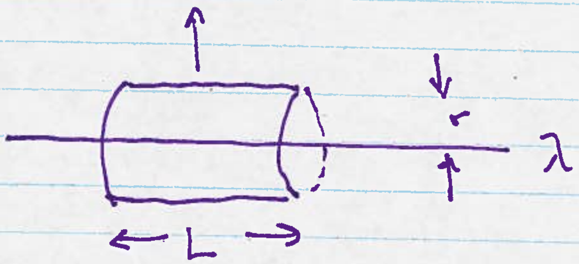
1 a) Electric field via Gauss's law

Midterm Solution

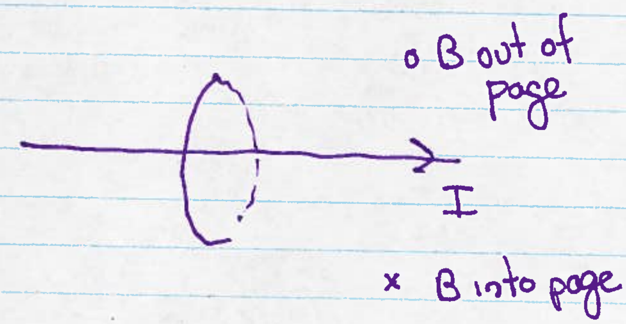


λ = charge/length

$$\int \vec{\nabla}\cdot\vec{E}\, d^3x = \oint \vec{E}\cdot\hat{n}\, da = 2\pi r L E_r$$

$$= \int 4\pi\rho\, d^3x = 4\pi\lambda L \Rightarrow \boxed{E_r = \frac{2\lambda}{r}}$$

Magnetic field via Ampére's law



$$\int \vec{\nabla}\times\vec{B}\cdot\hat{n}\, da = \oint \vec{B}\cdot d\vec{l}$$

$$= 2\pi r B_\phi = \frac{4\pi}{c}\int \vec{J}\cdot\hat{n}\, da$$

$$= \frac{4\pi}{c} I$$

$$\Rightarrow \boxed{B_\phi = \frac{2I}{rc}}$$

b) $J^\alpha = (c\lambda, I, 0, 0)\,\delta(y)\,\delta(z)$

$$J^{0\prime} = c\rho' = \gamma(J^0 - \beta J^1)$$

$$= c\gamma(\rho - \beta J_x/c)$$

or, dropping $\delta(y)\delta(z)$

$$\lambda' = \gamma(\lambda - \beta I/c)$$

$$I' = \gamma(I - \beta c \lambda)$$

$I' = 0$ for $\beta = I/c\lambda < 1$ by assumption ✓

$$E_r' = \gamma(E_r + (\vec{\beta}\times\vec{B})_r) \qquad \vec{\beta}\cdot\vec{E} = 0$$

$\vec{\beta}$ in x-direction $\quad \vec{\beta}\times\vec{B}$ in $-\hat{r}$ direction

$$E_r' = \gamma\left(\frac{2\lambda}{r} - \frac{2I}{rc}\frac{v}{c}\right) \qquad I = \beta c \lambda$$

$$= \gamma\,\frac{2\lambda}{r}(1-\beta^2) = \frac{2\lambda}{r}\frac{1}{\gamma} = \frac{1}{\gamma}E_r$$

c) $B'_\phi = \gamma\left(B_\phi - (\vec{\beta}\times\vec{E})_\phi\right)$ $\vec{\beta}\cdot\vec{B} = 0$

$\vec{\beta}\times\vec{E}$ in $+\hat{\phi}$ direction

$$B'_\phi = \gamma\left(\frac{2I}{cr} - \beta\frac{2\lambda}{r}\right)$$

$I = \beta c\lambda$ so $B'_\phi = \gamma\left(\frac{2\beta\lambda}{r} - \frac{2\beta\lambda}{r}\right) = 0$ ✓

