$$(ds)^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

↳ metric tensor

flat spacetime (of special relativity)

$g_{00} = 1$

$g_{11} = -1$

$g_{22} = -1$

$g_{33} = -1$

$g^{\alpha\beta} = g_{\alpha\beta}$ for flat spacetime

- geometry of spacetime is described by the metric tensor
- Lorentz group transformations leave the metric unchanged

$$dx^\alpha = S^\alpha{}_{\beta'}\, dx^{\beta'}$$

$$\frac{\partial x^\alpha}{\partial x'^\beta}$$

example

$$\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

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contracting contravariant vector with $g_{\alpha\beta}$ → covariant vector

$$A^\alpha = (A^0, \vec{A})$$

$$A_\alpha = g_{\alpha\beta} A^\beta = (A^0, -\vec{A})$$

$$A \cdot B = g_{\alpha\beta} A^\alpha B^\beta = A_\alpha B^\beta = A^0 B^0 - \vec{A} \cdot \vec{B}$$

$$\partial_\mu = \left(\frac{\partial}{\partial x^0}, \vec{\nabla}\right)$$

$$\partial^\mu = \left(\frac{\partial}{\partial x^0}, -\vec{\nabla}\right)$$

$$\partial_\mu A^\mu = \partial^\mu A_\mu = \frac{\partial A^0}{\partial x^0} + \vec{\nabla} \cdot \vec{A}$$

scalar ... ~~must be~~ is Lorentz invariant } 4-divergence

recall continuity equation $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$

define $J^\alpha = (c\rho, \vec{J})$ $\quad \partial_\alpha J^\alpha = \frac{1}{c}\frac{\partial}{\partial t} c\rho + \vec{\nabla} \cdot \vec{J} = 0$

back to the vector potential $\vec{A}$

Definition of $\vec{B}$ and $\vec{E}$ in terms of $\vec{A}$ and Φ automatically satisfies two of the homogeneous Maxwell equation.

$$\boxed{\vec{\nabla}\cdot\vec{B}=0} \Rightarrow \vec{\nabla}\times\vec{A}=\vec{B}$$

$$\boxed{\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A}) \Rightarrow \vec{\nabla}\times\left(E+\frac{\partial\vec{A}}{\partial t}\right)=0$$

$$\Rightarrow \vec{E}+\frac{\partial\vec{A}}{\partial t}=-\vec{\nabla}\Phi \qquad \therefore \vec{E}=-\vec{\nabla}\Phi-\frac{\partial\vec{A}}{\partial t}$$

gauge freedom $\vec{A}\to\vec{A}+\vec{\nabla}\lambda$ leaves $\vec{B}$ unchanged... but must also leave $\vec{E}$ unchanged

$$\Phi\to\Phi-\frac{\partial\lambda}{\partial t}$$

$$\boxed{\vec{\nabla}\cdot\vec{E}=\rho/\epsilon_0} \Rightarrow \vec{\nabla}\cdot\left(-\vec{\nabla}\Phi-\frac{\partial\vec{A}}{\partial t}\right)=-\nabla^2\Phi-\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A}=\rho/\epsilon_0$$

$$\boxed{\vec{\nabla}\times(\vec{\nabla}\times\vec{A})-\frac{1}{c^2}\frac{\partial}{\partial t}\left(-\vec{\nabla}\Phi-\frac{\partial\vec{A}}{\partial t}\right)=\mu_0\vec{J}}$$

$$-\nabla^2\vec{A}+\vec{\nabla}(\vec{\nabla}\cdot\vec{A})+\frac{1}{c^2}\frac{\partial}{\partial t}\vec{\nabla}\Phi+\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t}=\mu_0\vec{J}$$

if can set $\vec{\nabla}\cdot\vec{A}+\frac{1}{c^2}\frac{\partial\Phi}{\partial t}=0$ using gauge freedom then

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)\vec{A}=\mu_0\vec{J} \qquad \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)\Phi=\frac{\rho}{\epsilon_0}$$

