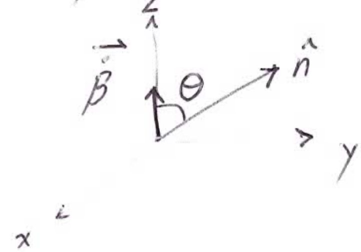
832 PS #4 Solutions

1 a) $z(t) = a\cos\omega_0 t$; $v(t) = a\omega_0(-\sin\omega_0 t)$; $\dot{v}(t) = -a\omega_0^2\cos\omega_0 t$

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c}\left(\frac{1}{4\pi\epsilon_0}\right)\left|\hat{n}\times(\hat{n}\times\dot{\vec{\beta}})\right|^2$$

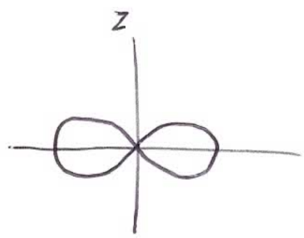
$$\left|\hat{n}\times(\hat{n}\times\dot{\vec{\beta}})\right|^2 = \dot{\beta}^2\sin\theta$$



$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c}\left(\frac{1}{4\pi\epsilon_0}\right)\frac{a^2\omega_0^4}{c^2}\sin^2\theta\,\cos^2\omega_0 t$$

problem asks for time-averaged: $\frac{1}{T}\int_0^T\cos^2\omega_0 t\,dt = \frac{1}{2}$

b)



typical dipole angular distribution of radiated power

total power (time-averaged) is $= \int[\quad]\,d\Omega$

$$\int_0^{2\pi}\int_0^{\pi}\sin^2\theta\,\sin\theta\,d\theta\,d\phi$$

$$= 2\pi\left(\frac{1}{3}(-2)-(-2)\right)$$

$$= 2\pi\left(\frac{4}{3}\right)$$

$$\therefore \boxed{\langle P\rangle = \frac{e^2}{3c^3}\left(\frac{1}{4\pi\epsilon_0}\right)a^2\omega_0^4}$$

1 b) circle radius R in x-y plane with angular frequency ω_0

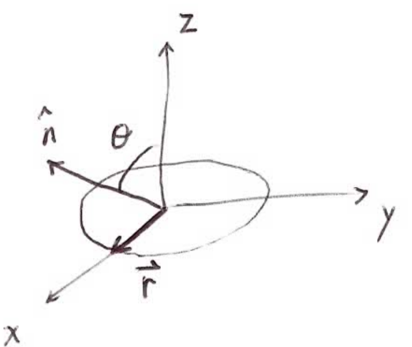
$$\vec{r} = R\left(\cos(\omega_0 t)\,\hat{x} + \sin(\omega_0 t)\,\hat{y}\right)$$

$$\vec{v} = \dot{\vec{r}} = R\omega_0\left(-\sin(\omega_0 t)\,\hat{x} + \cos(\omega_0 t)\,\hat{y}\right)$$

$$\dot{\vec{\beta}} = \frac{\dot{\vec{v}}}{c} = \frac{\ddot{\vec{r}}}{c} = -\frac{R\omega_0^2}{c}\left(\cos(\omega_0 t)\,\hat{x} + \sin(\omega_0 t)\,\hat{y}\right) = -\frac{R\omega_0^2}{c}\,\hat{r}$$

note: instantaneously, the radiation will have azimuthal dependence, but time-averaged it should not (from symmetry)

want to calculate $|\hat{n} \times (\hat{n} \times \dot{\vec{\beta}})|^2$



Can choose $t = 0$ to be when $\vec{r}$ (rotating) and $\hat{n}$ are in the x-z plane (i.e. defines the x-axis)

