Φ is from point charge $= \dfrac{q}{r}$ (singular at $r=0$)

$r \to$ distance from origin

$$\nabla^2\left(\frac{q}{r}\right) = \nabla^2\left(\frac{q}{|\vec{x}|}\right) = -4\pi\,\delta(\vec{x})\,q$$

$$\Phi(\vec{x}) = \frac{q}{r} = \frac{q}{|\vec{x}|}$$

$\rho(\vec{x}) = q\,\delta(\vec{x})$ point charge q at origin

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\,d^3x' = \frac{q}{4\pi\epsilon_0}\,\frac{1}{|\vec{x}|}$$

$$\nabla^2\psi = \underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)}_{\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)} + \frac{\partial}{\partial\theta}\text{ term} + \frac{\partial}{\partial\phi}\text{ term}$$

$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)$ is zero everywhere except at $r=0$

$\to$ clear that $\nabla^2\Phi(\vec{x}) = \nabla^2\left(\dfrac{q}{4\pi\epsilon_0}\dfrac{1}{|\vec{x}|}\right) = -\cancel{4\pi}\,\delta(\vec{x})\,\dfrac{q}{\cancel{4\pi}\epsilon_0} = -\dfrac{\rho(\vec{x})}{\epsilon_0}$

Poisson equation

$\to$ if localized discrete or continuous $\rho(\vec{x})$ then solve (including numerically)

$\Phi(\vec{x}) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \frac{\rho(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\,d^3x'$ and you're done

$\to$ but otherwise... region of space w/ or w/o charges inside, and boundary conditions

solve Poisson, Laplace using <u>Green's functions</u> $\to$ some methods simulate boundary conditions (e.g. method of images) but in general...

if $\nabla^2 G(\vec{x},\vec{x}\,') = -4\pi\,\delta(\vec{x}-\vec{x}\,')$

then $\Phi(\vec{x}) = \displaystyle\int d^3x'\, G(\vec{x},\vec{x}\,')\,\rho(\vec{x}\,')$

"impulse response of an inhomogeneous diff. eqn defined on a domain with specified initial or boundary conditions"

$L\,u(x) = f(x)$

$\uparrow$ differential operator on $u(x)$

what is $u(x)$

$\nabla^2\Phi(\vec{x}) = -\dfrac{\rho(\vec{x})}{\epsilon_0}$

what is $\Phi(\vec{x})$

$\to$ convolution of G with $f(x)$ is the solution to the inhom. diff. eqn for $f(x)$

$G(\vec{x},\vec{x})$ is ~~simply~~ $\frac{1}{|\vec{x}-\vec{x}'|}$ only one of a class of functions satisfying $\nabla^2 G(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}')$

$$\int \nabla^2\left(\frac{1}{r}\right) d^3x = \int_V \left(\vec{\nabla}\cdot\vec{\nabla}\left(\frac{1}{r}\right)\right) d^3x = \oint_S \vec{\nabla}\left(\frac{1}{r}\right)\cdot\hat{n}\, da$$

$$= \oint\left(-\frac{1}{r^2}\right) r^2\, d\Omega$$

$$= -4\pi$$

$$\cancel{\nabla^2 G(\vec{x},\vec{x}')} \; \nabla^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -4\pi\,\delta(\vec{x}-\vec{x}')$$

what good is that? haven't done anything yet...

Green's theorem

let $\vec{A} = \phi\vec{\nabla}\psi$ arbitrary scalar fields

$$\vec{\nabla}\cdot(\phi\vec{\nabla}\psi) = \phi\,\nabla^2\psi + \vec{\nabla}\phi\cdot\vec{\nabla}\psi$$

$$\phi\vec{\nabla}\psi\cdot\hat{n} = \phi\frac{\partial\psi}{\partial n}$$ where $\frac{\partial}{\partial n}$ is the normal derivative at the surface S

then from divergence theorem

$$\int_V \vec{\nabla}\cdot\vec{A}\, d^3x = \oint_S \vec{A}\cdot\vec{n}\, da$$

$$\int_V (\phi\nabla^2\psi + \vec{\nabla}\phi\cdot\vec{\nabla}\psi)\, d^3x = \oint_S \phi\frac{\partial\psi}{\partial n}\, da$$

write down above interchanging ϕ, ψ then subtract from

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, d^3x = \oint_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) da$$

