamma^\mu a_L^i W_\mu^-) + \frac{g}{2c_w} \sum_f \bar{\Psi}_f \gamma^\mu (I_f^3 - 2s_w^2 Q_f - I_f^3 \gamma_5)\Psi_f Z_\mu +$$

$$-\frac{1}{4}|\partial_\mu A_\nu - \partial_\nu A_\mu - ie(W_\mu^- W_\nu^+ - W_\mu^+ W_\nu^-)|^2 - \frac{1}{2}|\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+ +$$

$$-ie(W_\mu^+ A_\nu - W_\nu^+ A_\mu) + ig' c_w (W_\mu^+ Z_\nu - W_\nu^+ Z_\mu|^2 +$$

$$-\frac{1}{4}|\partial_\mu Z_\nu - \partial_\nu Z_\mu + ig' c_w (W_\mu^- W_\nu^+ - W_\mu^+ W_\nu^-)|^2 +$$

$$-\frac{1}{2}M_\eta^2 \eta^2 - \frac{g M_\eta^2}{8 M_W}\eta^3 - \frac{g'^2 M_\eta^2}{32 M_W}\eta^4 + |M_W W_\mu^+ + \frac{g}{2}\eta W_\mu^+|^2 +$$

$$+\frac{1}{2}|\partial_\mu \eta + i M_Z Z_\mu + \frac{ig}{2c_w}\eta Z_\mu|^2 - \sum_f \frac{g}{2}\frac{m_f}{M_W}\bar{\Psi}_f \Psi_f \eta$$

$$\frac{\partial \mathcal{L}}{\partial\, \partial^\alpha A^\beta} = -\frac{1}{4\pi}\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right) = -\frac{1}{4\pi} F_{\alpha\beta}$$

$$\frac{\partial \mathcal{L}}{\partial A^\beta} = -\frac{1}{c} J_\beta$$

so $$+\frac{1}{4\pi}\partial^\alpha F_{\alpha\beta} = +\frac{1}{c} J_\beta$$

Charge conservation is incorporated in eqn.

$$\partial^\beta \partial^\alpha F_{\alpha\beta} = 0 \Rightarrow \partial^\beta J_\beta = 0$$

Homogeneous Maxwell equations also taken into account since we write $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$

$$\mathcal{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}$$ defines dual.

$$\partial_\alpha \mathcal{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\partial_\alpha\left(\partial_\gamma A_\delta - \partial_\delta A_\gamma\right) = 0$$

since ϵ is completely antisymmetric

Proca Lagrangian – add a mass term for photon

$$\mathcal{L}_{Proca} = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} + \frac{\mu^2}{8\pi} A_\alpha A^\alpha - \frac{1}{c} J_\alpha A^\alpha$$

$$\partial_\alpha F^{\alpha\beta} + \mu^2 A^\beta = \frac{4\pi}{c} J^\beta$$

or, with $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$

$$\partial_\alpha \partial^\alpha A^\beta - \partial^\beta \partial_\alpha A^\alpha + \mu^2 A^\beta = \frac{4\pi}{c} J^\beta$$

NB. Current conservation requires $\partial_\beta J^\beta = 0$

But $\partial_\beta \partial_\alpha F^{\alpha\beta} = 0 \implies \partial_\beta A^\beta = 0$ Lorentz guage mandatory!

$$\implies \partial_\alpha \partial^\alpha A^\beta + \mu^2 A^\beta = \frac{4\pi}{c} J^\beta$$

Static limit $\quad (\nabla^2 - \mu^2) A^\alpha = -\frac{4\pi}{c} J^\alpha$

for charge at rest at origin $\quad J^\alpha = q\left(c\,\delta^3(\vec{x}), \vec{0}\right)$

and $\quad A^\alpha = (\Phi, \vec{A})$

so $\quad (\nabla^2 - \mu^2)\Phi = -4\pi q\, \delta^3(\vec{x})$

with $\mu = 0$ we have $\quad \Phi = \frac{q}{r}$

Recall $\quad \nabla^2 = \frac{1}{r}\frac{d^2}{dr^2} r \quad$ spherically symmetric case

so with $\mu \neq 0 \quad \Phi = \frac{q}{r} e^{-\mu r} \quad$ Yukawa potential

free field $\quad \hbar^2\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 - \mu^2\right) A^\alpha = 0$

$$\frac{\hbar}{c}\frac{\partial}{\partial t} \to \frac{E}{c} \qquad \hbar\vec{\nabla} \to p \qquad \hbar\mu \to mc$$

$$E^2 = p^2c^2 + m^2c^4$$

Superconductivity

Current $\vec{J} = Q n_Q \vec{v}_Q$

charge of charge carriers

number density of Q's

velocity of Q's

assume nonrelativistic!