2 a) $\vec{B} = \vec{\nabla} \times \vec{A}$ $\qquad$ $\vec{A}$ is in $\hat{\phi}$ direction

$$B_r = -\frac{\partial A_\phi}{\partial z} = -\frac{2A_0 z}{z_0^2} \frac{\ln(1 + r^2/r_0^2)}{r}$$

$$B_z = \frac{1}{r}\frac{\partial}{\partial r} r A_\phi = \frac{1}{r} A_0 (1 + z^2/z_0^2) \times \frac{2r}{r_0^2} \frac{1}{1 + r^2/r_0^2}$$

$$= \frac{2A_0}{r_0^2} \frac{1 + z^2/z_0^2}{1 + r^2/r_0^2}$$

b) $\omega_B = \dfrac{eB}{\gamma m c} = \dfrac{eBc}{U}$ $\qquad$ $U = \gamma m c^2$

$$a = \frac{v_\perp}{\omega_B} = \frac{\gamma m v_\perp}{\gamma m \omega_B} = \frac{p_\perp c}{eB} = \frac{(U^2 - m^2c^4)^{1/2}}{eB}$$

c) Magnetic bottle effect - use adiabatic invariant + cons. of energy to get

$$v_{\parallel}^2 = v_0^2 - v_{\perp 0}^2 \frac{B(z)}{B_0}$$

Since particle spirals along $r=0$ axis, consider only z-component

$$v_{\parallel}^2 = v_0^2 - v_{\perp 0}^2\left(1 + \frac{z^2}{z_0^2}\right)$$

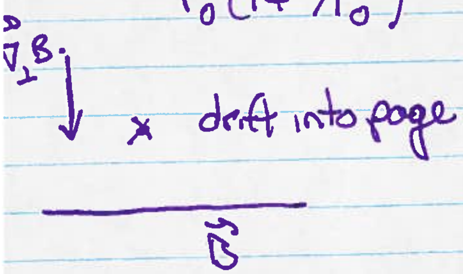
$$= v_0'^2 - v_{\perp 0}^2 \frac{z^2}{z_0^2} \qquad \text{i.e.} \quad \dot{z}^2 + \Omega^2 z^2 = 2E$$

SHO

$$\Omega = \frac{v_{\perp 0}}{z_0} = \omega_B\left(\frac{a}{z_0}\right) \qquad a, \omega_B \text{ as in b)}$$

d) gradient drift $\vec{v}_G = \dfrac{\omega_B a^2}{2} \dfrac{\vec{B} \times \vec{\nabla}_\perp B}{B^2}$

$$\vec{B} = \frac{2A_0}{r_0^2(1+r^2/r_0^2)}\hat{z} \qquad \vec{\nabla}_\perp B = -\frac{4A_0}{r_0^2}\frac{r}{r_0^2}\frac{1}{(1+r^2/r_0^2)^2}\hat{r}$$



$$v_G = \frac{\omega_B a^2}{2}\frac{R}{r_0^2(1+R^2/r_0^2)}(-\hat{\phi})$$

$$\dot{\phi} = \frac{v_G}{R} = \frac{1}{2}\omega_B\left(\frac{a}{r_0}\right)^2\frac{1}{1+R^2/r_0^2} \Rightarrow T = \frac{2\pi}{\dot{\phi}}$$

1. (10 points) An infinite, and infinitely thin, straight wire carries a charge per unit λ and current I as measured in frame K. If the wire is oriented along the x-axis, then the four-current may be written:

$$J^{\alpha} = (c\lambda,\, I,\, 0,\, 0)\, \delta(y)\delta(z) \tag{1}$$

   (a) Find the electric and magnetic fields in frame K as a function of the distance from the wire.

   (b) Show that if $\lambda > I/c$, one can find a frame K' in which the current is zero. (Hint: You may assume the relative motion of K and K' is along the x-axis.) Find the charge per unit length, λ', and the electric field, $\mathbf{E}'$, in K'.

   (c) Show that the magnetic field vanishes in K' by explicitly transforming the fields found in part (a).

2. (15 points) Consider the vector potential

$$\mathbf{A} = A_0 \left(1 + z^2/z_0^2\right) \frac{\ln\left(1 + r^2/r_0^2\right)}{r}\, \hat{\phi} \tag{2}$$

   where A_0, z_0, and r_0 are constants.

   (a) Show that the components of the magnetic field are

$$B_r = -\frac{2A_0 z}{z_0^2} \frac{\ln\left(1 + r^2/r_0^2\right)}{r} \tag{3}$$

$$B_z = \frac{2A_0}{r_0^2} \frac{1 + z^2/z_0^2}{1 + r^2/r_0^2} \tag{4}$$

   (b) A particle of charge q, mass m, and energy $U = \gamma m c^2$ orbits about the origin in the xy-plane. Express the gyration frequency, ω_B, and the gyration radius, a, in terms of these quantities, the constants which describe the magnetic field, and fundamental constants. Note that the relativistic momentum is $\mathbf{p} = \gamma m \mathbf{v}$ and that $U^2 = p^2c^2 + m^2c^4$. Assume $a \ll r_0$.

