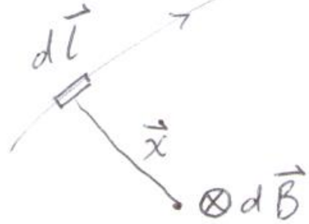
Magnetic Potential - Magnetostatics $\vec{J}, \vec{B}$ time-independent

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \quad \text{continuity equation}$$

$\therefore$ in magnetostatics $\vec{\nabla} \cdot \vec{J} = 0$

from Biot-Savart law $d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \vec{x}}{|\vec{x}|^3}$



$$I\, d\vec{l} = J(\vec{x})\, d^3x$$

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}') \times \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3}\, d^3x'$$

$$= \frac{\mu_0}{4\pi} \int \vec{\nabla}_x \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) \times \vec{J}(\vec{x}')\, d^3x'$$

$\ominus$ve sign reverses the order

$$\vec{B}(\vec{x}) = \vec{\nabla} \times \left(\frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}\, d^3x' \right)$$

like the scalar potential

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot \vec{\nabla} \times \vec{A} = 0$ automatically

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}\, d^3x' + \vec{\nabla}\psi$$

later

gradient of scalar (gauge freedom)

Consider $\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A})$

$$= \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) = \frac{\mu_0}{4\pi} \vec{\nabla} \int d^3x'\, J(\vec{x}') \cdot \vec{\nabla} \left(\frac{1}{\vec{x} - \vec{x}'} \right)$$

use $\vec{\nabla}\left(\frac{1}{\vec{x} - \vec{x}'} \right) = -\vec{\nabla}' \left(\frac{1}{\vec{x} - \vec{x}'} \right)$

$$= -\frac{\mu_0}{4\pi} \vec{\nabla} \int J(\vec{x}') \cdot \vec{\nabla}' \left(\frac{1}{\vec{x} - \vec{x}'} \right) d^3x' = \frac{\mu_0}{4\pi} \vec{\nabla} \int \frac{\overbrace{\vec{\nabla}' \cdot \vec{J}(\vec{x}')}^{0}}{|\vec{x} - \vec{x}'|}\, d^3x'$$

re-do $\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) = \frac{\mu_0}{4\pi}\vec{\nabla}\int d^3x'\ \vec{\nabla}_x\cdot\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$$= \frac{\mu_0}{4\pi}\vec{\nabla}\int d^3x'\ J(\vec{x}')\ \vec{\nabla}_x\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$$

$$= -\frac{\mu_0}{4\pi}\vec{\nabla}\int d^3x'\ \vec{J}(\vec{x}')\cdot\vec{\nabla}'_{x'}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$$

integration by parts $\int u v'_{dx} = uv - \int u'v_{dx}$

$$= \frac{\mu_0}{4\pi}\vec{\nabla}\int d^3x'\ \frac{\overbrace{\vec{\nabla}'\cdot\vec{J}(\vec{x}')}^{0}}{|\vec{x}-\vec{x}'|}$$

from continuity eqn in magnetostatics

+boundary term $\vec{J}(\vec{x}') * \frac{1}{|\vec{x}-\vec{x}'|}$

$$\therefore\ \vec{\nabla}\times\vec{B} = -\nabla^2\vec{A}$$

$$= -\frac{\mu_0}{4\pi}\int J(\vec{x}')\ \underbrace{\nabla^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)}_{-4\pi\delta(\vec{x}-\vec{x}')}\ d^3x'$$

$$\vec{\nabla}\times\vec{B} = \mu_0 J(\vec{x})$$

second term $\quad \nabla^2 \dfrac{1}{|\vec{x} - \vec{x}'|} = -4\pi\delta(\vec{x} - \vec{x}')$

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\cancelto{0}{\vec{\nabla}\cdot\vec{A}}) - \nabla^2\vec{A}$$

$$= -\frac{\mu_0}{4\pi}\int d^3x'\, J(\vec{x}')\,\nabla^2\left(\frac{1}{|\vec{x} - \vec{x}'|}\right)$$

