Jackson Chapter 12 Motion in $\vec{E}$ & $\vec{B}$ fields

Case 1: motion in uniform, static $\vec{B}$ field ($\vec{E}=0$)

$$\frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v}\times\vec{B}) \;;\quad \frac{d\,\text{Energy}}{dt} = q\,\vec{v}\cdot\vec{E}) = 0$$

Energy constant, γ constant

$$\vec{p} = \gamma m \vec{v}$$

$$\frac{d\vec{v}}{dt} = \cancel{q}\, \vec{v} \times \frac{q}{\gamma m}\vec{B}$$

$$= \vec{v} \times \vec{\omega}_B \text{, where } \vec{\omega}_B = \frac{q\vec{B}}{\gamma m}$$

gyration or precession frequency or cyclotron frequency

$$\boxed{\omega_B = \frac{qB}{\gamma m}}$$

circular motion perpendicular to $\vec{B}$

for unit vector $\hat{e}_3 \parallel$ to $\vec{B}$

$$\vec{v}(t) = v_\parallel \hat{e}_3 + \omega_B a(\hat{e}_1 - i\hat{e}_2)e^{-i\omega_B t}$$

$$\vec{v}(t) = \text{Re}\{\text{above expression}\}$$

RH system $\hat{e}_1 \times \hat{e}_2 = \hat{e}_3$

another integration

$$\vec{x}(t) = \vec{X}_0 + v_\parallel t\,\hat{e}_3 + ia(\hat{e}_1 - i\hat{e}_2)e^{-i\omega_B t}$$

Larmor radius

a = gyration radius

$$a = \frac{v_\perp}{\omega_B} = \frac{\gamma m v_\perp}{qB} = \frac{p_\perp}{qB}$$

in units $p_\perp\,[\text{MeV}/c] = 3.00\times10^{-4}\, Ba\,[\text{gauss-cm}]$

$\updownarrow 10^{4}\ \updownarrow 10^{2}$

(Tesla - m)

for q = electron charge $p_\perp(\text{MeV}/c) = \cancel{3.00}\ \ Ba$

3.00×10^{2}

examples in galaxy $B \simeq 3\times10^{-6}$ gauss

cosmic rays $p_\perp \simeq 10^{4} - 10^{15}$ [$\cancel{\text{MeV}}$ MeV/c]

$$a = 10^{13} - 10^{24}\ \text{cm}$$

$$\simeq 10^{-5}\ \text{pc} \sim 1\ \text{MPC}$$

$B = 1\text{T} = 10^{4}$ gauss

1 T magnet

1 m radius

$p_\perp = 300\ \text{MeV}/c$

Case 2: motion in $\vec{E}$ and $\vec{B}$ (not parallel, not necessarily $\perp$)

sub-case $\vec{E} \perp \vec{B}$ and $\frac{|\vec{E}|}{c} < |\vec{B}|$

S' moving with $\vec{u}$ w.r.t. S frame

<u>Choose</u> $\vec{u}$ reference frame moving $\perp$ to $\vec{E}$ and $\vec{B}$ (themselves $\perp$)

$$\vec{u} = \frac{\vec{E} \times \vec{B}}{B^2}$$

free normalization of how fast is reference frame

$$\vec{E}' = \gamma\left(\vec{E} + \frac{\vec{u}}{c} \times Bc\right) - \frac{\gamma^2}{\gamma+1}\frac{\vec{u}}{c}\left(\frac{\vec{u}}{c}\cdot\vec{E}\right)$$

chose $\perp$ $\vec{u}$

$$\vec{B}' = \gamma\left(\vec{B} - \frac{\vec{u}}{c} \times \frac{\vec{E}}{c}\right) - \frac{\gamma^2}{\gamma+1}\frac{\vec{u}}{c}\left(\frac{\vec{u}}{c}\cdot\vec{B}\right)$$

