832 PS # 3 Solutions

in SI units

1. $L = -\frac{mc^2}{\gamma} + e\vec{v}\cdot\vec{A} - e\Phi$

$A^\alpha = \left(\frac{\Phi}{c}, \vec{A}\right)$

$A^\alpha \to A^\alpha + \partial^\alpha \Lambda$

$\partial^\alpha = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right)$

$L' \to L - e\vec{v}\cdot\vec{\nabla}\Lambda - e\frac{\partial}{\partial t}\Lambda$

$\frac{d}{dt} = \left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)$

$$\boxed{L' \to L - e\frac{d}{dt}\Lambda}$$

this generates an "equivalent Lagrangian"; can show by showing L' yields same Euler-Lagrange equations of motion

$$\frac{\partial L'}{\partial q_i} = \frac{\partial L}{\partial q_i} - e\dot{q}_j\frac{\partial^2\Lambda}{\partial q_i q_j} - e\frac{\partial^2\Lambda}{\partial q_i\,\partial t}$$

same as

$$\frac{\partial L'}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i} - e\frac{\partial\Lambda}{\partial q_i} - e\frac{\partial^2\Lambda}{\partial t\,\partial \dot{q}_i}\to 0$$

since Λ has no $\dot{q}_i$ dependence

same as

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\underbrace{- e\frac{\partial^2\Lambda}{\partial q_i\,\partial q_j}\dot{q}_j - e\frac{\partial^2\Lambda}{\partial q_i\,\partial t}}_{\text{where } \frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla} \text{ was used}}$$

$$\underbrace{\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i}\right) - \frac{\partial L'}{\partial q_i}} = \underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}}$$

same Euler-Lagrange equations $\therefore$ equivalent Lagrangian

2. 4-divergence theorem

$$\int_{V^4} \partial_\beta \Theta^{\alpha\beta} d^4x = \oint_{V^3} \Theta^{\alpha\beta} n_\beta \, d^3x$$

4D "volume" — 3D "surface" bounding the 4D spacetime "volume"

n_β is a 4-vector normal to the "surface"

- the problem is source-free $\therefore \partial_\beta \Theta^{\alpha\beta} = 0$
- the problem has EM fields confined to a finite region of space

Let's consider the V^3 "surfaces" bounding the region of spacetime; there are 4*2 such 3D-surfaces

(e.g. $x = +1$, y, z, ct range over values;
$x = -1$, y, z, ct " " ";
$ct = t_b$, x, y, z range over values;
$ct = t_e$, x, y, z " ")
etc.

choose this bounding "box" to have these "surfaces" at values where the EM fields are zero (i.e. outside the finite region)

then $\oint_{V^3} \Theta^{\alpha\beta} n_\beta \, d^3x = \int_{ct = t_b} \Theta^{\alpha\beta} n_\beta \, d^3x + \int_{ct = t_e} \Theta^{\alpha\beta} n_\beta \, d^3x$

since the other bounding surfaces have $\Theta^{\alpha\beta} = 0$

n_β @ t_b is $(-1, 0, 0, 0)$
n_β @ t_e is $(+1, 0, 0, 0)$ $\Big\}$ normal to the V^3 "surfaces"

2. cont'd

since $\partial_\beta \Theta^{\alpha\beta} = 0$

$$\int_{V^4} \partial_\beta \Theta^{\alpha\beta} d^4x = 0 = \oint_{V^3} \Theta^{\alpha\beta} n_\beta \, d^3x$$

$$0 = \int_{ct = t_e} \Theta^{\alpha 0} d^3x - \int_{ct = t_b} \Theta^{\alpha 0} d^3x$$

t_b is time "begin"

t_e is time "end"

$\int d^3x$ is the 3-space integral over the full volume (V^3 "surface") where the EM fields are finite

and it is clear that $\underset{\text{@ time \underline{begin}}}{\int \Theta^{\alpha 0} d^3x} = \underset{\text{@ time\underline{end}}}{\int \Theta^{\alpha 0} d^3x}$

is a constant value (i.e. for all time)

and we already know

$\int \Theta^{\alpha 0} d^3x$ is a Lorentz 4-vector (one index α)

and we know $\int \Theta^{00} d^3x = \text{Energy} \leftarrow$ energy

$$\underbrace{\int \Theta^{i0} d^3x}_{\substack{\text{as I wrote it;} \\ \text{equivalent to} \\ \Theta^{0i} \text{ as} \\ \text{given in the} \\ \text{problem}}} = cP^i \leftarrow \text{momentum}$$

3. Liénard-Wiechert potential in 4-vector form

$$A^\alpha(x) = \left(\frac{\mu_0 c}{4\pi}\right) \frac{e\, u_s^\alpha(\tau)}{(x - r_s(\tau))\cdot u_s}\Bigg|_{\tau=\tau_0}$$

$x^\alpha = (ct, \vec{r})$; $r_s(\tau)$ is 4-vector position for moving source (point charge) as function of proper time along its world line

$u_s^\alpha = (\gamma c, \gamma \vec{\beta}_s c)$

$A^\alpha = (\frac{\Phi}{c}, \vec{A})$

eval. $(x - r_s(\tau))\cdot u_s$

define $R \equiv |\vec{r} - \vec{r}_s(\tau_0)|$

$$= \gamma c\,(\overset{ct}{\cancel{x}} - r_s^0(\tau_0)) - \gamma c \vec{\beta}_s \cdot (\vec{r} - \vec{r}_s(\tau_0))$$

$$= \gamma c R\,(1 - \vec{\beta}_s\cdot\hat{n}), \text{ where } \hat{n} = \frac{\vec{r} - \vec{r}_s(\tau_0)}{|\vec{r} - r_s(\tau_0)|}$$

note: r^0 and r_s^0 are time-coordinates of spacetime events separated along the light cone

$\therefore\ ct - r_s^0(\tau_0) = R$

$$\frac{\Phi}{c}(\vec{r}, t) = \frac{\mu_0 c}{4\pi}\, \frac{e\gamma c}{\gamma c R(1 - \vec{\beta}_s\cdot\hat{n})}$$

$$\boxed{\Phi(\vec{r}, t) = \left(\frac{1}{4\pi\epsilon_0}\right) \frac{e}{(1 - \vec{\beta}_s\cdot\hat{n})|\vec{r} - \vec{r}_s(\tau_0)|}\Bigg|_{\tau_0}}$$ ✓

← evaluated at τ_0 retarded time

$$\vec{A}(\vec{r}, t) = \frac{\mu_0 c}{4\pi}\, \frac{e\gamma c \vec{\beta}_s}{\gamma c R(1 - \vec{\beta}_s\cdot\hat{n})}$$

$$\boxed{\vec{A}(\vec{r}, t) = \left(\frac{\mu_0 c}{4\pi}\right) \frac{e\vec{\beta}_s}{(1 - \vec{\beta}_s\cdot\hat{n})|\vec{r} - \vec{r}_s(\tau_0)|}\Bigg|_{\tau_0}}$$ ✓

← evaluated at τ_0 retarded time