$F' = A F A^T$

$$\frac{\partial x'^{\alpha}}{\partial x^{\delta}} = A = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & 0 \\ & & 1 & 0 \\ & 0 & 0 & 1 \end{pmatrix}$$

for S' moving with velocity $\vec{V} = v\hat{x}$ relative to S

we find

$E_x' = E_x \qquad E_y' = \gamma(E_y - \beta B_z c) \qquad E_z' = \gamma(E_z + \beta B_y c)$

$B_x' = B_x \qquad B_y' = \gamma\left(B_y + \beta \frac{E_z}{c}\right) \qquad B_z' = \gamma\left(B_z - \beta \frac{E_y}{c}\right)$

or $E_{\parallel}' = E_{\parallel} \qquad \vec{E}_{\perp}' = \gamma(\vec{E}_{\perp} + \vec{\beta} \times \vec{B}c)$

$B_{\parallel}' = B_{\parallel} \qquad \vec{B}_{\perp}' = \gamma\left(\vec{B}_{\perp} - \vec{\beta} \times \frac{\vec{E}}{c}\right)$

(note: $\vec{\beta} = \frac{v}{c}\hat{x}$ in the above example)

$$\vec{E}_{\parallel} = \frac{\vec{\beta}(\vec{\beta}\cdot\vec{E})}{\beta^2}$$

So $\vec{E}' = \frac{\vec{\beta}(\vec{\beta}\cdot\vec{E})}{\beta^2} + \gamma(\vec{E}_{\perp} + \vec{\beta} \times \vec{B}c)$

$$\vec{E}_{\perp} = \vec{E} - \vec{E}_{\parallel} = \vec{E} - \frac{\vec{\beta}(\vec{\beta}\cdot\vec{E})}{\beta^2}$$

$$\therefore \vec{E}' = \gamma(\vec{E} + \vec{\beta} \times \vec{B}c) - \frac{(\gamma - 1)\vec{\beta}(\vec{\beta}\cdot\vec{E})}{\beta^2}$$

since $\gamma^2 = \frac{1}{1-\beta^2} \Rightarrow \frac{1}{\beta^2} = \frac{\gamma^2}{\gamma^2 - 1}$ $\quad \| \quad$ $\frac{\gamma - 1}{\beta^2} = \frac{\gamma^2(\gamma-1)}{\gamma^2 - 1} = \frac{\gamma^2}{\gamma + 1}$

and $\boxed{\vec{E}' = \gamma(\vec{E} + \vec{\beta} \times \vec{B}c) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{E})}$

similarly $\boxed{\vec{B}' = \gamma\left(\vec{B} - \vec{\beta} \times \frac{\vec{E}}{c}\right) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{B})}$

Illuminating Example

field of a charge in constant motion $\vec{v}$ (in x-direction)

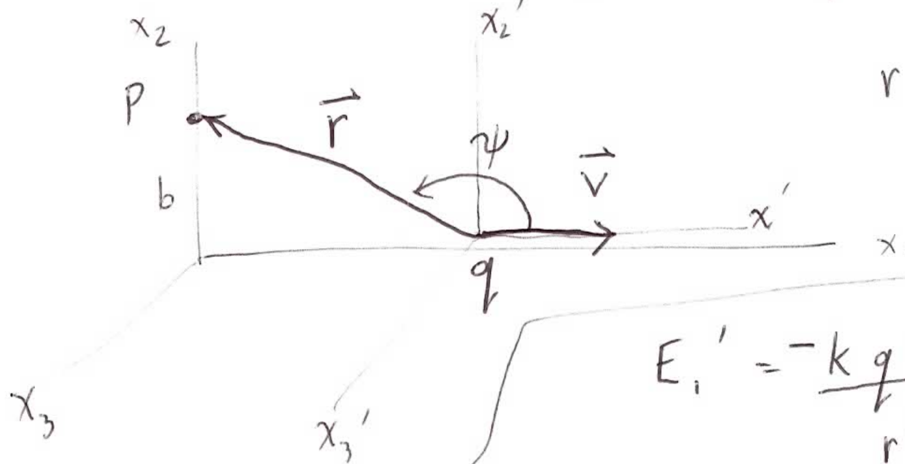
S' rest frame of charge $\qquad$ S rest frame of observer

$$k = \frac{1}{4\pi\epsilon_0}$$

observer at P $\qquad x_1 = x_3 = 0, \quad x_2 = b$

or $\qquad x_1' = -vt', \quad x_3' = 0 \quad x_2' = b$

$$r' = \sqrt{b^2 + (vt')^2}$$



$$E_1' = \frac{-kqvt'}{r'^3}, \quad E_2' = \frac{kqb}{r'^3}, \quad E_3' = 0$$

fields in S'

$$B_1' = B_2' = B_3' = 0$$

fields in S $\qquad E_1 = E_1', \quad E_2 = \gamma E_2'$

$$E_1 = \frac{-kqvt'}{r'^3}, \quad E_2 = \frac{\gamma kqb}{r'^3}, \quad E_3 = 0$$

