if $\left|\frac{\vec{E}}{c}\right| < \vec{B}$, $u = \frac{E}{B}$ ← what does this case represent?

if $\left|\vec{E}/c\right| > \vec{B}$, $u = \frac{B}{E}$ we can choose $\vec{u} = \frac{\vec{E}\times\vec{B}}{B^2}$ to have S' frame

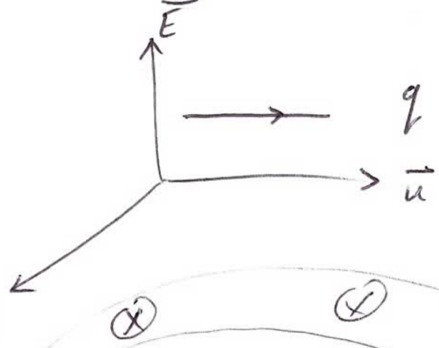
with $\vec{E}' = 0$, only $\vec{B}'$

but in S' if $v_\parallel = E/B$ $\therefore$ $v_\parallel' = 0$ then in S' $\vec{v}' = 0$

and $u = \frac{E}{B}$ $v_\perp = 0$ $\therefore$ $v_\perp' = 0$ and only $\vec{B}'$

$\Rightarrow$ no motion

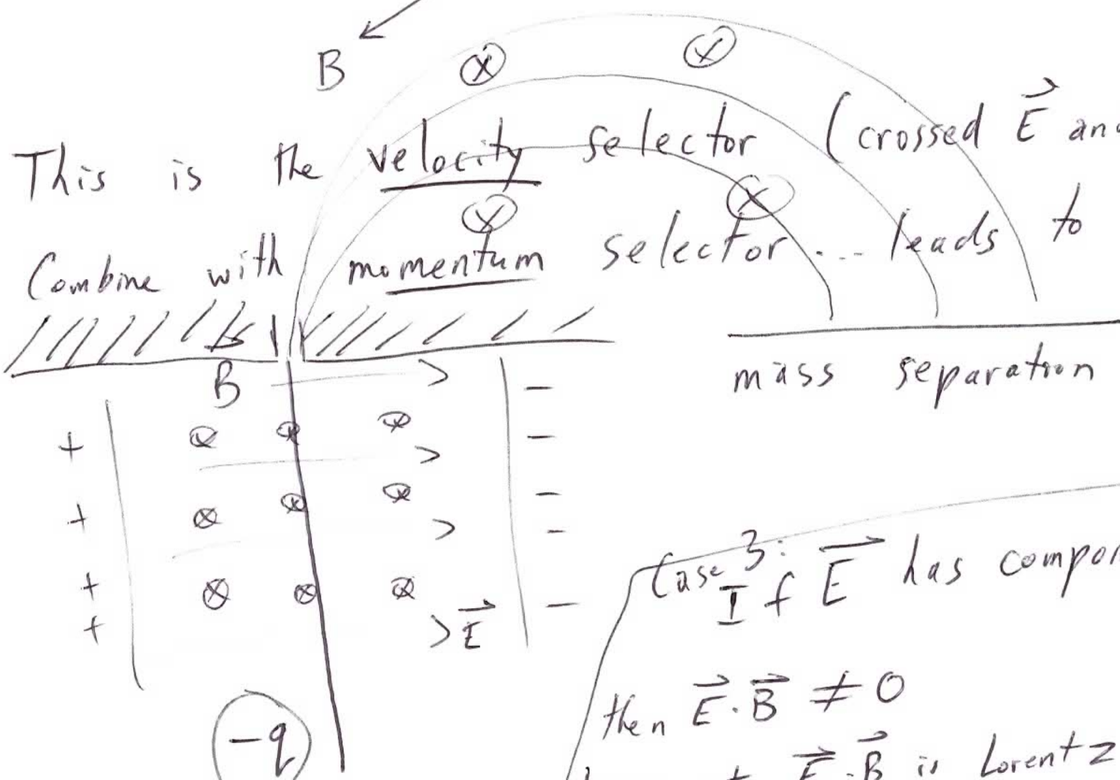
and in S



if $v_\parallel = \frac{E}{B}$

This is the <u>velocity</u> selector (crossed $\vec{E}$ and $\vec{B}$ fields).