Canonical momentum

$$\vec{P} = m_Q \vec{v} + \frac{Q}{c}\vec{A}$$

so $\vec{J} = Q n_Q \vec{v}_Q$

$$= \frac{Q}{m_Q} n_Q \vec{P} - \frac{Q^2}{m_Q c} n_Q \vec{A}$$

Superconducting state $\vec{P} = 0$ – wave function essentially uniform

so $\vec{J} = -\frac{Q^2}{m_Q c} n_Q \vec{A}$ i.e. $\vec{J} \propto \vec{A}$!

Equation for $\vec{A}$ is

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{A} = \frac{4\pi}{c}\vec{J}$$

$$= -\frac{4\pi Q^2}{m_Q c^2} n_Q A$$

i.e. Proca eqn $\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \mu^2\right)\vec{A} = 0$

$$\mu^2 = \frac{4\pi Q^2 n_Q}{m_Q c^2}$$

for static case, at bdy of ordinary + SC state we have

$$\frac{\partial^2}{\partial z^2}\vec{A} = \mu^2\vec{A} \qquad \text{i.e. } \vec{A} \propto e^{-\mu x}$$

$\lambda_L = \mu^{-1}$ London penetration depth

effective photon mass $(m_\gamma)_{eff} = \frac{\hbar\mu}{c} = \frac{\hbar}{c\lambda_L}$

$$m_\gamma c^2 = \hbar c \mu$$

$$= \left(\frac{4\pi \hbar^2 Q^2 n_Q}{m_Q} \right)^{1/2}$$

$$= \left| \frac{Q}{e} \right| \left(\frac{m_e}{m_Q} \right)^{1/2} \left(\frac{4\pi \hbar^2 e^2 n_Q}{m_e} \right)^{1/2}$$

Lets compare with $E_{Ryd} = \frac{1}{2} m_e c^2 \left(\frac{e^2}{\hbar c} \right)^2$

$$\frac{m_\gamma c^2}{E_{Ryd}} = \left| \frac{Q}{e} \right| \left(\frac{m_e}{m_Q} \right)^{1/2} \left(\frac{16\pi \hbar^6}{m_e^3 e^6} n_Q \right)^{1/2}$$

$$a_0 = \text{Bohr radius} = \frac{\hbar^2}{m_e e^2}$$

$$\frac{m_\gamma c^2}{E_{Ryd}} = \left| \frac{Q}{e} \right| \left(\frac{m_e}{m_Q} \right)^{1/2} \left(16\pi n_Q a_0^3 \right)^{1/2}$$

BCS $\quad Q = -2e \quad m_Q = 2m_e \quad n_Q = n_{eff}/2$

Stress tensor

$$T^{\alpha\beta} = \frac{1}{4\pi}\left(\eta^{\alpha\mu} F_{\mu\lambda} F^{\lambda\beta} + \frac{1}{4}\eta^{\alpha\beta} F^{\mu\lambda} F_{\mu\lambda}\right)$$

$$= \left(\begin{array}{c|c} \text{energy density} & \text{momentum density} \\ \frac{1}{8\pi}\left(E^2+B^2\right) & \frac{1}{4\pi}\left(\vec{E}\times\vec{B}\right) \\ \hline \text{energy flux} & \text{stress} \\ \frac{1}{4\pi}\left(\vec{E}\times\vec{B}\right) & -\frac{1}{4\pi}\left(E_iE_j + B_iB_j - \frac{1}{2}\delta_{ij}\left(E^2+B^2\right)\right) \end{array}\right)$$

$$\partial_\alpha T^{\alpha\beta} = \begin{cases} 0 & \text{vacuum} \\ -\frac{1}{c} F^{\beta\alpha} J_\alpha & \text{with particles} \end{cases}$$