$t = t'/\gamma$ and $r' = (b^2 + \gamma^2 v^2 t^2)^{1/2}$

$$\therefore E_1 = \frac{-kq\gamma vt}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}, \quad E_2 = \frac{k\gamma qb}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}$$

$$\vec{E} \text{ is } \frac{kq\gamma}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \underbrace{(-vt, b, 0)}_{\vec{r}}$$

vector from particle to observer

$B_1 = B_1' = 0$

$\vec{\beta} \times \vec{E}$

β in x-direction

so $\beta \times E_1$ component zero

$\beta \times E_3$ component zero
$E_3 = 0$

$\beta \times E_2$ component

$$\boxed{B_3 = \beta E_2/c}$$

all other components are zero

$$B_3 = \frac{k\beta\gamma q b}{c(b^2 + \gamma^2 v^2 t^2)^{3/2}}$$

---

observations: #1 as $\beta \to 1$ $B_3 \approx \frac{E_2}{c}$

magnetic field becomes "equal" to the transverse electric field

#2 non-relativistic $\gamma \approx 1$

$$B_3 = \frac{1}{4\pi\epsilon_0} \frac{v}{c^2} \frac{q b}{\cancel{\gamma} r^3} = \frac{\mu_0}{4\pi} \frac{q v b}{r^3}$$ Biot-Savart Law!

#3 at relativistic $\gamma >> 1$ then peak transverse

E_2 $(@t=0)$ $= \gamma$ times non-relativistic value of E_2

but measure of time the charged particle q is near observer at P

$$\Delta t \approx \frac{b}{\gamma v}$$

proper time

so transverse field goes up with γ

but duration goes down

#4 for relativistic $\beta \to 1$, observer sees "equal" transverse electric and magnetic fields

$\rightarrow$ this is like pulse of plane polarized radiation propagating in x-direction

#5 The parallel or Longitudinal E field points one way then back the other

→ if the observer's "detector" has any "inertia" does not get altered

∴ for practical purposes respond only to transverse field

same for relativistic charged particle as for pulse of EM radiation

---

back to $\vec{E}$ field $\quad \vec{E} = \dfrac{kq\gamma}{(\underbrace{b^2+\gamma^2v^2t^2}){}^{3/2}}(-vt, b, 0)$

$$\gamma^2 r^2 = \gamma^2 v^2 t^2 + \gamma^2 b^2$$

$$\gamma^2 r^2 + (1-\gamma^2) b^2 = \gamma^2 v^2 t^2 + b^2$$

$$= \gamma^2 r^2 \left(1 - \frac{\gamma^2 - 1}{\gamma^2 r^2} b^2\right)$$

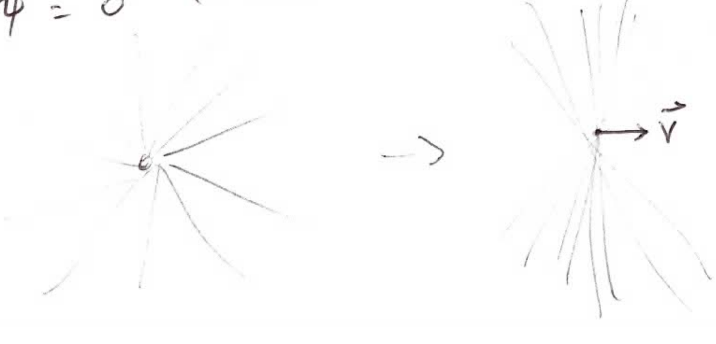
$$\frac{1}{\gamma^2} = 1 - \beta^2 \qquad \sin\psi = \frac{b}{r}$$

$$= \gamma^2 r^2 (1 - \beta^2 \sin^2\psi)$$

$$\therefore \vec{E} = \frac{kq\,\vec{r}}{\gamma^2 r^3 (1-\beta^2 \sin\psi)^{3/2}}$$

for $\psi = \pi/2$ (particle crosses $x=0$) $\vec{E} = \dfrac{kq\,\hat{y}}{b^2} \times \gamma$

for $\psi = 0$ (observer on x-axis) $\vec{E} = \dfrac{kq\,\hat{x}}{\gamma^2 (vt)^2}$



"whiskbroom" pattern of force lines