Combine with <u>momentum</u> selector ... leads to mass spec.



mass separation occurs

Case 3: If $\vec{E}$ has component parallel to $\vec{B}$

then $\vec{E}\cdot\vec{B} \neq 0$

turns out $\vec{E}\cdot\vec{B}$ is Lorentz scalar

$$\mathcal{F}^{\alpha\beta}F_{\alpha\beta} = -4\vec{E}\cdot\vec{B}$$

if $\vec{E}\cdot\vec{B} \neq 0$ then impossible to go to an inertial frame with $\vec{E}$ or $\vec{B}$ zero $\therefore$ if $\vec{E}\cdot\vec{B} \neq 0$, $\vec{E}$ and $\vec{B}$ exist simultaneously in all Lorentz frames we did uniform, static $\vec{E}$ field only trivial

$\vec{B}$ " " 1^{st} year physics

$\vec{E}, \vec{B}$ fields $\perp$ considered ✓

$\vec{E}, \vec{B}$ with parallel components → discussed

Non-uniform, static $\vec{B}$ field

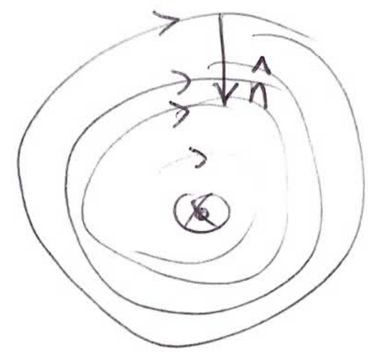
consider gentle variation (distance over which $\vec{B}$ changes is large compared to the gyration radius a)

∴ spiralling along field lines with gyration radius set by local B-field strength

with next-order perturbation: orbit undergoes slow change described as drifting of guiding centre and slow variations of gyration (freq.)

case: gradient of spatial varition of $\vec{B}$ is $\perp$ to $\vec{B}$

example ~~B~~ field around wire



motion parallel to $\vec{B}$ remains uniform

$$\vec{B} = B(r)\,\hat{\phi}$$

gradient is in $\hat{\rho}$ direction

1^{st} order expansion

$$\vec{\omega}_B(\vec{x}) = \frac{q}{\gamma m c}\vec{B}(\vec{x}) \simeq \vec{\omega}_0\left[1 + \frac{1}{B_0}\left(\frac{\partial B}{\partial \xi}\right)_0 \hat{n}\cdot\vec{x}\right]$$

($\omega_B = \omega_0$ at the origin)

ξ coordinate along $\hat{n}$ - direction of gradient

$$\frac{d\vec{v}_\perp}{dt} = \vec{v}_\perp \times \vec{\omega}_B(\vec{x})$$

assume no motion along field lines

$$\vec{v}_\perp = \vec{v}_0 + \vec{v}_1$$

$$\frac{d\vec{v}_0}{dt} = \vec{v}_0 \times \vec{\omega}_0 \quad 0^{th} \text{ order}$$

$$\frac{d\vec{v}_1}{dt} \simeq \left[\vec{v}_1 + \vec{v}_0(\vec{n}\cdot\vec{x}_0)\frac{1}{B_0}\left(\frac{\partial B}{\partial \xi}\right)_0\right] \times \vec{\omega}_0$$

keep only 1^{st} order

recall $\vec{v}_0 = \omega_0 a(\hat{e}_1 - i\hat{e}_2)e^{-i\omega_0 t} + v_\parallel \hat{e}_3$

$$\vec{x}_0 = v_\parallel t\,\hat{e}_3 + ia(\hat{e}_1 - i\hat{e}_2)e^{-i\omega_0 t}$$

$$\vec{\omega}_0 \times \hat{e}_3 = 0$$

$$\vec{\omega}_0 \times \hat{e}_2 = -\omega_0 \hat{e}_1$$

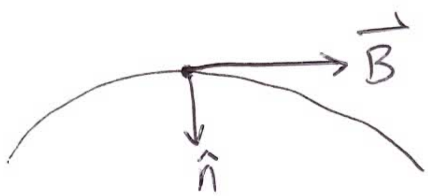
$$\vec{\omega}_0 \times \hat{e}_1 = \omega_0 \hat{e}_2$$

gives $\vec{v}_0 = -\vec{\omega}_0 \times \vec{x}_0$

$$\frac{d\vec{v}_1}{dt} = \left[\vec{v}_1 - \frac{1}{B_0}\left(\frac{\partial B}{\partial \xi}\right)_0 \vec{\omega}_0 \times \vec{x}_0\,(\hat{n}\cdot\vec{x}_0)\right] \times \vec{\omega}_0$$

want to know about non-oscillating terms - net motion averaged over cycles

$$\langle \vec{v}_1 \rangle \equiv \vec{V}_G = \frac{1}{B_0}\left(\frac{\partial B}{\partial \xi}\right)_0 \vec{\omega}_0 \times \underbrace{\langle \vec{x}_0 (\hat{n}\cdot\vec{x}_0)\rangle}$$



orbits in plane containing $\hat{n}$, $\perp$ to $\vec{B}$
$\hat{e}_1$ out of page

$$\vec{x}_0 = a(\sin\omega_B t\,\hat{e}_1 + \cos\omega_B t\,\underbrace{\hat{e}_2}_{\hat{n}})$$