Suppose $\vec{\nabla}\cdot\vec{A}+\frac{1}{c^2}\frac{\partial\Phi}{\partial t}=f(\vec{x},t)$

in a different gauge $\vec{\nabla}\cdot\vec{A}'+\nabla^2\lambda+\frac{1}{c^2}\frac{\partial\Phi'}{\partial t}-\frac{1}{c^2}\frac{\partial^2\lambda}{\partial t^2}=f(\vec{x},t)$

if find λ so that $\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\lambda=f(\vec{x},t)$

then $\vec{\nabla}\cdot\vec{A}'+\frac{1}{c^2}\frac{\partial\Phi'}{\partial t}=0$ conclude free to choose λ to make recall Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$, then $\nabla^2 \Phi = -\rho/\epsilon_0$

$$\boxed{\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0}$$ is the Lorenz gauge

Ludvig Lorenz Danish not

~~Lorentz gauge~~ Hendrik Lorentz Dutch (transformation, Lorentz force law)

suggest $\partial_\mu A^\mu = 0$

where A^μ is $\left(\frac{\Phi}{c}, \vec{A}\right)$

$J^\alpha = (c\rho, \vec{J})$

$\Box$ is $\left(\frac{1}{c^2}\frac{\partial}{\partial t^2} - \nabla^2\right)$ 4-Laplacian called the $\rightarrow$ D'Alembertian

$\Box = \partial^\mu \partial_\mu$

$$\boxed{\Box A^\mu = \mu_0 J^\mu}$$ this is Maxwell's equations in $\partial_\alpha A^\alpha = 0$ Lorenz gauge

note for $A^0 = \frac{\Phi}{c}$ $J^0 = c\rho$

$$\Box \frac{\Phi}{c} = \mu_0 c\rho = \frac{\rho}{\epsilon_0}$$ b/c $\epsilon_0 \mu_0 = \frac{1}{c^2}$

Explicitly, the fields $\partial^\alpha = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right)$

$$E_x = -\frac{\partial \Phi}{\partial x} - \frac{\partial A_x}{\partial t} = \partial^1 cA^0 - \partial^0 cA^1$$

$$B_x = (\vec{\nabla}\times\vec{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \partial^3 A^2 - \partial^2 A^3$$

in general $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$ anti symmetric rank two tensor, has 6 components $\frac{\vec{E}}{c}, \vec{B}$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

$\boxed{\text{Electromagnetic Field tensor or Faraday tensor}}$

$$\partial_\alpha F^{\alpha\beta} = \partial_\alpha \partial^\alpha A^\beta - \cancel{\partial_\alpha \partial^\beta A^\alpha} \to 0$$

Lorenz gauge $\partial_\alpha A^\alpha = 0$

$$\partial_\alpha F^{\alpha\beta} = \Box A^\beta = \mu_0 J^\beta$$

note: $\partial_\beta \partial_\alpha F^{\alpha\beta} = \mu_0 \partial_\beta J^\beta = 0$ continuity equation

$$\boxed{\text{note: } F_{\mu\nu} F^{\mu\nu} = 2\left(B^2 - \frac{E^2}{c^2}\right) \quad \text{Lorentz invariant quantity}}$$

for completeness $F_{\alpha\beta} = g_{\alpha\mu} g_{\beta\nu} F^{\mu\nu}$ $\quad \vec{E} \to -\vec{E}$, $\vec{B} \to \vec{B}$

$$\epsilon^{\alpha\beta\gamma\delta} = \begin{cases} +1 & \text{for } \alpha=0, \beta=1, \gamma=2, \delta=3 \text{ and any even permutation} \\ -1 & \text{for any odd permutation} \\ 0 & \text{if any two indices are equal} \end{cases}$$

$$\epsilon_{\alpha\beta\gamma\delta} = -\epsilon^{\alpha\beta\gamma\delta}$$

define dual field-strength tensor

$$\mathcal{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & \frac{E_z}{c} & -\frac{E_y}{c} \\ B_y & -\frac{E_z}{c} & 0 & \frac{E_x}{c} \\ B_z & \frac{E_y}{c} & -\frac{E_x}{c} & 0 \end{pmatrix}$$

and then $\partial_\alpha \mathcal{F}^{\alpha\beta} = 0 \left(\begin{array}{l} \text{for } \beta = 0 \Rightarrow \vec{\nabla}\cdot\vec{B} = 0 \\ \text{for } \beta = 1,2,3 \Rightarrow \vec{\nabla}\times\vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0 \end{array} \right.$