$=0$ $\therefore$ in S' we deliberately chose to have $\vec{E}' = 0$

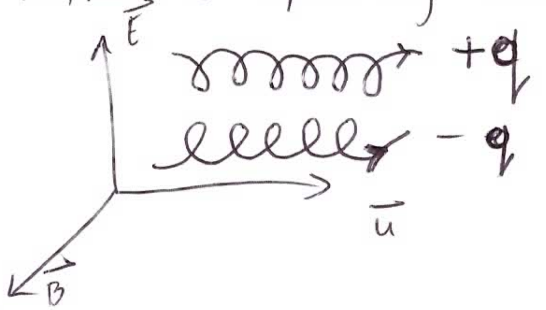
$$\vec{B}'_\perp = \gamma\left(\vec{B} - \frac{E^2\vec{B}}{B^2c^2}\right) = \gamma\left(1 - \frac{E^2}{B^2c^2}\right)\vec{B}$$

$$\gamma = \frac{1}{(1-\beta^2)^{1/2}} = \frac{1}{\left(1-\left(\frac{E}{cB}\right)^2\right)^{1/2}}$$

$$\therefore \vec{B}'_\perp = \frac{1}{\gamma}\vec{B} = \left(1 - \frac{E^2}{B^2c^2}\right)^{1/2}\vec{B} = \left(\frac{B^2 - E^2/c^2}{B^2}\right)^{1/2}\vec{B}$$

in S' only field acting is static B' which points in same direction as $\vec{B}$ by is weaker by $\frac{1}{\gamma}$

in S' motion is spiraling around ~~lines of force~~ $\vec{B}$



$\vec{E} \times \vec{B}$ drift

what if $|\vec{E}/c| > |\vec{B}|$?

$$\vec{u} = \frac{\vec{E}\times\vec{B}}{B^2} \Rightarrow u = \frac{E}{B}$$

works only for $E/c < B$ for that means $u < c$

if $E/c \geq B$ would imply $\cancel{u \geq c}$

So transform to S'' in which $\vec{B} = 0$

$$\vec{u} = \frac{\vec{E}\times\vec{B}}{E^2}c^2$$

no $\vec{E}_{\parallel}''$

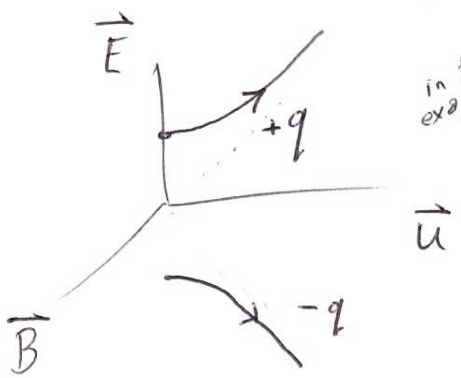
no $\vec{B}''$

$$\vec{E}'' = \vec{E}_{\perp}'' = \gamma\left(\vec{E} + \frac{\vec{E}\times\vec{B}c^2}{\cancel{c}\,E^2}\times\vec{B}\cancel{c}\right)$$

$$= \gamma\left(\vec{E} - \frac{B^2c^2\vec{E}}{E^2}\right)$$

$$\vec{E}'' = \gamma\left(1 - \frac{B^2}{(E/c)^2}\right)\vec{E} = \frac{1}{\gamma}\vec{E}$$

in this reference frame, charge sees pure $\vec{E}$ field, accelerates



in this example ← consider $\boxed{V_{\parallel} = u} \Rightarrow V_{\parallel}'' = 0$

$$V_{\perp} = \frac{V_{\perp}''}{\gamma}$$

remember $F_{\mu\nu}F^{\mu\nu} = 2(B^2 - E^2/c^2)$ becomes < 0 if $|\vec{E}/c| > |\vec{B}|$ ∴ can't find frame where only $\vec{B}$ field; but can find one with only $\vec{E}$ field!