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0 J(\vec{x})}$$

as expected

in electrostatics
Gauss's law is integral form of $\boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}}$

$\boxed{\vec{\nabla}\times\vec{E} = 0}$

in magnetostatics
Ampère's law is the equivalent

$$\oint_S (\vec{\nabla}\times\vec{B})\cdot\hat{n}\,da = \mu_0\oint_S \vec{J}\cdot\hat{n}\,da$$

Stokes's theorem

$$= \oint_C \vec{B}\cdot d\vec{l} = \mu_0 I$$

$\boxed{\vec{\nabla}\cdot\vec{B} = 0}$

Magnetic Vector Potential

if $\vec{J} = 0$ in a region, the $\vec{\nabla}\times\vec{B} = 0$, can define magnetic <u>Scalar</u> potential Φ_M, $\vec{B} = -\vec{\nabla}\Phi_M$

the $\nabla^2\Phi_M = 0$ Laplace equation and techniques for handling problems similar to electrostatics
→ won't consider further!

more generally, if $\vec{\nabla}\cdot\vec{B} = 0$

B is the curl of $\vec{A}$ b/c $\vec{\nabla}\cdot(\vec{\nabla}\times A) = 0$ automatically

$$\vec{A} \rightarrow \vec{A} + \vec{\nabla}\psi$$

gauge transformation

b/c $\vec{\nabla}\times\vec{\nabla}\psi = 0$ so $\vec{B} = \vec{\nabla}\times\vec{A}$ unchanged

$$\vec{B}(\vec{x}) = \vec{\nabla}\times\vec{A}(\vec{x})$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J}$$

if $A \to A + \vec{\nabla}\psi$

then $\vec{\nabla}\cdot\vec{A}$ has first term zero since $\vec{\nabla}'\cdot\vec{J} = 0$ in $A(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{J(\vec{x}')}{|\vec{x}-\vec{x}'|}d$

thus reduces to $\nabla^2\psi = 0$ if $\underline{\vec{\nabla}\cdot\vec{A} = 0}$

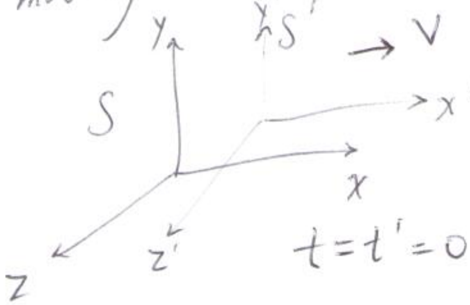
this choice of gauge is the "Coulomb gauge" and then

$\nabla^2\vec{A} = -\mu_0\vec{J}$ is a Poisson equation for <u>each</u> rectangular component of $\vec{A}$

but ... on to relativity where $\vec{A}$ becomes more complete

Special Relativity

- laws of physics are the same (in all) independent of reference frames moving with constant, relative velocity wrt each other



$t = t' = 0$

prior to 1905 Galilean transformation

$x' = x - vt$ $u_x' = u_x - V$

$y' = y$ $u_y' = u_y$

$z' = z$ $u_z' = u_z$

$t' = t$

<u>velocities add</u>

Newton's Law $\vec{F} = m\vec{a}$

$\vec{a} = \frac{d\vec{u}}{dt}$

$a' = \frac{d\vec{u}'}{dt} = \frac{d\vec{u}}{dt}$ so $\vec{F}' = \vec{F}$, everybody is happy

Maxwell's equations in vacuum $(\vec{J} = 0)$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = -\nabla^2\vec{B} + \vec{\nabla}(\overbrace{\vec{\nabla}\cdot\vec{B}}^{0}) = \frac{1}{c^2}\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

$$-\nabla^2\vec{B} = -\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}$$