$\hat{n}$ defines x-z plane

then, azimuthal dependence comes from the "$\cos(\omega_0 t)$" rotation part of $\hat{r}$

$$\hat{n} = \cos\theta\,\hat{z} + \sin\theta\,\hat{x}$$

$$\hat{n} \times \dot{\vec{\beta}} = -\frac{R\omega_0^2}{c}\left[\cos\theta\cos(\omega_0 t)\,\hat{y} - \cos\theta\sin(\omega_0 t)\,\hat{x} + \sin\theta\sin(\omega_0 t)\,\hat{z}\right]$$

$$\hat{n} \times (\hat{n} \times \dot{\vec{\beta}}) = -\frac{R\omega_0^2}{c}\left[-\cos^2\theta\cos(\omega_0 t)\,\hat{x} - \cos^2\theta\sin(\omega_0 t)\,\hat{y} - \sin^2\theta\sin(\omega_0 t)\,\hat{y} + \sin\theta\cos\theta\cos(\omega_0 t)\,\hat{z}\right]$$

$$= \frac{R\omega_0^2}{c}\left[\cos^2\theta\cos(\omega_0 t)\,\hat{x} + \sin(\omega_0 t)\,\hat{y} - \sin\theta\cos\theta\cos(\omega_0 t)\,\hat{z}\right]$$

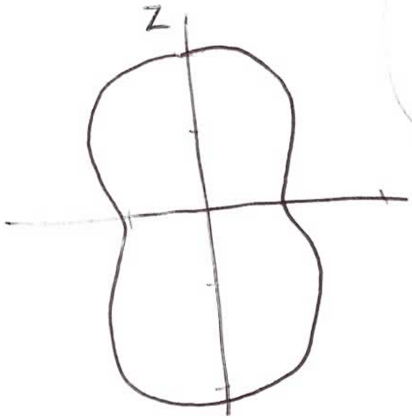
$$|\hat{n} \times (\hat{n} \times \dot{\vec{\beta}})|^2 = \frac{R^2\omega_0^4}{c^2}\left[\left[(\cos^2\theta)^2 + \sin^2\theta\cos^2\theta\right]\cos^2(\omega_0 t) + \sin^2(\omega_0 t)\right]$$

$$= \frac{R^2\omega_0^4}{c^2}\left[\cos^2\theta\,(\overbrace{\cos^2\theta + \sin^2\theta}^{1})\cos^2(\omega_0 t) + \sin^2(\omega_0 t)\right]$$

1 b) cont'd

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c}\left(\frac{1}{4\pi\epsilon_0}\right)\frac{R^2\omega_0^4}{c^2}\left[\cos^2\theta \underbrace{\cos^2(\omega_0 t)}_{\text{time-averaged}} + \underbrace{\sin^2(\omega_0 t)}_{1/2}\right]$$

$$\boxed{\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{e^2 R^2 \omega_0^4}{8\pi c^3}\left(\frac{1}{4\pi\epsilon_0}\right)\left[1+\cos^2\theta\right]}$$



"typical" $(1+\cos^2\theta)$ angular distribution

$$\langle P \rangle = \int \left\langle \frac{dP}{d\Omega} \right\rangle d\Omega = 2\pi[\;]\underbrace{\int_{-1}^{+1}(1+\cos^2\theta)\,d(\cos\theta)}_{=8/3}$$

$$\boxed{\langle P \rangle = \frac{2}{3}\frac{e^2R^2\omega_0^4}{c^3}\left(\frac{1}{4\pi\epsilon_0}\right)}$$

2. Synchrotron ~~radius~~ radiation from the Crab

energy $E = 10^{13}\ eV$ electrons

$B = 3 \times 10^{-4}$ gauss $= 3 \times 10^{-8}\ T$

a) $\rho = \frac{p_\perp}{qB} \approx \frac{E/c}{|q|B}$; $\omega_0 = \frac{qB}{\gamma m} = \frac{qBc^2}{E}$

$= \frac{10^{13}\,(1.6 \times 10^{-19})/(3 \times 10^8)}{(1.6 \times 10^{-19})\ 3 \times 10^{-8}}$

