if $A \to A + \vec\nabla\psi$

then $\vec\nabla \cdot \vec A$ has first term zero since $\vec\nabla' \cdot \vec J = 0$ in $A(\vec x) = \frac{\mu_0}{4\pi}\int \frac{J(\vec x')}{|\vec x - \vec x'|} d$

thus reduces to $\nabla^2 \psi = 0$ if $\vec\nabla \cdot \vec A = 0$

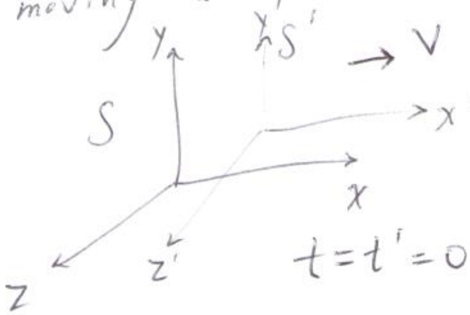
this choice of gauge is the "Coulomb gauge" and then

$\nabla^2 \vec A = -\mu_0 \vec J$ is a Poisson equation for <u>each</u> rectangular component of $\vec A$

but ... on to relativity where $\vec A$ becomes more complete

Special Relativity

- laws of physics are the same independent (in all) of reference frames moving with constant, relative velocity wrt each other



$t = t' = 0$

prior to 1905 Galilean transformation

$x' = x - vt$ $\quad u_x' = u_x - V$

$y' = y$ $\quad u_y' = u_y$

$z' = z$ $\quad u_z' = u_z$

$t' = t$

velocities add

Newton's Law $\vec F = m\vec a$

$\vec a = \frac{d\vec u}{dt}$

$a' = \frac{d\vec u'}{dt} = \frac{d\vec u}{dt}$ so $\vec F' = \vec F$, everybody is happy

Maxwell's equations in vacuum $(\vec J = 0)$

$$\vec\nabla \times \vec E = -\frac{\partial \vec B}{\partial t} \qquad \vec\nabla \times \vec B = \frac{1}{c^2}\frac{\partial \vec E}{\partial t}$$

$$\vec\nabla \times (\vec\nabla \times \vec B) = -\nabla^2 \vec B + \vec\nabla(\overbrace{\vec\nabla \cdot \vec B}^{0}) = \frac{1}{c^2}\frac{\partial}{\partial t}(\vec\nabla \times \vec E)$$

$$-\nabla^2 \vec B = -\frac{1}{c^2}\frac{\partial^2 \vec B}{\partial t^2}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{B} = 0$$ wave equation for light (EM radiation)

now $\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} + \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} = \frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}$$

so under Galileon transformation

$$\left[\frac{\partial^2}{\partial x'^2} - \frac{1}{c^2}\left(\frac{\partial^2}{\partial t'^2} - 2v\frac{\partial^2}{\partial t'\partial x'} + v^2\frac{\partial^2}{\partial x'^2}\right)\right]\vec{B} = 0$$

wave equation <u>NOT</u> invariant under Galilean transformation

→ this is in vacuum!

wave equation for sound wave propagating isn't invariant under Galileon transformation... but that's OK, interpret rest frame of medium (air) as the proper one for evaluating wave equation

how about Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\nabla^2 - i\hbar\frac{\partial}{\partial t}\right)\psi = V\psi$$

$$\rightarrow \left(-\frac{\hbar^2}{2m}\nabla'^2 - i\hbar\frac{\partial}{\partial t'} + i\hbar v\frac{\partial}{\partial x'}\right)\psi = V\psi$$

subst. $\psi = \psi' e^{i\frac{m}{\hbar}vx' + i\frac{mv^2}{2\hbar}t'}$ ← same physics since (absorb "boost") in the phase of ψ

with $V = V'$

gives $\left(-\frac{\hbar^2}{2m}\nabla'^2 - i\hbar\frac{\partial}{\partial t'}\right)\psi = V'\psi'$

Lorentz transformation $\quad \beta = \frac{v}{c}$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

$$x' = \gamma(x - \beta c t)$$
$$y' = y$$
$$z' = z$$
$$ct' = \gamma(ct - \beta x)$$

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

like a rotation matrix
only with hyperbolic cosh and sinh

$$\cosh\theta = \gamma \qquad \text{and } \cosh^2\theta - \sinh^2\theta = 1$$
$$\sinh\theta = \beta\gamma \qquad = \gamma^2 - \beta^2\gamma^2$$
$$= \gamma^2(1-\beta^2) = 1$$

it's the relative $\ominus$ve sign between space and time

show Maxwell's EM wave eqn invariant under Lorentz transform

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} \quad ; \quad \frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} + \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'}$$

then $\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{B} = 0$ in S is what in S'?