---

choose ψ to be $1/r$ or $\frac{1}{|\vec{x}-\vec{x}'|}$ ψ is a $G(\vec{x},\vec{x}')$

choose ϕ to be $\Phi(\vec{x})$ and $\nabla^2\Phi(\vec{x}) = -\frac{\rho(\vec{x})}{\epsilon_0}$

$$\nabla^2 G(\vec{x},\vec{x}') = -4\pi\,\delta(\vec{x}-\vec{x}')$$

then

$$\int_V \left[-4\pi\, \Phi(\vec{x}\,')\, \delta(\vec{x}-\vec{x}\,') + \frac{1}{r}\frac{\rho(\vec{x}\,')}{\epsilon_0} \right] d^3x' =$$

$$\oint_S \left[\Phi(\vec{x}\,') \frac{\partial}{\partial n'}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial \Phi(\vec{x}\,')}{\partial n'} \right] da'$$

if $\vec{x}$ lies inside the surface

$$-4\pi\, \Phi(\vec{x}) + \int_V \frac{\rho(\vec{x}\,')}{r\,\epsilon_0} d^3x' = \oint_S [\quad] da'$$

$$* \quad \boxed{\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{x}\,')}{r} d^3x' + \frac{1}{4\pi}\oint_S \left[\frac{1}{r}\frac{\partial \Phi(\vec{x}\,')}{\partial n'} - \Phi(\vec{x}\,')\frac{\partial}{\partial n'}\left(\frac{1}{r}\right) \right] da'}$$

if $\vec{x}$ lies outside the surface then ~~$\oint [\ ] = 0$~~
first term in $\int_V [\quad] d^3x'$ is zero
and interpret $\oint_S$ as surface-charge density $\sigma = \frac{\cancel{4}}{\cancel{4\pi}}\epsilon_0 \frac{\partial \Phi}{\partial n'}$ and
a dipole layer of $-\frac{\cancel{4}\epsilon_0}{\cancel{4\pi}}\Phi$ ... see Jackson

1) if surface S goes to infinity and electric field on S falls off faster than $1/r$, the surface integral vanishes and you get the usual

2) if charge-free volume $\rho(\vec{x}\,')$ for all $\vec{x}\,'$ then potential in the volume $\Phi(\vec{x})$ given by potential and normal derivative on the surface ... solution of Laplace equation with boundary conditions $\nabla^2 \Phi(\vec{x}) = 0$

of Φ is $\vec{E}$

Jackson notes:
Cauchy boundary conditions
overspecify ... not arbitrary
$\Phi(\vec{x})$, $\frac{\partial \Phi}{\partial n}$ on boundary, Φ on boundary (Dirichlet)
(Neumann)

Formal Solution of Electrostatic BVP with Green's function

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + F(\vec{x},\vec{x}')$$

with F satisfying

$$\nabla'^2 F(\vec{x},\vec{x}') = 0$$

over all $\vec{x}'$

$$\Phi(\vec{x}) = \int_V \frac{\rho(\vec{x}')}{4\pi\epsilon_0} G(\vec{x},\vec{x}') d^3x' + \frac{1}{4\pi}\oint_S \left[G(\vec{x},x')\frac{\partial\Phi(\vec{x}')}{\partial n'} - \Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}\right] da$$

for Dirichlet boundary conditions (Φ known on boundary)

demand $G_D(\vec{x},\vec{x}') = 0$ for $\vec{x}'$ on S

then

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}') G_D(\vec{x},\vec{x}') d^3x' - \frac{1}{4\pi}\oint_S \Phi(\vec{x}')\frac{\partial G_D}{\partial n'} da'$$

