832 PS # 1 Solutions

1a. use Jackson Section 2.5

$$\Phi = -E_0 \left(r - \frac{a^3}{r^2} \right) \cos\theta$$

where E_0 is applied $\vec{E}$ field
a is radius

$$\frac{\sigma}{\epsilon_0} = -\frac{\partial \Phi}{\partial n}\Big|_a = -\frac{\partial \Phi}{\partial r}\Big|_a = E_0 \cos\theta \left(1 - -2\frac{a^3}{r^3}\Big|_{r=a} \right)$$

$$\sigma = 3\epsilon_0 E_0 \cos\theta$$

Force on patch da of surface charge is $d\vec{F} = \frac{\sigma^2}{2\epsilon_0} d\vec{a}$ (from Jackson Fig. 2.4). This represents the "external" field acting on σ surface charge density (which is $\frac{1}{2}\frac{\sigma}{\epsilon_0}$) which results in the conductor having $E = 0$ inside and $E = \frac{\sigma}{\epsilon_0}$ at the surface.

$\vec{F}$ on upper hemisphere: consider z-component only (by symmetry other components net force is zero)

$$\vec{F} = \int \frac{\sigma^2}{2\epsilon_0} d\vec{a} = \int_0^{2\pi} \int_0^1 \underbrace{\frac{(3\epsilon_0 E_0 \cos\theta)^2}{2\epsilon_0} \cos\theta\, a^2 d(\cos\theta)\, d\phi}_{dF_z} \hat{z}$$

$$= \frac{2\pi}{2\epsilon_0} 9 \epsilon_0^2 E_0^2 a^2 \underbrace{\int_0^1 \cos^3\theta \, d(\cos\theta)}_{=\frac{1}{4}} \hat{z}$$

$$\boxed{\vec{F} = \frac{9}{4}\pi \epsilon_0 E_0^2 a^2 \hat{z}}$$

$\vec{F}$ on lower hemisphere is equal and opposite.

The force required to prevent the hemispheres from separating is $\frac{9\pi}{4}\epsilon_0 E_0^2 a^2$ applied from both top and bottom.

1b. The sphere is now charged with Q. A conducting sphere spreads the charge evenly with $\frac{Q}{4\pi a^2}$ charge density.

$$\sigma = \underbrace{3\epsilon_0 E_0 \cos\theta}_{\text{induced}} + \frac{Q}{4\pi a^2}$$

The force on upper hemisphere (similar to part a):

$$\vec{F} = \int \frac{\sigma^2}{2\epsilon_0} d\vec{a} = \frac{1}{2\epsilon_0}(2\pi)\int_0^1 \left[9\epsilon_0^2 E_0^2 \cos^2\theta + \frac{Q^2}{16\pi^2 a^4} + \frac{6\epsilon_0 E_0 \cos\theta\, Q}{4\pi a^2}\right] \cos\theta\, a^2\, d(\cos\theta)\ \hat{z}$$

for z-component

$$\vec{F} = \left\{ \underbrace{\frac{9}{4}\pi\epsilon_0 E_0^2 a^2}_{\text{induced charge force, like part a}} + \frac{2\pi Q^2 a^2}{32\epsilon_0 \pi^2 a^4} \underbrace{\int_0^1 \cos\theta\, d(\cos\theta)}_{=\frac{1}{2}} + \frac{3QE_0}{2} \underbrace{\int_0^1 \cos^2\theta\, d(\cos\theta)}_{=\frac{1}{3}} \right\} \hat{z}$$

Now, if calculate force on lower hemisphere with $\int_{-1}^{0} d(\cos\theta)$

you get $\int_{-1}^{0} \cos^3\theta\, d(\cos\theta) = -\frac{1}{4}$ $\quad \int_{-1}^{0} \cos\theta\, d(\cos\theta) = -\frac{1}{2}$ $\quad \underbrace{\int_{-1}^{0} \cos^2\theta\, d(\cos\theta) = +\frac{1}{3}}$

The force on the lower hemisphere is not equal to the upper. The last term is $+\frac{QE_0}{2}\hat{z}$ upper and $+\frac{QE}{2}\hat{z}$ lower. This is just QE_0 force of charge Q in $E_0\hat{z}$ external field. Does not add to separation force.

