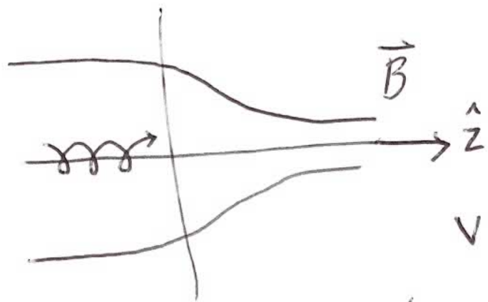
Adiabatic Invariance of Flux (previously $\vec{\nabla} B$ was $\perp$; now add $\parallel$

(no $\vec{E}$ field
static)

$$\frac{d\,\text{Energy}}{dt} = q\,\vec{E}\cdot\vec{v} = 0$$



then energy conserved

$$v^2 = v_{\parallel}^2 + v_{\perp}^2 = \text{constant}$$

if change is slow (many cycles over time for field to change)

there is an adiabatic invariant

flux threading particle orbit

$B a^2$

or $\omega_B a^2$ since $\omega_B = \dfrac{qB}{\gamma m}$ (γ is constant)

or $\dfrac{v_\perp^2}{\omega_B}$ or $\dfrac{v_\perp^2}{B}$ (from $v_\perp = a\,\omega_B$)

or $\dfrac{p_\perp^2}{B}$

see Jackson 12.5 for discussion of action integral $\oint p_i\, dq_i$ ← over complete cycle of q_i establishing adiabatic invariant

@ $z=0$, axial field B_0; $\vec{v}_{\perp 0}$; $v_{\parallel 0}$; $v_0^2 = v_{\parallel 0}^2 + v_{\perp 0}^2$

$$\frac{v_\perp^2}{B} = \frac{v_{\perp 0}^2}{B_0}$$

$$\therefore\ v_\parallel^2 = v_0^2 - v_\perp^2 = v_0^2 - \frac{v_{\perp 0}^2\, B(z)}{B_0}$$

$v_\parallel \to 0$ when ↗ $= 0$ { translational energy converted into rotational energy until $v_\parallel = 0$, then reflected

above determined using $\dfrac{v_\perp^2}{B}$ adiabatic invariance

show the same using Lorentz force

use cylindrical coordinates)

$\boxed{\vec{B} \text{ static so } \gamma \text{ is constant}}$

$$\vec{B} = \vec{B}(\rho, z)$$

along z-axis $\vec{B} = B(z)\,\hat{z}$

near z-axis $\vec{\nabla}\cdot\vec{B} = 0$ implies

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\, B_\rho(\rho, z)\right] = -\frac{\partial B}{\partial z}$$

$$\therefore\ B_\rho = -\frac{1}{2}\rho\frac{\partial B}{\partial z}$$

① $$\ddot{\rho} - \rho\dot{\phi}^2 = \frac{q}{\gamma m}(\vec{v}\times\vec{B})_\rho$$

② $$\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi} = \frac{q}{\gamma m}(\vec{v}\times\vec{B})_\phi$$

③ $$\ddot{z} = \frac{q}{\gamma m}(\vec{v}\times\vec{B})_z$$

0^{th} order: $\rho = a$ and ① gives $-\rho\dot{\phi}^2 = \frac{q}{\gamma m}\rho\dot{\phi} B_z$ or $\dot{\phi} = \frac{-qB}{\gamma m}$

$\boxed{\oplus\ \omega_B \text{ implies } \ominus\ \dot{\phi}}$

② gives $\rho^2\dot{\phi}$ const.

$$= -a^2\omega_{B,0} = \frac{-v_{\perp 0}^2}{\omega_{B,0}} \qquad \omega_{B,0} = \frac{qB}{\gamma m}$$

at 0^{th} order $\ddot{z} = 0$ but small $B_\rho \times v_\phi$ implies z force

$$\ddot{z} = \frac{q}{2\gamma m}\rho^2\dot{\phi}\frac{\partial B}{\partial z}$$

$$\simeq -\frac{v_{\perp 0}^2}{2}\frac{q}{\gamma m}\frac{1}{\omega_{B,0}}\frac{\partial B}{\partial z} = \frac{-v_{\perp 0}^2}{2B_0}\left(\frac{\partial B}{\partial z}\right)$$

using $$\frac{d\dot{z}}{dt} = k\,\frac{dB}{dz} = k\,\frac{dB}{dt}\cdot\frac{dt}{dz}$$

$$\therefore \int \dot{z}\,d\dot{z} = \int k\,dB$$

$$\frac{1}{2}\dot{z}^2 - \frac{1}{2}v_0^{\,2} = \frac{-v_{\perp 0}^{\,2}}{2B_0}\,B(z)$$

and thus $\dot{z}^2$

$$\boxed{v_\parallel^2 = v_0^{\,2} - \frac{v_{\perp 0}^{\,2}}{B_0}\,B(z)}$$