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for Neumann boundary conditions can't just choose $\frac{\partial G_N(\vec{x},\vec{x}')}{\partial n'} = 0$ for $\vec{x}'$ on S

see Jackson

choose $\frac{\partial G_N(\vec{x},\vec{x}')}{\partial n'} = -\frac{4\pi}{S}$ for $\vec{x}'$ on S

then

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}') G_N(\vec{x},\vec{x}') d^3x' + \frac{1}{4\pi}\oint_S \frac{\partial\Phi(\vec{x}')}{\partial n'} G_N(\vec{x},\vec{x}') da$$

$$+ \langle\Phi\rangle_S$$ average value of potential over the whole surface

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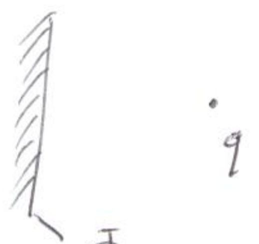
Jackson notes: the classic Neumann problem "exterior problem"
volume bounded by two surfaces: closed and finite, the other at infinity → surface area S is infinite and $\langle\Phi\rangle_S$ vanishes

meaning of $F(\vec{x},\vec{x})$: solution $\nabla^2 F = 0$ system of charges external to V
"external distribution of charges so chosen as to satisfy homogeneous boundary conditions $\Phi = 0$ or $\frac{\partial\Phi}{\partial n} = 0$ when combined with potential of a point charge at $\vec{x}'$ (both potential at $\vec{x}$ and external distribution depend on source point $\vec{x}'$, hence $F(\vec{x},\vec{x}')$
e.g. method of images is equivalent to finding $F(\vec{x},\vec{x}')$ to satisfy boundary conditions

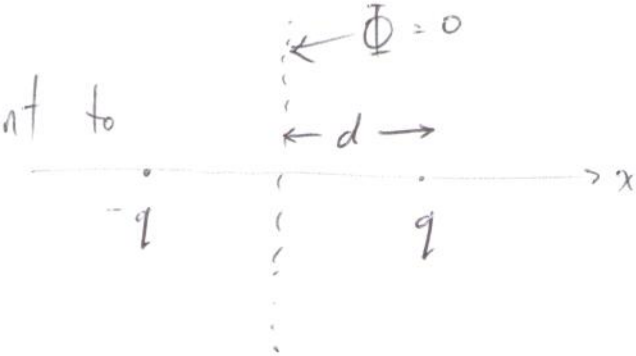
e.g. Dirichlet problem with conductors
$F(\vec{x}, \vec{x}\,')$ is potential due to surface-charge distribution induced on the conductors by point charge at $\vec{x}\,'$

Method of images



$\Phi = 0$ conductor, grounded

equivalent to



$G_D(\vec{x}, \vec{x}\,') = 0$ on S

$= \cancel{\frac{q}{|d\hat{x} - \vec{x}\,'|} + \frac{q}{|d\hat{x} - \vec{x}\,'|}}$

$G_D(\vec{x}, \vec{x}\,') = \frac{1}{|\vec{x} - \vec{x}\,'|} - \frac{1}{|\vec{x} - -\vec{x}\,'|}$

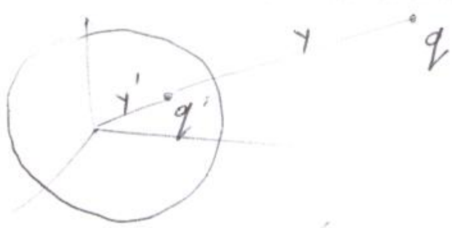
$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\vec{x} - d\hat{x}|} + \frac{-q}{|\vec{x} - -d\hat{x}|}\right]$$

$$\oint_S \Phi(\vec{x}\,') \frac{\partial G_D}{\partial n'} da' = 0$$

$\rho(\vec{x}) = q\,\delta(\vec{x} - d\hat{x})$

$F(\vec{x}, \vec{x}\,') = \frac{-1}{|\vec{x} - -\vec{x}\,'|}$

$\nabla^2 F = 0$ ✓



point charge q located at $\vec{y}$

find $\Phi(\vec{x})$ with $\Phi(|\vec{x}| = a) = 0$

gnd spherical conductor

$$\Phi(\vec{x}) = \left(\frac{q}{|\vec{x} - \vec{y}|} + \frac{q'}{|\vec{x} - \vec{y}\,'|}\right)\frac{1}{4\pi\epsilon_0}$$