The force one needs to keep the hemispheres from separating is

$$\boxed{F = \frac{9}{4}\pi\epsilon_0 E_0^2 a^2 + \frac{Q^2}{32\epsilon_0 \pi a^2}}$$

applied to upper and lower hemispheres.

2. show wave equation invariant under Lorentz transformation

$$\boxed{\Box = \left(\frac{1}{c^2} \frac{\partial}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right)}$$

wave equation operator

with loss in generality consider boost in x-direction

$x' = \gamma(x - \beta c t)$

$y' = y$

$z' = z$

$ct' = \gamma(ct - \beta x) \therefore t' = \gamma(t - \frac{\beta}{c} x)$

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} + \cancel{\frac{\partial y'}{\partial x}}^{\to 0}\frac{\partial}{\partial y'} + \cancel{\frac{\partial z'}{\partial x}}^{\to 0}\frac{\partial}{\partial z'}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial y'} \quad ; \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z'}$$

$$\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} + \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} + \cancel{\frac{\partial y'}{\partial t}}^{0}\frac{\partial}{\partial y'} + \cancel{\frac{\partial z'}{\partial t}}^{0}\frac{\partial}{\partial z'}$$

$$\frac{\partial^2}{\partial x^2} = \left(\gamma \frac{\partial}{\partial x'} - \frac{\beta\gamma}{c}\frac{\partial}{\partial t'}\right)\left(\gamma\frac{\partial}{\partial x'} - \frac{\beta\gamma}{c}\frac{\partial}{\partial t'}\right)$$

$$= \gamma^2 \frac{\partial^2}{\partial x'^2} + \frac{\gamma^2\beta^2}{c^2}\frac{\partial^2}{\partial t'^2} - 2\frac{\beta\gamma^2}{c}\frac{\partial^2}{\partial x' \partial t'}$$

$$\frac{\partial^2}{\partial t^2} = \left(\gamma\frac{\partial}{\partial t'} - \beta\gamma c \frac{\partial}{\partial x'}\right)\left(\gamma\frac{\partial}{\partial t'} - \beta\gamma c\frac{\partial}{\partial x'}\right)$$

$$= \gamma^2\frac{\partial^2}{\partial t'^2} + \beta^2\gamma^2 c^2 \frac{\partial^2}{\partial x'^2} - 2\beta\gamma^2 c \frac{\partial^2}{\partial x' \partial t'}$$

2. cont'd

putting all together

$$\Box = \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right)$$

$$= \left[\frac{1}{c^2} \left(\gamma^2 \frac{\partial^2}{\partial t'^2} + \beta^2 \gamma^2 c^2 \frac{\partial^2}{\partial x'^2} - \cancel{2\beta\gamma^2 c \frac{\partial^2}{\partial x' \partial t'}} \right) \right.$$

$$\left. - \left(\gamma^2 \frac{\partial^2}{\partial x'^2} + \frac{\gamma^2 \beta^2}{c^2} \frac{\partial^2}{\partial t'^2} - \cancel{\frac{2\beta\gamma^2}{c} \frac{\partial^2}{\partial x' \partial t'}} \right) - \frac{\partial^2}{\partial y'^2} - \frac{\partial^2}{\partial z'^2} \right.$$

$$= \frac{1}{c^2} \underbrace{\gamma^2 (1-\beta^2)}_{=1} \frac{\partial^2}{\partial t'^2} - \underbrace{\gamma^2 (1-\beta^2)}_{=1} \frac{\partial^2}{\partial x'^2} - \frac{\partial^2}{\partial y'^2} - \frac{\partial^2}{\partial z'^2}$$

$$= \frac{1}{c^2} \frac{\partial^2}{\partial t'^2} - \frac{\partial^2}{\partial x'^2} - \frac{\partial^2}{\partial y'^2} - \frac{\partial^2}{\partial z'^2}$$

$\therefore$ wave equation operator (and hence Maxwell's wave equation for EM waves) is invariant under Lorentz transformation.

