Lagrangian of Relativistic Charged Particles in external EM fields

Lorentz force $\boxed{\dfrac{du^\alpha}{d\tau} = \dfrac{q}{m} F^{\alpha\beta} u_\beta}$

$u^\alpha = (\gamma c, \gamma \vec{v})$

$u^\alpha = \dfrac{p^\alpha}{mc}$

note: my particle physics convention for $p^\alpha = (E, \vec{p}c)$

what about Lagrangian or Hamiltonian mechanics to deduce equations of motion?

$L[q_i(t), \dot{q}_i(t), t]$ generalized coordinates $q, \dot{q}$

$A = \int_{t_1}^{t_2} L \, dt \rightarrow$ path is an extremum for A (principle of least action)

$\delta A = 0$

leads to Euler-Lagrange equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

$\boxed{L = T - V}$ kinetic energy, potential $\Big\}$ non-relativistic only

~~what~~

want A to be Lorentz invariant scalar

change to $A = \int_{\tau_1}^{\tau_2} \gamma L \, d\tau$ proper time

$\therefore \gamma L$ must also be Lorentz invariant

$p_i = \dfrac{\partial L}{\partial \dot{q}_i}$ generalized momentum

when L does not depend on q_i then

$\dfrac{d}{dt}(p_i) = 0$ conservation of generalized momentum coordinate for free particle $\quad p_i = \gamma m v_i \Rightarrow$ want $\dfrac{\partial L_{free}}{\partial v_i} = \gamma m v_i$

L_{free} is the T part
kinetic energy not suitable for relativistic

if $\quad L_{free} = -\dfrac{mc^2}{\gamma} = -mc^2\sqrt{1-\beta^2} = -\dfrac{mc^2}{\gamma}$

then $\quad \dfrac{\partial L_{free}}{\partial v} = -\dfrac{1}{2}mc^2 \left(\dfrac{1}{\sqrt{1-\beta^2}}\right)\left(-\dfrac{2v}{c^2}\right) = \gamma m v$

and clear that $\gamma L = -mc^2$ is Lorentz invariant

now add the V part for EM

$V = e\Phi$ in non-relativistic limit

$L_{int} \rightarrow L_{int}^{NR} = -e\Phi$

for relativistic and Covariant formulation:

$$L_{int} = -\frac{e}{\gamma} U_\alpha A^\alpha \qquad \cancel{A^\alpha = (\Phi, \vec{A})}$$

$$A^\alpha = \left(\frac{\Phi}{c}, \vec{A}\right)$$

$$L_{int} = -e\Phi + e\,\vec{v}\cdot\vec{A} \qquad U_\alpha = (\gamma c, -\gamma\vec{v})$$

full Lagrangian relativistic charged particle in EM field is:

$$L = -\frac{mc^2}{\gamma} - e\Phi + e\,\vec{v}\cdot\vec{A}$$

$\cancel{\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}}\right) - \dfrac{\partial L}{\partial q} = 0}$

does this generate the correct equations of motion?

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

$$\frac{\partial L}{\partial q_i} = e\vec{\nabla}\left[\vec{v}\cdot\vec{A} - \phi\right] = e\left[(\vec{v}\cdot\vec{\nabla})\vec{A} + \vec{v}\times(\vec{\nabla}\times\vec{A})\right] - e\vec{\nabla}\phi$$

$$\vec{\nabla}(\vec{a}\cdot\vec{b}) = (\vec{a}\cdot\vec{\nabla})\vec{b} + (\vec{b}\cdot\vec{\nabla})\vec{a} + \vec{a}\times(\vec{\nabla}\times\vec{b}) + \vec{b}\times(\vec{\nabla}\times\vec{a})$$

and $\vec{v}$ is just ~~at~~ a vector, does not depend on q_i

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}}\right) = \frac{d}{dt}\left[\gamma m\vec{v} + \frac{\partial}{\partial\vec{v}}\left(e\vec{v}\cdot\vec{A}\right)\right] = \frac{d}{dt}(\gamma m\vec{v}) + e\frac{d\vec{A}}{dt}$$

note: $\dfrac{d\vec{A}}{dt} = \dfrac{\partial\vec{A}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{A}$ "convective derivative"

$$\therefore \frac{d}{dt}(\gamma m\vec{v}) = e\left[\underbrace{-\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}}_{\vec{E}} + \vec{v}\times\underbrace{(\vec{\nabla}\times\vec{A})}_{B}\right]$$

Lagrangian for charged particle in EM field has canonical momentum or conjugate momentum or generalized to position $\vec{x}$

$$\vec{P}_i = \frac{\partial L}{\partial v_i} = \gamma m v_i + e A_i = p_i + eA_i$$

$$\boxed{\vec{P} = \vec{p} + e\vec{A}}$$