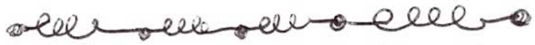
Lagrangian for the EM field - Classical Field Theory



q_i are displacements of masses from equil$^{\underline{m}}$

i label mass $\rightarrow$ then setup $L(q_i, \dot{q}_i, t)$ as normal

in continuum limit: $i \rightarrow x^\alpha$

$\phi(x^\alpha)$ are the displacements at each position x^α

$\hookrightarrow$ in general can be k fields in a problem $\phi_k(x^\alpha)$

$\dot{q}_i \rightarrow \dfrac{\partial \phi_k}{\partial x_\alpha} = \partial^\alpha \phi_k$ (including time derivative ... full 4-vector treatment!)

$A = \int L\, dt$ becomes $\int \mathcal{L}\, d^3x\, dt$ where $\mathcal{L}$ is Lagrangian density

$A = \int \mathcal{L}\, d^4x$ $\leftarrow$ Lorentz invariant

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad \longrightarrow \quad \boxed{\partial^\beta \frac{\partial \mathcal{L}}{\partial(\partial^\beta \phi_k)} - \frac{\partial \mathcal{L}}{\partial \phi_k} = 0}$$

$\mathcal{L}(\phi_k, \partial^\alpha \phi_k)$

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for EM field the "coordinates" are $A^\alpha(x^\alpha)$ $\leftarrow$ vector field

the "velocities" are $\partial^\beta A^\alpha(x^\alpha)$ $\leftarrow$ tensor field

Since $\mathcal{L}$ is Lorentz scalar, it must be quadratic in the velocities

e.g. $L_{free} = -\dfrac{mc}{\gamma}\sqrt{U_\alpha U^\alpha} = -\dfrac{mc^2}{\gamma}$

4-velocity

L depends on x^α, U^α

the $\mathcal{L}$ depends on $\partial^\beta A^\alpha$ or $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$

so expect $\mathcal{L}$ to depend on $F_{\alpha\beta} F^{\alpha\beta}$ $= 2(B^2 - E^2/c^2)$

The interaction term must involve source densities $J^\alpha(x^\alpha)$

then expect $J_\alpha A^\alpha$ (scalar) to appear in $\mathcal{L}$

The EM $\mathcal{L}$ is:

$$\boxed{\mathcal{L} = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{c} J_\alpha A^\alpha}$$ in Gaussian units

$$\boxed{\mathcal{L} = -\frac{1}{4\mu_0} F_{\alpha\beta} F^{\alpha\beta} - J_\alpha A^\alpha}$$ in SI units

$$\mathcal{L} = -\frac{1}{4\mu_0} g_{\mu\alpha} g_{\nu\beta} (\partial^\alpha A^\beta - \partial^\beta A^\alpha)(\partial^\mu A^\nu - \partial^\nu A^\mu) - J_\alpha A^\alpha$$

$\dfrac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)}$ has four terms $\quad +\dfrac{1}{4\mu_0} F_{\alpha\beta}$

$$g_{\mu\beta} g_{\nu\alpha} \, \partial^\beta A^\alpha \, F^{\mu\nu} = -\frac{1}{4\mu_0} F_{\beta\alpha} = +\frac{1}{4\mu_0} F_{\alpha\beta}$$

and so it goes with all four terms ...

$$\partial^\beta \left(\frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)} \right) = \partial^\beta \left(\frac{1}{\mu_0} F_{\alpha\beta} \right) \qquad \left| \frac{\partial \mathcal{L}}{\partial A^\alpha} = -J_\alpha \right.$$

how we wrote Maxwell's eqn

$$\partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta$$

$\Uparrow$ as a tensor equation before

$$\therefore \partial^\beta \left(\frac{1}{\mu_0} F_{\alpha\beta} \right) = -J_\alpha \implies \boxed{\partial^\beta \left(\frac{1}{\mu_0} F_{\beta\alpha} \right) = J_\alpha}$$