   (c) Now assume that the particle has a small component to its velocity along the z-axis. Show that the particle executes simple harmonic motion about $z = 0$ and find the period for this motion.

   (d) Assume, instead, that the particle moves in the $z = 0$ plane but at a radius $R \gg a$. Determine the magnitude and direction of the drift velocity. Does the particle drift back to its original position? If so, calculate the period for this motion.

Vector Formulas

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$
$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$
$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$
$$\nabla\times\nabla\psi = 0$$
$$\nabla\cdot(\nabla\times\mathbf{a}) = 0$$
$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$$
$$\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$$
$$\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}$$
$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$
$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$$
$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$

If $\mathbf{x}$ is the coordinate of a point with respect to some origin, with magnitude $r = |\mathbf{x}|$, $\mathbf{n} = \mathbf{x}/r$ is a unit radial vector, and $f(r)$ is a well-behaved function of r, then

$$\nabla\cdot\mathbf{x} = 3 \qquad \nabla\times\mathbf{x} = 0$$
$$\nabla\cdot[\mathbf{n}f(r)] = \frac{2}{r}f + \frac{\partial f}{\partial r} \qquad \nabla\times[\mathbf{n}f(r)] = 0$$
$$(\mathbf{a}\cdot\nabla)\mathbf{n}f(r) = \frac{f(r)}{r}[\mathbf{a} - \mathbf{n}(\mathbf{a}\cdot\mathbf{n})] + \mathbf{n}(\mathbf{a}\cdot\mathbf{n})\frac{\partial f}{\partial r}$$
$$\nabla(\mathbf{x}\cdot\mathbf{a}) = \mathbf{a} + \mathbf{x}(\nabla\cdot\mathbf{a}) + i(\mathbf{L}\times\mathbf{a})$$

where $\mathbf{L} = \frac{1}{i}(\mathbf{x}\times\nabla)$ is the angular-momentum operator.

In the following ϕ, ψ, and $\mathbf{A}$ are well-behaved scalar or vector functions, V is a three-dimensional volume with volume element d^3x, S is a closed two-dimensional surface bounding V, with area element da and unit outward normal $\mathbf{n}$ at da.

$$\int_V \nabla\cdot\mathbf{A}\, d^3x = \int_S \mathbf{A}\cdot\mathbf{n}\, da \qquad \text{(Divergence theorem)}$$
$$\int_V \nabla\psi\, d^3x = \int_S \psi\mathbf{n}\, da$$
$$\int_V \nabla\times\mathbf{A}\, d^3x = \int_S \mathbf{n}\times\mathbf{A}\, da$$
$$\int_V (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)\, d^3x = \int_S \phi\mathbf{n}\cdot\nabla\psi\, da \qquad \text{(Green's first identity)}$$
$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, d^3x = \int_S (\phi\nabla\psi - \psi\nabla\phi)\cdot\mathbf{n}\, da \qquad \text{(Green's theorem)}$$

In the following S is an open surface and C is the contour bounding it, with line element $d\mathbf{l}$. The normal $\mathbf{n}$ to S is defined by the right-hand-screw rule in relation to the sense of the line integral around C.

$$\int_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\, da = \oint_C \mathbf{A}\cdot d\mathbf{l} \qquad \text{(Stokes's theorem)}$$
$$\int_S \mathbf{n}\times\nabla\psi\, da = \oint_C \psi\, d\mathbf{l}$$

Explicit Forms of Vector Operations

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian $(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical (ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}\right) + \mathbf{e}_3 \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical (r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1 \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right] + \mathbf{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi).\right]$$

$$\left.\begin{aligned} x_0' &= \gamma(x_0 - \beta x_1) \\ x_1' &= \gamma(x_1 - \beta x_0) \\ x_2' &= x_2 \\ x_3' &= x_3 \end{aligned}\right\} \tag{11.16}$$

where we have introduced the suggestive notation $x_0 = ct$, $x_1 = z$, $x_2 = x$, $x_3 = y$ and also the convenient symbols,

$$\begin{aligned} \boldsymbol{\beta} &= \frac{\mathbf{v}}{c}, \qquad \beta = |\boldsymbol{\beta}| \\ \gamma &= (1 - \beta^2)^{-1/2} \end{aligned} \tag{11.17}$$