$\rho \approx 10^{12}\ m$

$= \frac{(3 \times 10^{-8})(3 \times 10^8)^2}{10^{13}}$

$\omega_0 = 2.7 \times 10^{-4}\ s^{-1}$

also $\omega_0 = \frac{c}{\rho}$ gives ↑

critical frequency $\omega_c = \frac{3}{2}\frac{c}{\rho}\gamma^3 = \frac{3}{2}\omega_0\gamma^3 = 3 \times 10^{18}\ s^{-1}$

$\gamma \approx \frac{10^{13}}{0.5 \times 10^6} = 2 \times 10^7$

and $\hbar\omega_c = 2\ keV$

$\hbar = 6.58 \times 10^{-16}\ eV \cdot s$

b) $$\frac{dI}{d\omega} \simeq \begin{cases} \frac{e^2}{c}\left(\frac{\omega\rho}{c}\right)^{1/3} = \frac{e^2}{c}\left(\frac{\omega}{\omega_0}\right)^{1/3}, & \text{for } \omega \ll \omega_c \\ \sqrt{\frac{3\pi}{2}}\,\frac{e^2}{c}\,\gamma\left(\frac{\omega}{\omega_c}\right)^{1/2} e^{-\omega/\omega_c}, & \text{for } \omega \gg \omega_c \end{cases}$$

is the intensity or energy radiated per unit angular frequency spectrum

$P = \frac{\omega_0}{2\pi}\left(\frac{dI}{d\omega}\right)$ (energy per second) per unit frequency spectrum

for $\omega \ll \omega_c$, $P(E,\omega)$ goes as $\left(\omega^{1/3} \cdot \omega_0^{2/3}\right)$

ω_0 goes as $\frac{1}{E}$ so $P(E,\omega)$ goes as $\left(\frac{\omega}{E^2}\right)^{1/3}$

2b) cont'd

for $\omega \gg \omega_c$; $P(E,\omega) = \frac{\omega_0}{2\pi}\sqrt{\frac{3\pi}{2}}\frac{e^2}{c}\gamma\left(\frac{\omega}{3/2\, c/\rho\, \gamma^3}\right)^{1/2} e^{-\omega/\omega_c}$

$= \frac{\sqrt{\pi}}{2\pi}\frac{e^2}{c}\gamma\omega_0\left(\frac{\omega}{\omega_0\gamma^3}\right)^{1/2} e^{-\omega/\omega_c}$

γ goes as E, ω_0 goes as $1/E$, $\omega_c = \frac{3}{2}\omega_0\gamma^3 \cos\theta$ goes as E^2

$\uparrow$ pitch angle

$\therefore P(E,\omega)$ goes as $\left(\frac{\omega}{E^2}\right)^{1/2} e^{-\omega/\omega_c}$

which can be written $\left(\frac{\omega}{E^2}\right)^{1/3}\left(\frac{\omega}{E^2}\right)^{1/6} e^{-\omega/\omega_c}$

choose $f\left(\frac{\omega}{\omega_c}\right) = \left(\frac{\omega}{\omega_c}\right)^{1/6} e^{-\omega/\omega_c}$ and since ω_c goes as E^2

$\therefore P(E,\omega)$ goes as const. $\left(\frac{\omega}{E^2}\right)^{1/3} f\left(\frac{\omega}{\omega_c}\right)$

2c) if $N(E)\,dE \propto E^{-n}\,dE$ (electron energy spectrum)

$$P(\omega) = \frac{\int P(E,\omega)\,N(E)}{\int N(E)\,dE} = \text{const.}\int\left(\frac{\omega}{E^2}\right)^{1/3} f\left(\frac{\omega}{\omega_c}\right) E^{-n}\,dE$$

integrated over the range of electron energies that are relevant

$f\left(\frac{\omega}{\omega_c}\right) \approx 1$ for $\omega/\omega_c \ll 1$, for $\frac{\omega}{\omega_c} > 1$, the function decreases rapidly to zero

