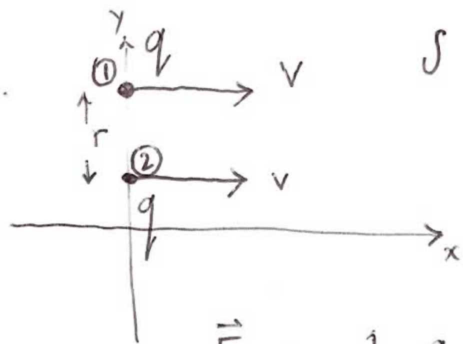
832 PS #2 Solutions

1. 

S a) what are $\vec{E}$ and $\vec{B}$ at ① due to ②?

We saw in class the $\vec{E}$ field from a moving (relativistic) charge [and $\vec{B}$ field]

$$\vec{E} = \frac{1}{4\pi\epsilon_0} q \frac{\gamma}{r^2} \hat{y}$$

$$\vec{B} = \frac{\beta|E|}{c}\hat{z} = \frac{1}{4\pi\epsilon_0}\frac{\beta}{c}\frac{q\gamma}{r^2}\hat{z} = \frac{1}{4\pi\epsilon_0}\frac{v}{c^2}\frac{q\gamma}{r^2}\hat{z} = \boxed{\frac{\mu_0}{4\pi}\frac{qvr}{r^3}\gamma\hat{z}}$$

relativistic "Biot-Savart"

$$\vec{F}_{\text{on 1 by 2}} = q(\vec{E} + \vec{v}\times\vec{B})$$

$$= \left[\frac{1}{4\pi\epsilon_0} q^2 \frac{\gamma}{r^2} - \frac{1}{4\pi\epsilon_0} q^2 \frac{\gamma}{r^2}\frac{v\beta}{c}\right]\hat{y}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2}\gamma\left[1 - \frac{v^2}{c^2}\right]\hat{y} \qquad \gamma^2 = \frac{1}{1 - v^2/c^2}$$

$$\boxed{\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2}\left(\frac{1}{\gamma}\right)}$$

b) in S' rest frame of moving charges, there is only the $\vec{E}$ field

$$\vec{F}'_{\text{on 1 by 2}} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2}\hat{y}' \left(\begin{matrix}\text{note:}\\ r = r'\\ \text{also } \hat{y} = \hat{y}'\end{matrix}\right)$$

S' moving with velocity $\vec{V} = v\hat{x}$ relative to S

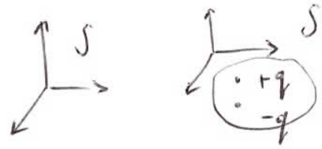
how to get $\vec{F}$ from $\vec{F}'$?

$$\vec{F} \equiv \frac{d\vec{p}}{dt} \quad \therefore \quad F_y' = \frac{dp_y'}{dt'} = \frac{dp_y'}{dt}\frac{dt}{dt'} = \cancel{\frac{dp_y}{dt}\frac{dt}{dt'}} = F_y\left(\frac{dt}{dt'}\right)$$

now $p_y = p_y'$ (that's how transverse momentum transforms under Lorentz boost)

$ct = \gamma(ct' + \beta x')$ and $x' = 0$ for all t $\therefore$ $\frac{dt}{dt'} = \gamma$ $\longrightarrow$ $\boxed{F_y' = F_y\gamma}$

2.



electric dipole moment $\vec{p}$; S' moving with $\vec{\beta}$ velocity with $\vec{\beta}$, $\vec{p}$ pointing arbitrary directions

$J'^{\mu} = (c\rho', 0)$ no current density in S' and $\int \rho'(\vec{x}')\, d^3x' = 0$

$J^{\mu} = (\gamma c \rho', \gamma \vec{\beta} c \rho')$

$$\rho(\vec{x}) = \gamma \rho'(\vec{x}')$$
$$\vec{J}(\vec{x}) = \gamma \vec{v} \rho'(\vec{x}')$$

where $\rho'(\vec{x}')$ could be δ-fn point charges in a dipole configuration, limit as $a \to 0$

$$\rho'(\vec{x}') = \lim_{a \to 0} \left[q\, \delta(\vec{x}' - a\hat{y}') - q\, \delta(\vec{x}' + a\hat{y}') \right]$$

without loss in generality

replace $\vec{x}'$ with $\vec{x}$

$$x^0 = \gamma(x'^0 + \beta x_\parallel') = \gamma(x'^0 + \vec{\beta} \cdot \vec{x}')$$
$$x_\parallel = \gamma(x_\parallel' + \beta x^{0\prime})$$
$$x_\perp = x_\perp'$$

