Midterm Sol$^{\underline{n}}$

1. $\Box A^\mu = \mu_0 J^\mu$, where $\partial_\alpha A^\alpha = 0$ Lorenz gauge

2. a) $\rho(r, \phi, z) = \frac{Q}{2\pi R} \delta(r-R)\,\delta(z)$

Note $\int \rho\, d^3x = \int \rho\, r\, d\theta\, dr\, dz$

$= \frac{Q(2\pi)}{2\pi R}\int \delta(r-R)\delta(z)\, r\, dr\, dz$

$= \frac{QR}{R} = Q$ ✓

$\vec{J}(r, \phi, z) = \rho R \omega_B (-\hat{\phi})$

$= -\frac{Q}{2\pi}\omega_B\, \delta(r-R)\,\delta(z)\,\hat{\phi}$

$\omega_B = \frac{QB}{\gamma m}$

b) choose $\vec{\beta}$ such that $\vec{\beta}\times\vec{B}c = -\vec{E}$

$\vec{\beta} = \frac{\frac{\vec{E}}{c}\times\vec{B}}{B^2} = \frac{-EB}{cB^2}\hat{y} = \frac{-E}{cB}\hat{y}$

then in S' $\vec{E}' = \gamma(\vec{E} + \vec{\beta}\times\vec{B}c)$

$\vec{E}' = \gamma(\vec{E} - \frac{EB^2c}{cB^2}\hat{x}) = 0$

note: $\vec{\beta}\cdot\vec{E} = 0 = \vec{\beta}\cdot\vec{B}$ since $\vec{\beta}\perp\vec{E}$, $\vec{\beta}\perp\vec{B}$

in S' $\vec{B}' = \gamma(\vec{B} - \vec{\beta}\times\frac{\vec{E}}{c})$

$\vec{B}' = \gamma(B\hat{z} - \frac{E}{cB}\left(\frac{E}{c}\right)\hat{z})$

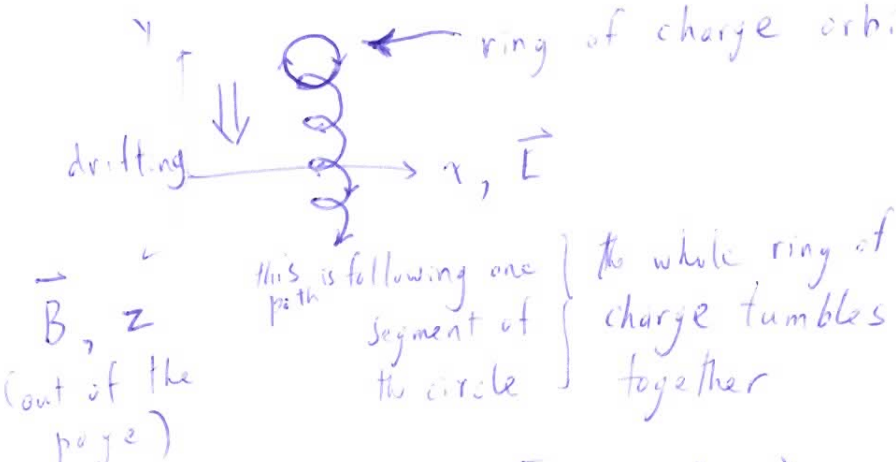
$\vec{B}' = \gamma\left[B - \left(\frac{E}{c}\right)^2\left(\frac{1}{B}\right)\right]\hat{z}$

$= \gamma B\left[1 - \left(\frac{E}{c}\right)^2\left(\frac{1}{B}\right)^2\right]\hat{z}$

$\vec{B}' = \gamma B[1-\beta^2]\hat{z} = \frac{1}{\gamma}B\hat{z}$

2c) In S' frame, one still observes the ring of charge orbiting $\vec{B}$ field (which is weaker) with lower ω_B.

The S' frame is moving in the $-\hat{y}$ direction with respect to the S frame, so the orbiting charge ring drifts in that direction as seen in the S frame.



2d) $J_\alpha A^\alpha = c\rho \frac{\Phi}{c} - \vec{J} \cdot \vec{A}$

since no $\vec{E}$ field, choose $\Phi = 0$ everywhere

find $\vec{A}$ using Stokes theorem $\int_S (\vec{\nabla} \times \vec{A}) \cdot \hat{n} \, da = \oint_C \vec{A} \cdot d\vec{l}$

$$B_z \pi r^2 = A_\phi 2\pi r$$

$$\therefore A_\phi = \frac{1}{2} r B_z$$

verify $\vec{\nabla} \times \vec{A} = \frac{1}{r}\left(\frac{\partial}{\partial r}(r A_\phi)\right)\hat{z} = B\hat{z}$ ✓