$$\langle \vec{x}_0(\hat{n}\cdot\vec{x}_0)\rangle = a^2\langle \cancelto{0}{\sin\omega_B t\cos\omega_B t}\,\hat{e}_1 + \cos^2\omega_B t\,\hat{n}\rangle$$

$$= \frac{a^2}{2}\hat{n}$$

$$\vec{V}_G = \frac{a^2}{2}\frac{1}{B_0}\left(\frac{\partial B}{\partial \xi}\right)_0 \vec{\omega}_0 \times \hat{n}$$

gradient drift velocity

$$\vec{\omega}_B = \omega_B \frac{\vec{B}}{B}$$

$$\vec{\nabla}_\perp B = \hat{n}\,\frac{\partial B}{\partial \xi}$$

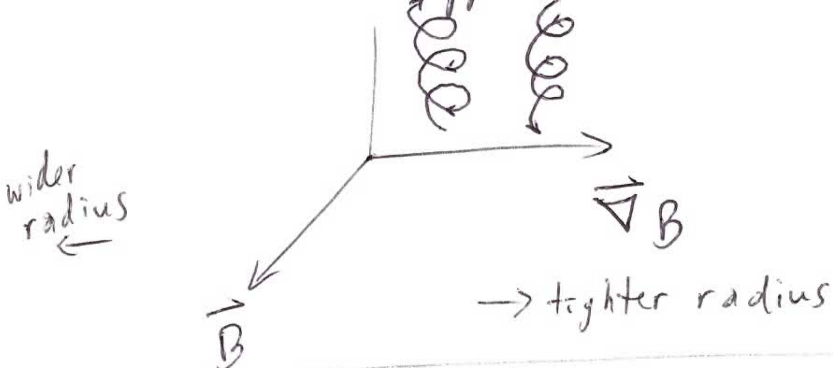
$$\frac{\vec{V}_G}{a\omega_B} = \frac{a}{2B^2}\vec{B}\times\vec{\nabla}_\perp B$$

if $a \left| \frac{\vec{\nabla} B}{B} \right| \ll 1$ distance over which $\vec{B}$ changes is large compared to gyration radius a

v_G drift velocity is small compared to orbital velocity $\omega_B a$

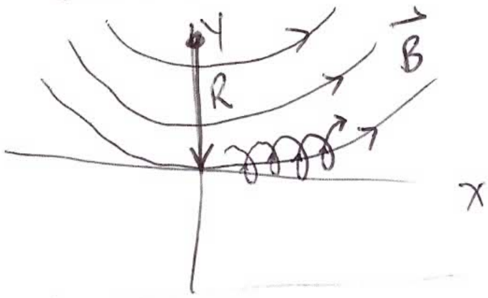
$\therefore$ particle spirals rapidly around $\vec{B}$ with its center of rotation slowly moving perpendicular to both $\vec{B}$ and $\vec{\nabla} B$

→ direction is opposite for $\oplus$ and $\ominus$ [it's in definition for $\omega_B = \frac{qB}{\gamma m}$]



Curvature of $\vec{B}$ also causes drift

radius of curvature $R \gg a$ gyration radius



field of infinite straight current-carrying wire

$$\vec{B} = B(\rho)\hat{\phi}$$

cylindrical coordinates ρ, ϕ, z

$\vec{\nabla} \cdot \vec{B} = 0$

$$\vec{\nabla} \times \vec{B} = \hat{z} \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho B_\phi = 0 \Rightarrow^{\text{implies } \vec{J}=0,\text{ static}} \quad B_\phi \propto \frac{1}{\rho}$$

(e.g. $\frac{\mu_0 I}{2\pi r}$)

$$\vec{B} = B_0 \frac{R}{\rho} \hat{\phi} \qquad \vec{\omega}_B = \omega_0 \frac{R}{\rho} \hat{\phi}$$

Lorentz force $\vec{a} = \vec{v} \times \vec{\omega}_B$

acceleration

$$\omega_0 = \frac{qB_0}{\gamma m}$$

Force law $\rightarrow$ equations of motion $\rightarrow$ full solution

$V_{\parallel}$ motion in $\hat{\phi}$ direction

$V_{\perp}$ in ρ, z plane

$$\vec{r} = \rho\,\hat{\rho} + z\,\hat{z}$$

$$\vec{V} = \dot{\rho}\,\hat{\rho} + \rho\dot{\phi}\,\hat{\phi} + \dot{z}\,\hat{z}$$

$$\vec{a} = \vec{V}\times\vec{\omega}_B = \frac{\omega_0 R}{\rho}\dot{\rho}\,\hat{z} - \frac{\omega_0 R}{\rho}\dot{z}\,\hat{\rho}$$

$$= (\ddot{\rho} - \rho\dot{\phi}^2)\,\hat{\rho} + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\,\hat{\phi} + \ddot{z}\,\hat{z}$$