The inverse Lorentz transformation is

$$\left.\begin{aligned} x_0 &= \gamma(x_0' + \beta x_1') \\ x_1 &= \gamma(x_1' + \beta x_0') \\ x_2 &= x_2' \\ x_3 &= x_3' \end{aligned}\right\} \tag{11.18}$$

$$\left.\begin{aligned} x_0' &= \gamma(x_0 - \boldsymbol{\beta} \cdot \mathbf{x}) \\ \mathbf{x}' &= \mathbf{x} + \frac{(\gamma - 1)}{\beta^2}(\boldsymbol{\beta} \cdot \mathbf{x})\boldsymbol{\beta} - \gamma\boldsymbol{\beta}x_0 \end{aligned}\right\} \tag{11.19}$$

$$\left.\begin{aligned} A_0' &= \gamma(A_0 - \boldsymbol{\beta} \cdot \mathbf{A}) \\ A_\parallel' &= \gamma(A_\parallel - \beta A_0) \\ \mathbf{A}_\perp' &= \mathbf{A}_\perp \end{aligned}\right\} \tag{11.22}$$

where the parallel and perpendicular signs indicate components relative to the velocity $\mathbf{v} = c\boldsymbol{\beta}$. The invariance from one inertial frame to another embodied through the second postulate in (11.15) has its counterpart for any 4-vector in the invariance,

$$A_0'^2 - |\mathbf{A}'|^2 = A_0^2 - |\mathbf{A}|^2 \tag{11.23}$$

$$\begin{aligned} E_1' &= E_1 & B_1' &= B_1 \\ E_2' &= \gamma(E_2 - \beta B_3) & B_2' &= \gamma(B_2 + \beta E_3) \\ E_3' &= \gamma(E_3 + \beta B_2) & B_3' &= \gamma(B_3 - \beta E_2) \end{aligned} \tag{11.148}$$

Here and below, the subscripts 1, 2, 3 indicate ordinary Cartesian spatial components and are not covariant indices. The inverse of (11.148) is found, as usual, by interchanging primed and unprimed quantities and putting $\beta \to -\beta$. For a general Lorentz transformation from K to a system K' moving with velocity $\mathbf{v}$ relative to K, the transformation of the fields can be written

$$\begin{aligned} \mathbf{E}' &= \gamma(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}) - \frac{\gamma^2}{\gamma + 1}\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{E}) \\ \mathbf{B}' &= \gamma(\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}) - \frac{\gamma^2}{\gamma + 1}\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{B}) \end{aligned} \tag{11.149}$$

We begin with the charge density $\rho(\mathbf{x}, t)$ and current density $\mathbf{J}(\mathbf{x}, t)$ and the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{11.127}$$

From the discussion at the end of Section 11.6 and especially (11.77) it is natural to postulate that ρ and $\mathbf{J}$ together form a 4-vector J^α:

$$J^\alpha = (c\rho, \mathbf{J}) \tag{11.128}$$

Then the continuity equation (11.127) takes the obviously covariant form,

$$\partial_\alpha J^\alpha = 0 \tag{11.129}$$

In the Lorentz family of gauges the wave equations for the vector potential $\mathbf{A}$ and the scalar potential Φ are

$$\begin{aligned} \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{dt^2} - \nabla^2 \mathbf{A} &= \frac{4\pi}{c}\mathbf{J} \\ \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi &= 4\pi\rho \end{aligned} \tag{11.130}$$

with the Lorentz condition,

$$\frac{1}{c}\frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{A} = 0 \tag{11.131}$$

The differential operator form in (11.130) is the invariant four-dimensional Laplacian (11.78), while the right-hand sides are the components of a 4-vector. Obviously, Lorentz covariance requires that the potentials Φ and $\mathbf{A}$ form a 4-vector potential,

$$A^\alpha = (\Phi, \mathbf{A}) \tag{11.132}$$

Then the wave equations and the Lorentz condition take on the manifestly covariant forms,

$$\Box A^\alpha = \frac{4\pi}{c} J^\alpha$$

and

$$\partial_\alpha A^\alpha = 0 \tag{11.133}$$

The fields $\mathbf{E}$ and $\mathbf{B}$ are expressed in terms of the potentials as

$$\begin{aligned} \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\Phi \\ \mathbf{B} &= \nabla \times \mathbf{A} \end{aligned} \tag{11.134}$$