Homogeneous Maxwell's equations are handled too:

$$\tilde{\mathcal{F}}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}$$ defines dual tensor

$$\partial_\alpha \tilde{\mathcal{F}}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\,\partial_\alpha\left(\partial_\gamma A_\delta - \partial_\delta A_\gamma\right)$$

antisymmetric with interchange of two indices

$$= \underbrace{\epsilon^{\alpha\beta\gamma\delta}}_{\text{antisymmetric}}\,\underbrace{\partial_\alpha \partial_\gamma}_{\text{symmetric}} A_\delta$$

$$= 0$$

automatically satisfied!

$$\underbrace{\partial^\alpha \Big(\partial^\beta}_{\text{symmetric}} \underbrace{F_{\beta\alpha} \Big]}_{\text{antisymmetric}} = \partial^\alpha \Big[\mu_0 J_\alpha\Big]$$

$$= 0 = \mu_0 \partial^\alpha J_\alpha$$ continuity equation

Equations of motion for EM field absent J_α (no sources)

$$\partial^\beta F_{\beta\alpha} = 0 = \partial^\beta\left(\partial_\beta A_\alpha - \partial_\alpha A_\beta\right)$$

$$= \Box A_\alpha - \partial^\beta \partial_\alpha A_\beta = 0$$ in Lorenz gauge

The equations of motion is the wave equation.

with source current density J_α then this give inhomogeneous Maxwell's equations

Proca Lagrangian - add photon mass term to Lagrangian

$$\mathcal{L}_{Proca} = -\frac{1}{4\mu_0} F_{\alpha\beta} F^{\alpha\beta} + \frac{\mu^2}{\cancel{8\pi}\, 2\mu_0} A_\alpha A^\alpha - J_\alpha A^\alpha$$

A^α - vector field of EM is spin-1 photon (vector quantum field)

μ has dimensions inverse length (reciprocal Compton wavelength of photon)

$$\mu = \frac{m_\gamma c}{\hbar}$$

Equations of motion: $\partial^\beta F_{\beta\alpha} + \mu^2 A_\alpha = \mu_0 J_\alpha$

$$\partial^\beta(\partial_\beta A_\alpha - \partial_\alpha A_\beta) + \mu^2 A_\alpha = \mu_0 J_\alpha$$

$$\cancel{\Box A_\alpha - \partial^\beta \partial_\alpha A_\beta + \mu^2 A_\alpha = \mu_0 J_\alpha}$$

note: $\boxed{\mu_0 \partial^\alpha J_\alpha = \underbrace{\partial^\alpha \partial^\beta (\partial_\beta A_\alpha - \partial_\alpha A_\beta)}_{\text{zero}} + \mu^2 \partial^\alpha A_\alpha = 0}$

requires ← Lorenz gauge

$$\therefore \Box A_\alpha + \mu^2 A_\alpha = \mu_0 J_\alpha$$

$$(\Box + \mu^2) A_\alpha = \mu_0 J_\alpha$$ ← like Klein-Gordon equation with source term

photon rest mass

static limit: $\nabla^2 A_\alpha - \mu^2 A_\alpha = -\mu_0 J_\alpha$

for point charge at rest at origin: $J^\alpha = J_\alpha = (cq\,\delta^3(\vec{x}), \vec{0})$

$A^\alpha = \left(\frac{\Phi}{c}, \vec{A}\right)$

$$\left(\nabla^2 - \mu^2\right)\frac{\Phi}{c} = -\mu_0 c q\, \delta^3(\vec{x})$$

with $\mu \to 0$, this is $\boxed{\nabla^2 \Phi = -\frac{q}{\epsilon_0}\delta^3(\vec{x}) = -\frac{\rho(\vec{x})}{\epsilon_0}}$

Poisson equation

but with $\mu \neq 0$

recall $\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2} r$, we find $\boxed{\Phi(\vec{x}) = \frac{q}{4\pi\epsilon_0}\frac{e^{-\mu r}}{r}}$

Yukawa potential

$$\nabla^2 \Phi(\vec{x}) = \frac{\mu^2}{4\pi\epsilon_0} q \frac{e^{-\mu r}}{r}$$

and $\left(\nabla^2 - \mu^2\right)\frac{\Phi}{\cancel{c}} = -\frac{q}{\epsilon_0}\delta^3(\vec{x})$

↳ $= 0$ everywhere except at $r = 0$

↓ potential falls off with $e^{-\mu r}$

→ short range force with range $\sim 1/\mu$