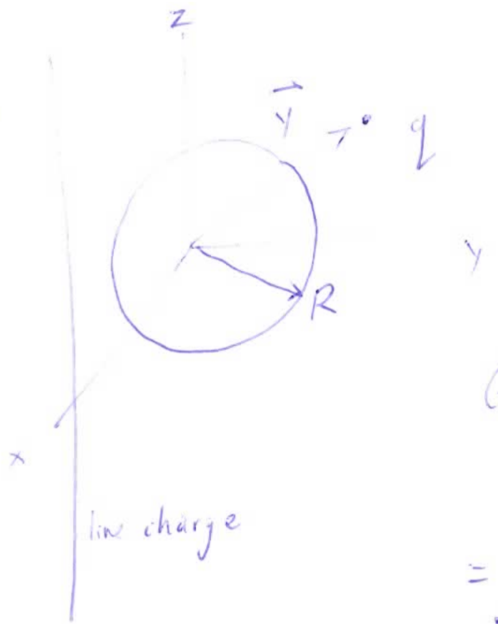
$$\vec{A} = \frac{1}{2} r B \, \hat{\phi}$$

$$\vec{J} = -\frac{Q}{2\pi} \omega_B \, \delta(r-R)\delta(z)\,\hat{\phi}$$

$$\therefore J_\alpha A^\alpha = + \frac{Q}{4\pi} r B \omega_B \, \delta(r-R)\delta(z) = J'_\alpha A'^\alpha$$

since this quantity is a Lorentz scalar

3.



a) $G(\vec{x},\vec{x}') = \dfrac{1}{|\vec{x}-\vec{x}'|} - \dfrac{R/x'}{\left|\vec{x}-\frac{R^2}{x'^2}\vec{x}'\right|}$

b) for all $\vec{x} = R\hat{r}$

$$G(\vec{x},\vec{x}') = \frac{1}{|R\hat{r}-\vec{x}'|} - \frac{R/x'}{\left|R\hat{r}-\frac{R^2}{x'^2}\vec{x}'\right|} = \frac{1}{|R\hat{r}-\vec{x}'|} - \frac{1}{\left|x'\hat{r}-\frac{R}{x'}\vec{x}'\right|}$$

$$= \frac{1}{[R^2+x'^2-2R\hat{r}\cdot\vec{x}']^{1/2}} - \frac{1}{[x'^2+R^2-2R(\hat{r}\cdot\vec{x}')]^{1/2}}$$

$$= 0$$

c) $\rho(x,y,z) = \lambda\,\delta(x-2R)\,\delta(y)$

where λ is the charge per unit length

then with Dirichlet boundary conditions

$$\oint_S \cancel{G(\vec{x},\vec{x}')}^{\,0}\,\frac{\partial\Phi}{\partial n'}\,da'$$ term

and with $\Phi(\vec{x}) = 0$ on the boundary $\oint_S \Phi(\vec{x}')^{\,0}\,\frac{\partial G}{\partial n'}\,da'$

$$\therefore \Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \lambda\,\delta(x'-2R)\,\delta(y')\left[\frac{1}{|\vec{x}-\vec{x}'|} - \frac{R/|\vec{x}|'}{\left|\vec{x}-\frac{R^2}{|\vec{x}'|^2}\vec{x}'\right|}\right]d^3x'$$

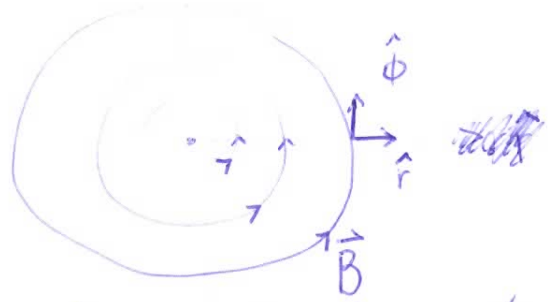
to avoid confusion

$\vec{x}'$ is the vector with components x', y', z'

integrate over all space over all $\vec{x}'$

previously x' denoted the magnitude of $\vec{x}'$ but that's confusing so write $|\vec{x}'|$

4. $\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$



$\omega_B = \frac{eB}{\gamma m}$

In this region, charged particle dominant motion is spiraling around $\vec{B}$ field lines, with gyroradius a

Any parallel velocity component is not affected by $\overset{*}{\vec{B}}$ (to first order)

There are both gradient and curvature drift terms.

$$\boxed{\frac{\vec{V}_G}{\omega_B a} = \frac{a}{2B^2}\left(\vec{B} \times \vec{\nabla}_\perp \vec{B}\right)}$$ from formula sheet

$$\vec{\nabla}_\perp B_\phi = \frac{-\mu_0 I}{2\pi r^2} = -\frac{1}{r} B \hat{r}$$

$$\vec{B} \times \vec{\nabla}_\perp \vec{B} = \frac{B^2}{r}\hat{z}$$

$$\frac{\vec{V}_G}{\omega_B a} = \frac{a}{2r}\hat{z} = \frac{V_\perp^2}{2\omega_B r}\hat{z}$$ $\boxed{\text{note: } \omega_B a \text{ is } V_\perp}$

There is $$\boxed{\vec{V}_c = \frac{V_\parallel^2}{\omega_B R}\left(\frac{\vec{R} \times \vec{B}_0}{R B_0}\right)}$$ also in the $\hat{z}$

from formula sheet

$$\vec{V}_c = \frac{V_\parallel^2}{\omega_B r}\hat{z}$$

∴ total drift velocity $$\vec{V}_D = \frac{\left(V_\parallel^2 + \frac{1}{2}V_\perp^2\right)}{\omega_B r}\hat{z}$$

* charged particles with some parallel velocity component are guided along the field lines, with the lines as guiding centres.