The x components of $\mathbf{E}$ and $\mathbf{B}$ are explicitly

$$E_x = -\frac{1}{c}\frac{\partial A_x}{\partial t} - \frac{\partial \Phi}{\partial x} = -(\partial^0 A^1 - \partial^1 A^0)$$
$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -(\partial^2 A^3 - \partial^3 A^2) \tag{11.135}$$

where the second forms follow from (11.132) and $\partial^\alpha = (\partial/\partial x_0, -\nabla)$. These equations imply that the electric and magnetic fields, six components in all, are the elements of a *second-rank, antisymmetric field-strength tensor*,

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \tag{11.136}$$

Explicitly, the field-strength tensor is, in matrix form,

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{11.137}$$

For reference, we record the field-strength tensor with two covariant indices,

$$F_{\alpha\beta} = g_{\alpha\gamma}F^{\gamma\delta}g_{\delta\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \tag{11.138}$$

The elements of $F_{\alpha\beta}$ are obtained from $F^{\alpha\beta}$ by putting $\mathbf{E} \to -\mathbf{E}$. Another useful quantity is the *dual field-strength tensor* $\mathcal{F}^{\alpha\beta}$. We first define the totally antisymmetric fourth-rank tensor $\epsilon^{\alpha\beta\gamma\delta}$:

$$\epsilon^{\alpha\beta\gamma\delta} = \begin{cases} +1 & \text{for } \alpha = 0, \beta = 1, \gamma = 2, \delta = 3, \text{ and} \\ & \text{any even permutation} \\ -1 & \text{for any odd permutation} \\ 0 & \text{if any two indices are equal} \end{cases} \tag{11.139}$$

Note that the nonvanishing elements all have one time and three (different) space indices and that $\epsilon_{\alpha\beta\gamma\delta} = -\epsilon^{\alpha\beta\gamma\delta}$. The tensor $\epsilon^{\alpha\beta\gamma\delta}$ is a *pseudotensor* under spatial inversions. This can be seen by contracting it with four different 4-vectors and examining the space inversion properties of the resultant rotationally invariant quantity. The dual field-strength tensor is defined by

$$\mathcal{F}^{\alpha\beta} = \tfrac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} \tag{11.140}$$

The elements of the dual tensor $\mathcal{F}^{\alpha\beta}$ are obtained from $F^{\alpha\beta}$ by putting $\mathbf{E} \to \mathbf{B}$ and $\mathbf{B} \to -\mathbf{E}$ in (11.137). This is a special case of the duality transformation (6.151).

$$\nabla \cdot \mathbf{E} = 4\pi\rho$$
$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{J}$$

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c}J^\beta$$

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0$$

can be written in terms of the dual field-strength tensor as

$$\partial_\alpha \mathcal{F}^{\alpha\beta} = 0 \tag{11.142}$$

In terms of $F^{\alpha\beta}$, rather than $\mathcal{F}^{\alpha\beta}$, these homogeneous equations are the four equations

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \tag{11.143}$$

$$\frac{d\mathbf{p}}{dt} = e\left[\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B}\right]$$

$$\frac{dE}{dt} = e\mathbf{u} \cdot \mathbf{E}$$

$$\frac{d\mathbf{v}}{dt} = \mathbf{v} \times \boldsymbol{\omega}_B$$

$$\boldsymbol{\omega}_B = \frac{e\mathbf{B}}{\gamma mc} = \frac{ec\mathbf{B}}{E}$$

Thus the gradient drift velocity is given by

$$\mathbf{v}_G = \frac{a^2}{2}\frac{1}{B_0}\left(\frac{\partial B}{\partial \xi}\right)_0 (\boldsymbol{\omega}_0 \times \mathbf{n})$$

An alternative form, independent of coordinates, is

$$\frac{\mathbf{v}_G}{\omega_B a} = \frac{a}{2B^2}\,(\mathbf{B} \times \nabla_\perp B)$$

$$\mathbf{v}_C \simeq c\,\frac{\gamma m}{e}\,v_\parallel^2\,\frac{\mathbf{R} \times \mathbf{B}_0}{R^2 B_0^2} \tag{12.57}$$

With the definition of $\omega_B = eB_0/\gamma mc$, the curvature drift can be written

$$\mathbf{v}_C = \frac{v_\parallel^2}{\omega_B R}\left(\frac{\mathbf{R} \times \mathbf{B}_0}{RB_0}\right) \tag{12.58}$$

$$v_\parallel^2 = v_0^2 - v_{\perp 0}^2\,\frac{B(z)}{B_0}$$