① $$\ddot{\rho} - \rho\dot{\phi}^2 = -\frac{\omega_0 R}{\rho}\dot{z}$$

② $$\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi} = 0 \Rightarrow \rho^2\dot{\phi} = L = \text{constant} \quad \dot{L} = 0$$

$$\equiv V_{\parallel} R \quad \text{(motion along field lines)}$$

③ $$\ddot{z} = \frac{\omega_0 R}{\rho}\dot{\rho}$$

integrate $\Rightarrow$ $$\dot{z} = \omega_0 R \ln(\rho/R) + V_0$$

for helix with radius small compared to R

$$\rho = R + x \longrightarrow \rho/R = 1 + x/R$$

expand $(\rho/R)^n$ and $\ln(\rho/R)$ in powers of x/R (to 1st order)

$\Rightarrow$ ① $$\ddot{x} - \frac{L^2}{R^3(\rho/R)^3} = \ddot{x} - \frac{L^2}{R^3}\left(1 - 3\left(\frac{x}{R}\right)\right) = -\frac{\omega_0 R}{\rho}\left[\omega_0 R\left(\frac{x}{R}\right) + V_0\right]$$

$$= -\omega_0 \cancel{R}\left[1 - \left(\frac{x}{R}\right)\right]\left[\omega_0\left(\frac{x}{\cancel{R}}\right) + V_0\right]$$

$$\ddot{x} - \frac{L^2}{R^3}\left[1 - 3\left(\frac{x}{R}\right)\right] = -\omega_0^2 x - \omega_0 \cancel{R} V_0\left[1 - \left(\frac{x}{R}\right)\right]$$

$$\boxed{\ddot{x} + \left(\omega_0^2 + 3\frac{V_{\parallel}^2}{R^2}\right)x \simeq \frac{V_{\parallel}^2}{R} - \omega_0 V_0}$$

SHM around a displaced equil$^{\underline{m}}$

$$\langle x \rangle = \frac{V_{\parallel}^2}{\omega_0^2 R} - \frac{V_0}{\omega_0}$$

where $V_{\parallel} << \omega_0 R$ is assumed

$$\langle \dot{z} \rangle \simeq V_0 + \omega_0 R \left(\frac{x}{R} \right) \simeq V_0 + \omega_0 x \simeq \frac{V_{\parallel}^2}{\omega_0 R}$$

curvature drift (average z-component velocity)

V_c is $\frac{V_{\parallel}^2}{\omega_0 R}$ in the $+z$ direction or $\vec{R} \times \vec{B}$

so write $\vec{V}_c = \frac{V_{\parallel}^2}{\omega_0 R} \frac{\vec{R} \times \vec{B}}{RB}$

---

same direction as gradient drift

$V_{\perp} = \omega_B a$

$$\vec{V}_G = \frac{a^2 \omega_B}{2B^2} \vec{B} \times \vec{\nabla}_{\perp} B$$

for this example $\vec{B} = B_0 \frac{R}{\rho} \hat{\phi}$; $\vec{\nabla}_{\perp} B = -\frac{\vec{R}}{R^2} B$

$$\text{so } \vec{V}_G = \frac{a^2 \omega_B}{2B^2} \frac{\vec{R} \times \vec{B}}{R^2} (B) = \frac{\omega_B a^2}{2R} \left(\frac{\vec{R} \times \vec{B}}{RB} \right)$$

$$= \frac{V_{\perp}^2}{2\omega_B R} \left(\frac{\vec{R} \times \vec{B}}{RB} \right)$$

total drift velocity

$$\vec{V}_D = \vec{V}_c + \vec{V}_G = \frac{1}{\omega_B R} \left(V_{\parallel}^2 + \frac{1}{2} V_{\perp}^2 \right) \left(\frac{\vec{R} \times \vec{B}}{RB} \right)$$

numbers (example) from plasma physics

$$v_{\parallel}^2 + \frac{1}{2} v_{\perp}^2 \simeq \frac{kT}{m} \quad ; \quad \omega_B = \frac{eB}{\gamma m}$$

Jackson gives:

$$v_D \,[\mathrm{cm/s}] = \frac{172\; T\,[K]}{R\,[m]\; B\,[\mathrm{gauss}]}$$

$R = 1$ meter

$B = 10^3$ gauss $\Rightarrow v_D \simeq 2\times 10^3$ cm/s

$T = 10^4$ K (1 eV energy) $\simeq O(10)$ m/s

$$v_{\parallel}, v_{\perp} \simeq 100-1000 \text{ m/s @ } 10^4 K$$

$\therefore$ drift velocity is 1-10% of thermal velocity (in this curved $\vec{B}$ field geometry/example)