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put Lorentz force law in manifestly covariant form

$$\vec{F} = \frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v}\times\vec{B})$$

$$\frac{d\vec{p}}{d\tau} = \frac{d\vec{p}}{dt}\left(\frac{dt}{d\tau}\right) = q(\vec{E}+\vec{v}\times\vec{B})\frac{dt}{d\tau}$$

$$\frac{dt}{d\tau} = \gamma = \frac{u^0}{c}$$

$$\frac{dp^k}{d\tau} = q\left(\frac{u^0}{c}E^k + \underbrace{u^i B^j \epsilon_{ij}{}^k}_{\text{Levi-Civita tensor}}\right)$$

Levi-Civita tensor
$\left(\begin{array}{l} +1 \text{ even permutation } 123 \\ -1 \text{ odd } \quad '' \\ 0 \text{ if two indices same} \end{array} \right.$

4-velocity

$$u^\alpha = (\gamma c, \gamma \vec{v})$$

where $\vec{v} = \dfrac{d\vec{x}}{dt}$

$$\boxed{u^\alpha = \frac{dx^\alpha}{d\tau}} \qquad \frac{dt}{d\tau} = \gamma$$

proper time

tangent 4-vector of a world line

$$p^\alpha \equiv mcu^\alpha$$
$$= (\gamma mc^2, \gamma m\vec{v}c)$$
$$p^\alpha = (E, \vec{p}c)$$

$$F^{\alpha\beta}u_\beta = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} \cancel{\gamma c} \\ \cancel{-\gamma v_x} \\ \cancel{-\gamma v_y} \\ \cancel{-\gamma v_z} \end{pmatrix} \begin{pmatrix} u^0 \\ -u^1 \\ -u^2 \\ -u^3 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma \frac{\vec{E}\cdot\vec{u}}{c} \\ \gamma E_x + \gamma u_y B_z - \gamma u_z B_y \\ \gamma E_y + \gamma u_z B_x - \gamma u_x B_z \\ \gamma E_z + \gamma u_x B_y - \gamma u_y B_x \end{pmatrix} = \begin{pmatrix} \frac{\vec{E}}{c}\cdot\vec{u} \\ \frac{u^0}{c}E_x + u^2 B_z - u^3 B_y \\ \frac{u^0}{c}E_y + u^3 B_x - u^1 B_z \\ \frac{u^0}{c}E_z + u^1 B_y - u^2 B_x \end{pmatrix}$$

$$\frac{dp^\alpha}{c\,d\tau} = q F^{\alpha\beta} u_\beta$$

what means $\frac{dp^0}{c\,d\tau} = q F^{0\beta} u_\beta = \frac{q}{c} \vec{E} \cdot \vec{u} = \frac{q}{c} \gamma \vec{E} \cdot \vec{v}$

$$\frac{dE}{d\tau} = \frac{dE}{dt}\left(\frac{dt}{d\tau}\right) = q\gamma \vec{E} \cdot \vec{v}$$

$$\boxed{W = \frac{dE}{dt} = q \vec{E} \cdot \vec{v}}$$

particle energy p^0

How do $\vec{E}$ and $\vec{B}$ transform under Lorentz boost?

as $F^{\alpha\beta}$ (rank two tensor)

$$F'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{\partial x'^\beta}{\partial x^\delta} F^{\gamma\delta}$$

or by matrix multiplication

$$F' = A F A^T$$

e.g. $A_{\text{boost in } x} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$