→ try image charge q' at $\vec{y}\,'$

$$= \left(\frac{q}{|x\hat{n} - y\hat{n}'|} + \frac{q'}{|x\hat{n} - y'\hat{n}'|}\right)\frac{1}{4\pi\epsilon_0}$$

such that $\Phi(|\vec{x}| = a) = 0$

factor out x, y' then $x = a$

$$\Phi(x = a) = \left(\frac{q}{a\left|\hat{n} - \frac{y}{a}\hat{n}'\right|} + \frac{q'}{y'\left|\hat{n}' - \frac{a}{y'}\hat{n}\right|}\right)\frac{1}{4\pi\epsilon_0}$$

choose $\frac{q}{a} = -\frac{q'}{y'}$

$\frac{y}{a} = \frac{a}{y'}$

for all possible values $\hat{n} \cdot \hat{n}'$

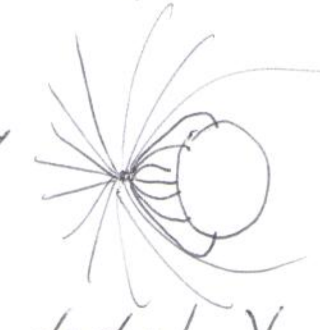
$$q' = -\frac{a}{y} q \quad ; \quad y' = \frac{a^2}{y}$$

as q brought closer to sphere, image charge grows in magnitude and moves out from centre

charge q outside grounded conducting sphere

$$\sigma = -\epsilon_0 \left.\frac{\partial \Phi}{\partial x}\right|_{x=a}$$

surface charge density



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point charge at $\vec{y}$ near conducting sphere radius a at fixed potential V

$$\Phi(\vec{x}) = \left(\frac{q}{|\vec{x}-\vec{y}|} - \frac{a q}{y \left|\vec{x} - \frac{a^2}{y^2}\vec{y}\right|} \right)\frac{1}{4\pi\epsilon_0} + \frac{V a}{|\vec{x}|}$$

$\uparrow$ put charge Va at the origin

in Green's function approach

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')\, G_D(\vec{x}, \vec{x}')\, d^3x'$$

$$\rho(\vec{x}) = q\,\delta(\vec{x}-\vec{y})$$

$$-\frac{1}{4\pi}\oint_S \underline{\Phi(\vec{x}')}\, \frac{\partial G_D}{\partial n'}\, da'$$

this term?

General solution

$$G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{a}{x' \left|\vec{x} - \frac{a^2}{x'^2}\vec{x}'\right|}$$

in spherical coordinates

$$= \frac{1}{(x^2 + x'^2 - 2xx'\cos\gamma)^{1/2}} - \frac{1}{\left(\frac{x^2 x'^2}{a^2} + a^2 - 2xx'\cos\gamma\right)^{1/2}}$$

if either x or $x' = a$, $G = 0$

need $\frac{\partial G_D}{\partial n'}$ $\quad \hat{n}'$ unit normal <u>outwards</u> from volume of interest (ie. inwards along $\vec{x}'$ toward the origin)

$$\left.\frac{\partial G_D}{\partial n'}\right|_{x'=a} = -\frac{(x^2 - a^2)}{a(x^2 + a^2 - 2ax\cos\gamma)^{3/2}}$$

if $\Phi(x'=a) \neq 0$ this term contributes to $\Phi(\vec{x})$