3a. in S' frame, have uniform linear charge density q_0

$\vec{E}'$ in S' frame from Gauss's law

$$\int \frac{q_0\,dx'}{\epsilon_0} = \oint_S \vec{E}' \cdot 2\pi r'\,dx'$$

$\vec{B}' = 0$ (no moving charges) ; $\vec{E}' = \dfrac{q_0}{2\pi\epsilon_0 r'}\,\hat{r}'$ $\qquad \vec{\beta} = \beta\hat{x}$

---

in S frame, $\vec{E}_\perp = \gamma \vec{E}_\perp'$ (since $\vec{B}' = 0$)

$$E_x = E_x' = 0$$

$$B_x = B_x' = 0$$

$$\vec{B}_\perp = \gamma\left(\vec{B}_\perp' + \vec{\beta} \times \frac{\vec{E}'}{c}\right) = \gamma\,\frac{\vec{\beta} \times \vec{E}_\perp'}{c}$$

note: equals $\vec{B}_\perp = \vec{\beta} \times \dfrac{\vec{E}_\perp}{c}$ as it should

note: $r' = r$ (transverse to direction of boost)

$\therefore\ \vec{E} = \dfrac{\gamma q_0}{2\pi\epsilon_0 r}\,\hat{r}$ and $\vec{B} = \dfrac{\gamma q_0}{2\pi\epsilon_0 c\, r}\left(\vec{\beta} \times \hat{r}\right)\left(\beta\hat{\phi}\right)$

3b. $\rho'(\vec{x}') = \rho'(r') = \dfrac{q_0\,\delta(r')}{2\pi r'}$ $\Big\}$ must integrate to linear charge density q_0 ✓ $\int_0^{2\pi}\int_0^R R\,d\theta\,dR$

$$\vec{J}'(\vec{x}') = 0$$

$$J'^{\mu} = (c\rho', 0) \rightarrow J^{\mu} = (\gamma c\rho', \gamma c\vec{\beta}\rho')$$

in lab frame $\rho(r) = \dfrac{\gamma q_0\,\delta(r)}{2\pi r}$

$$\vec{J}(r) = \gamma c\vec{\beta}\,\frac{q_0\,\delta(r)}{2\pi r}$$

3c. In S frame, linear charge density is γq_0 $\left[\text{from } \rho(r) = \frac{\gamma q_0 \delta(r)}{2\pi r}\right]$.

Note: this can be deduced also as Lorentz contracted linear distance along x. Since x distance is shorter, linear charge density is higher by γ!

Directly from Gauss' Law again:

$$\vec{E} = \frac{\gamma q_0}{2\pi\epsilon_0 r}\hat{r}$$

Ampère's Law $\oint_C \vec{B}\cdot d\vec{l} = \mu_0 \oint_S \vec{J}\cdot d\vec{S}$

$$B_\phi(2\pi a) = \mu_0 \gamma c \beta q_0 \underbrace{\oint_S \frac{\delta(r)}{2\pi r} r\, d\theta\, dr}_{=1}$$

$$\int_0^{2\pi}\int_0^a \frac{\delta(r)}{2\pi \cancel{r}} \cancel{r}\, d\theta\, dr$$

$$B_\phi = \frac{\mu_0 \gamma c^2 \beta q_0}{2\pi a \, c}$$

$$B_\phi = \frac{\gamma q_0 \beta}{2\pi\epsilon_0 c a} \quad ✓$$

at radius a

to avoid symbol confusion