... Problem Set #1

$$\gamma^2 - (\beta\gamma)^2 = 1$$

$$\cosh^2\theta - \sinh^2\theta = 1$$

we know from special relativity the spacetime interval

$$c^2(t_A - t_B)^2 - (\vec{x}_A - \vec{x}_B)^2 \text{ is invariant}$$

(same as magnitude of vector under rotation of basis)

magnitude of spacetime 4-vector : Lorentz invariant

$$(ct')^2 - (x')^2 - (y')^2 - (z')^2$$

$$= (\gamma ct - \beta\gamma x)^2 - (-\beta\gamma ct + \gamma x)^2 - y^2 - z^2$$

$$= (\gamma ct)^2 - 2\beta\gamma^2 ctx + (\beta\gamma x)^2 - (\beta\gamma ct)^2 + 2\beta\gamma^2 ctx - (\gamma x)^2 - y^2 - z^2$$

$$= \gamma^2(1-\beta^2)(ct)^2 - \gamma^2(1-\beta^2)x^2 - y^2 - z^2$$

$$= ct^2 - x^2 - y^2 - z^2$$

4-vector notation

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$$

general 4-vector

$$A^\mu = (A^0, \vec{A})$$

$$A^{0\prime} = \gamma(A^0 - \beta A^1)$$
$$A^{1\prime} = \gamma(A^1 - \beta A^0)$$
$$A^{2\prime} = A^2$$
$$A^{3\prime} = A^3$$

$$\boxed{(A^{0\prime})^2 - \vec{A}' \cdot \vec{A}' = A^{0\,2} - \vec{A}\cdot\vec{A}}$$

Lorentz invariant

Energy Momentum $p^\mu = (E, \vec{p}c)$

in rest frame of a particle $p^\mu = (mc^2, 0)$

$(mc^2)^2$ is Lorentz invariant $= E^2 - \vec{p}\cdot\vec{p}\,c^2$

$$\longrightarrow E^2 = p^2c^2 + (mc^2)^2$$

in group theory: transformations acting on 4-vectors that leave s^2 invariant for a group $s^2 = x^{0\,2} - x^{1\,2} - x^{2\,2} - x^{3\,2}$

- Lorentz group (includes ordinary rotations and Lorentz boost)
- Poincaré group includes translations and reflections in both space and time

Tensors of rank k are defined by their transformation properties $x \to x'$

Scalar (rank zero) is a number whose value is unchanged by the transformation

s^2 Lorentz invariant scalar
$m^2 \to m$ " " "
$d\tau$ proper time is Lorentz invariant scalar

vector (rank one)

contravariant
Covariant

$\mathbf{A}$ is $A^i \hat{e}_i$ Einstein summation notation (repeated indices summed over)

components	basis vectors
contra vary	covary

example 1 $m \to cm$
components ↑ by 100

example 2 rotate basis vectors
components vary inversely

units [m] for example

units $[m^{-1}]$ for example

$$A'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} A^{\beta}$$

e.g. $$A'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^0} A^0 + \frac{\partial x'^{\alpha}}{\partial x^1} A^1 + \frac{\partial x'^{\alpha}}{\partial x^2} A^2 + \frac{\partial x'^{\alpha}}{\partial x^3} A^3$$
contravariant

$$B'_{\alpha} = \underbrace{\frac{\partial x^{\beta}}{\partial x'^{\alpha}}}_{} B_{\beta}$$
covariant

inverse of

dual or covariant vector (3-gradient) $\vec{\nabla}$ units $[m^{-1}]$

$$\nabla_i = \frac{\partial}{\partial x^i} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

4-gradient
$$\partial_{\mu} \equiv \frac{\partial}{x^{\mu}} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla} \right)$$

rank two tensor

$$F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} F^{\gamma\delta}$$

$$G'_{\alpha\beta} = \frac{\partial x^{\gamma}}{\partial x'^{\alpha}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} G_{\gamma\delta}$$

$$H'^{\alpha}{}_{\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} H^{\gamma}{}_{\delta}$$

inner product $B \cdot A \equiv B_{\alpha} A^{\alpha}$

$$B' \cdot A' = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} B_{\beta} A^{\gamma} = \frac{\partial x^{\beta}}{\partial x^{\gamma}} B_{\beta} A^{\gamma} = \delta^{\beta}_{\gamma} B_{\beta} A^{\gamma} = B \cdot A$$

inner product one index is contravariant, one index covariant