$$\vec{x}_\parallel' = \frac{\vec{\beta}(\vec{\beta} \cdot \vec{x})}{\beta^2}$$
$$\vec{x}_\perp' = \vec{x}' - \vec{x}_\parallel$$

$$\vec{x} = \vec{x}_\parallel + \vec{x}_\perp = \gamma\left(\frac{\vec{\beta}(\vec{\beta} \cdot \vec{x}')}{\beta^2} + \vec{\beta} x^{0\prime}\right) + \vec{x}' - \frac{\vec{\beta}(\vec{\beta} \cdot \vec{x})}{\beta^2}$$

$$\vec{x} = \vec{x}' + \frac{(\gamma - 1)}{\beta^2} \vec{\beta}(\vec{\beta} \cdot \vec{x}') + \gamma \vec{\beta} x^{0\prime}$$

and $\vec{x}' = \vec{x} + \frac{\gamma - 1}{\beta^2} \vec{\beta}(\vec{\beta} \cdot \vec{x}) - \underbrace{\gamma \vec{\beta} x^0}_{\gamma \vec{v} t}$

note: $\rho(\vec{x})$ and $\vec{J}(\vec{x})$ do have time dependence since the dipole is moving in S frame

$$\vec{m} \equiv \frac{1}{2c} \int \vec{x} \times \vec{J}(\vec{x})\, d^3x$$

note: Gaussian units here | SI would be $\vec{m} = \frac{1}{2} \int \vec{x} \times \vec{J}\, d^3x$

$\int d^3x = \frac{1}{\gamma} \int d^3x'$ Lorentz contracted volume integral

$$\vec{m} = \frac{1}{2c\gamma} \int d^3x' \left[\vec{x}' + \frac{\gamma - 1}{\beta^2} \vec{\beta}(\vec{\beta} \cdot \vec{x}') + \gamma \vec{\beta} x^0 \right] \times \gamma \vec{\beta} c \rho'(\vec{x}')$$

2. cont'd

wait, to be exactly correct (doesn't matter for $\vec{m}$, but...)

the volume integral is done at a specific time x^0

but corresponding $x^{0\prime}$ in the integral $x^{0\prime} = \gamma(x^0 - \vec{\beta}\cdot\vec{x})$ has space dependence

or put another way $x^0 = \gamma(x^{0\prime} + \vec{\beta}\cdot\vec{x}\,')$ and $\gamma x^{0\prime} = x^0 - \gamma(\vec{\beta}\cdot\vec{x}\,')$

at fixed x^0 different point in $\vec{x}\,'$ have different $x^{0\prime}$

thus $$\vec{x} = \vec{x}\,' + \frac{\gamma-1}{\beta^2}\vec{\beta}(\vec{\beta}\cdot\vec{x}\,') + \gamma\vec{\beta}x^{0\prime}$$

$$= \vec{x}\,' + \frac{\gamma-1}{\beta^2}\vec{\beta}(\vec{\beta}\cdot\vec{x}\,') + \vec{\beta}\left[x^0 - \gamma(\vec{\beta}\cdot\vec{x}\,')\right]$$

$$\frac{\gamma-1}{\beta^2} = \frac{\gamma^2}{\gamma+1} \Rightarrow \vec{x} = \vec{x}\,' + \left[\frac{\gamma^2-\gamma(\gamma+1)}{\gamma+1}\right]\vec{\beta}(\vec{\beta}\cdot\vec{x}\,') + \beta x^0$$

$$\vec{x} = \vec{x}\,' - \frac{\gamma}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{x}\,') + \beta x^0$$

$$\vec{m} = \frac{1}{2c\gamma}\int d^3x'\left[\vec{x}\,' - \frac{\gamma}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{x}\,') + \vec{\beta}x^0\right]\times\gamma\vec{\beta}\,c\rho'(\vec{x}\,')$$

$$= \frac{1}{2}\int d^3x'\left[\vec{x}\,'\times\vec{\beta}\rho'(\vec{x}\,')\right]$$ since $\vec{\beta}\times\vec{\beta}$ terms $= 0$

$$= \frac{1}{2}\left[\underbrace{\int d^3x'\,\vec{x}\,'\rho'(\vec{x}\,')}_{\equiv\ \vec{p}\,'}\right]\times\vec{\beta}$$ → constant

$\vec{p}\,'$ electric dipole moment in S' frame

note: problem called $\vec{p}$ dipole moment in S' frame (w/o prime) but clearer to call it $\vec{p}\,'$

$$\boxed{\therefore\ \vec{m} = \frac{1}{2}(\vec{p}\times\vec{\beta})}$$

and in SI units would be $\vec{m} = \frac{1}{2}(\vec{p}\times\vec{v})$

$$\vec{p} \equiv \int\vec{x}\rho(\vec{x})d^3x = \frac{1}{\gamma}\int d^3x'\left[\vec{x}\,' - \frac{\gamma}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{x}\,') + \beta x^0\right]\gamma\rho'(\vec{x}\,')$$

$$= \int d^3x'\left[\underbrace{\vec{x}\,'\rho'(\vec{x}\,')}_{\equiv\ \vec{p}\,'} - \frac{\gamma}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{x}\,')\rho'(\vec{x}\,') + \underbrace{\beta x^0\rho'(\vec{x}\,')}_{=0}\right]$$