$\therefore$ approximate by integrating over energies so $\omega \ll \omega_c$

since ω_c goes as E^2 ($\omega_c = kE^2$), we want $E = \sqrt{\omega_c/k}$

(i.e. E must be larger than $\sqrt{\omega/k}$ so that $\omega_c > \omega$)

$$P(\omega) = \text{const.}\int_{\sqrt{\omega/k}}^{\infty} \omega^{1/3} E^{-(n+2/3)}\,dE$$

2c) cont'd

$$P(\omega) = \text{const.}\ \omega^{1/3} \left. \frac{E^{-(n-1/3)}}{-(n-1/3)} \right|_{\sqrt{\omega/k}}^{\infty} \quad \text{assuming } n > 1/3$$

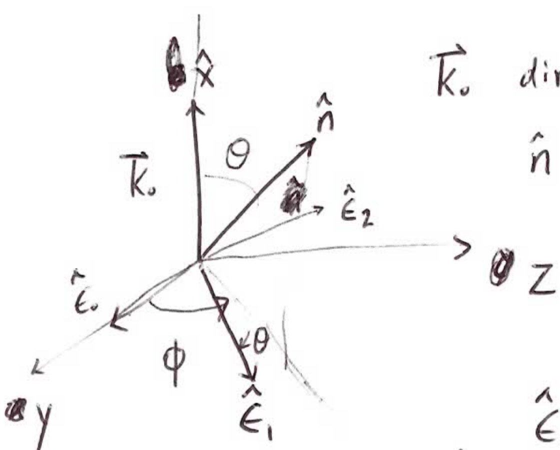
$$= \text{const.}\ \omega^{1/3} \sqrt{\omega}^{-(n-1/3)}$$

$$P(\omega) = \text{const.}\ \omega^{-n/2}\, \omega^{1/3}\, \omega^{1/6}$$

$$= \text{const.}\ \omega^{-(n-1)/2} = \text{const.}\ \omega^{-\alpha} \quad \text{where } \alpha = \frac{n-1}{2}$$

3. classical differential scattering cross section for Thomson scattering is $\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2 |\hat{\epsilon}^* \cdot \hat{\epsilon}_0|^2$ Jackson Eq. 14.124

where $\frac{e^2}{mc^2} = r_e$ (classical electron radius in Gaussian units)



$\vec{k}_0$ direction of propagation of wave

$\hat{n}$ direction of scattered radiation

$\hat{\epsilon}_1$ polarization vectors for scattered radiation is in $\hat{n}$ - $\vec{k}_0$ plane

$\hat{\epsilon}_2 \perp$ to $\hat{\epsilon}_1$ and $\hat{n}$

$\hat{\epsilon}_0$ polarization vector of incoming radiation

$\hat{\epsilon}^*$ polarization of outgoing radiation

- if radiation incident with $\vec{k}_0$ pointing in $+\hat{x}$ direction and $\hat{n}$ is in $+\hat{z}$ direction
  $\therefore$ $\hat{\epsilon}_1$ is in $-\hat{x}$ direction, $\hat{\epsilon}_2$ is in $-\hat{y}$ direction
  and only the incoming (unpolarized) radiation with $\hat{\epsilon}_0$ $\hat{y}$ component gets scattered
- if radiation incident with $\vec{k}_0$ pointing in $-\hat{y}$ direction only the incoming (unpolarized) radiation with $\hat{\epsilon}_0$ $\hat{x}$ component gets scattered

3. cont'd

So this problem is trivial ... just to get you to study Thomson scattering and understand what the polarization vectors mean

∴ if there is quadrupole anisotropy, and radiation from the -x-direction is say 3x as intense as radiation from the +y-direction, both incident radiation being unpolarized (i.e. equal amount of radiation with all linear polarizations as appropriate for each incident wave) the scattered radiation at 90° (in the +z direction) is polarized with the degree of (linear) polarization being set by the relative intensities.