0 since $\int d^3x'\rho'(\vec{x}\,') = 0$ for any x^0

$$\boxed{\vec{p} = \vec{p}\,' - \frac{\gamma}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{p}\,')}$$

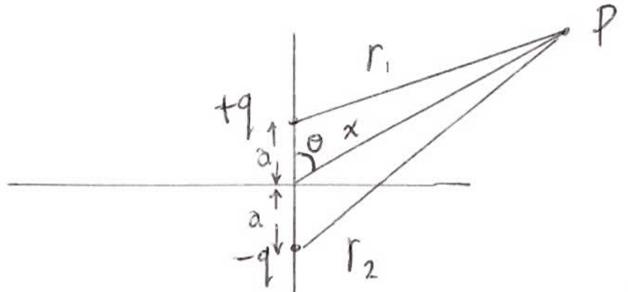
3. In S' frame, find $\Phi'(\vec{x}')$ and $\vec{A}(\vec{x}')$

since $\vec{J}'(\vec{x}') = 0$ then $\vec{A}'(\vec{x}') = 0$

$$\Phi'(\vec{x}') = \frac{1}{4\pi\epsilon_0} \frac{\vec{p}' \cdot \vec{x}'}{|\vec{x}'|^3}$$, where the dipole is at the origin in S'

Derivation:

$$\Phi_{\text{at } P} = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r_1} - \frac{q}{r_2}\right]$$



$$r_1^2 = a^2 + x^2 - 2ax\cos\theta$$
$$r_2^2 = a^2 + x^2 - 2ax\cos(\pi - \theta)$$
$$= a^2 + x^2 + 2ax\cos\theta$$

$$\frac{1}{r_{1 \text{ or } 2}} = \frac{1}{x}\left[1 + \left(\frac{a}{x}\right)^2 \mp 2\left(\frac{a}{x}\right)\cos\theta\right]^{-1/2}$$

for $a \ll x$ $$\approx \frac{1}{x}\left[1 - \frac{1}{2}\left(\frac{a}{x}\right)^2 \pm \left(\frac{a}{x}\right)\cos\theta\right]$$

$$\Phi_{\text{at } P} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{x}\right)\left[1 - \frac{1}{2}\left(\frac{a}{x}\right)^2 + \frac{a}{x}\cos\theta - 1 + \frac{1}{2}\left(\frac{a}{x}\right)^2 + \frac{a}{x}\cos\theta\right]$$

$$= \frac{q}{4\pi\epsilon_0 x} 2\left(\frac{a}{x}\right)\cos\theta$$

$p = q2a \implies \Phi = \frac{1}{4\pi\epsilon_0} \frac{p\cos\theta}{x^2} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{x}}{|\vec{x}|^3}$

transform into S frame

$$A'^{\mu} = \left(\frac{\Phi'}{c}, 0\right)$$

$$A^{\mu} = \left(\gamma\frac{\Phi'}{c}, \gamma\vec{\beta}\frac{\Phi'}{c}\right)$$

$$\vec{x}' = \vec{x} + \frac{\gamma - 1}{\beta^2}\vec{\beta}(\vec{\beta}\cdot\vec{x}) - \gamma\vec{\beta}x^0$$

$$\Phi(\vec{x}) = \gamma\Phi'(\vec{x}') = \gamma\left(\frac{1}{4\pi\epsilon_0}\right)\frac{\vec{p}'\cdot\vec{x}'}{|\vec{x}'|^3} = \gamma\left(\frac{1}{4\pi\epsilon_0}\right)\left[\frac{\vec{p}'\cdot\vec{x} + \frac{\gamma-1}{\beta^2}\vec{p}'\cdot\vec{\beta}(\vec{\beta}\cdot\vec{x}) - \gamma\vec{p}'\cdot\vec{\beta}x^0}{|\vec{x}'|^3}\right]$$

to first order in β

$\gamma = (1-\beta^2)^{-1/2} \approx 1 + \frac{1}{2}\beta^2 \approx 1$ (to first order in β)

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot(\vec{x} - \vec{v}t)}{|\vec{x} - \vec{v}t|^3}; \quad \vec{A}(\vec{x}) = \frac{\vec{\beta}}{c}\Phi(\vec{x})}$$