Hamiltonian $\longrightarrow$ Canonical Stress-Energy Tensor
for a particle   for a field

$$\boxed{H = \int \mathcal{H}\, d^3x} \longrightarrow \text{should be like energy (time-like component)}$$

$\uparrow$ Hamiltonian density

$\int \mathcal{H}\, d^4x \rightarrow$ one time-derivative of this $\int \mathcal{H}\, d^4x$ } gives H

$\therefore$ $\mathcal{H}$ must be time-time (^{00} component) of rank-two tensor

$$H = \sum_i p_i \dot{q}_i - L \quad \Rightarrow \quad \mathcal{H} = \sum_k \underbrace{\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi_k}{\partial t}\right)}}_{\text{"canonical momentum of the field"}} \underbrace{\frac{\partial \phi_k}{\partial t}}_{\text{"velocities"}} - \mathcal{L}$$

Covariant generalization of $\mathcal{H}$

$$\boxed{T^{\alpha\beta} = \sum_k \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_k)} \partial^\beta \phi_k - g^{\alpha\beta}\mathcal{L}}$$

for

$$\mathcal{L}_{EM\text{-}free} = -\frac{1}{4\mu_0} F_{\mu\nu}F^{\mu\nu} = \frac{-1}{4\mu_0}\left[2\left(B^2 - \frac{E^2}{c^2}\right)\right] = -\frac{1}{2}\left[\frac{B^2}{\mu_0} - \epsilon_0 E^2\right]$$

$$\boxed{\mathcal{L}_{EM\text{-}free} = \frac{1}{2}\left[\epsilon_0 E^2 - \frac{B^2}{\mu_0}\right]}$$

then

$$T^{\alpha\beta} = -\frac{1}{\mu_0} g^{\alpha\mu} F_{\mu\lambda}\partial^\beta A^\lambda - g^{\alpha\beta}\mathcal{L}_{EM\text{-}free}$$

Jackson notes "deficiencies" with $T^{\alpha\beta}$
- not gauge invariant [involves potentials explicitly]
- trace is non-zero
- T^{00}, T^{0i} differ from "usual" expressions for energy and momentum densities
- $T^{0i} \neq T^{i0}$ (required for angular momentum conservation of the field)

$$T^{00} = \frac{1}{2\mu_0}(E^2 + B^2) + \frac{1}{\mu_0}\left(\vec{\nabla}\cdot \phi \frac{\vec{E}}{c^2}\right)$$

$$T^{0i} = \frac{1}{\mu_0}\left(\frac{\vec{E}}{c} \times \vec{B}\right)_i + \frac{1}{\mu_0}\vec{\nabla}\cdot\left(A_i \frac{\vec{E}}{c}\right)$$

$$T^{i0} = \frac{1}{\mu_0}\left(\frac{\vec{E}}{c} \times \vec{B}\right)_i + \frac{1}{\mu_0}\left[(\vec{\nabla} \times \Phi \vec{B})_i - \frac{\partial}{\partial x_0}\left(\frac{\phi E_i}{c^2}\right)\right]$$

$\vec{\nabla}\cdot$ terms integrate over all space to zero (transform into surface integrals at infinity where fields and potentials $\to 0$)

Construct $\Theta^{\alpha\beta}$ symmetric stress tensor that's gauge invariant

"Belinfante's method combined with Noether's" / described in Landau Lifshitz Classical Theory of Fields

$$\Theta^{\alpha\beta} = \frac{1}{\mu_0}\left[g^{\alpha\mu}F_{\mu\lambda}F^{\lambda\beta} + \frac{1}{4}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}\right]$$

$$\left[\begin{array}{c|c} \overset{\text{energy density}}{\frac{1}{2\mu_0}\left(\frac{E^2}{c^2}+B^2\right)} & \overset{\text{momentum density times } c}{\frac{1}{\mu_0}\left(\frac{\vec{E}}{c}\times\vec{B}\right)} \\ \hline \overset{\text{energy flux} \div \text{by } c}{\frac{1}{\mu_0}\left(\frac{\vec{E}}{c}\times\vec{B}\right)} & \overset{\text{stress}}{-\frac{1}{\mu_0}\left(\frac{E_iE_j}{c^2}+B_iB_j-\frac{1}{2}\delta_{ij}\left(\frac{E^2}{c^2}+B^2\right)\right)} \end{array}\right]$$

$$\partial_\alpha \Theta^{\alpha\beta} = \left\{\begin{array}{l} 0 \\ -F^{\beta\lambda}J_\lambda \end{array}\right.$$

$\to$ Poynting's theorem and conservation of momentum for free EM field

with sources

$\Theta^{\alpha\beta}$ α-component (row) of momentum in β direction (column)

00	01	02	03
10	11	12	13
20	21	22	23
30	31	32	33

$$\Theta^{00} = \frac{1}{2\mu_0}\left(\frac{E^2}{c^2}+B^2\right)$$

$$\Theta^{0i} = \frac{1}{4\pi}\left(\frac{\vec{E}}{c}\times\vec{B}\right)$$

$$\Theta^{ij} = -\frac{1}{\mu_0}\left[\frac{E_iE_j}{c^2}+B_iB_j-\frac{1}{2}\delta_{ij}\left(\frac{E^2}{c^2}+B^2\right)\right]$$

momentum density

$$\Theta^{\alpha\beta} = \left(\begin{array}{c|c} u & c\vec{g} \\ \hline c\vec{g} & -T_{ij}^{(M)} \end{array}\right)$$

Maxwell stress tensor

Poynting's theorem

rate of doing work (on point charge) by external EM field

$q\vec{v}\cdot\vec{E}$ ($\vec{B}$ does no work)

$\rightarrow \int_V \vec{J}\cdot\vec{E}\, d^3x$ is work done on continuous distribution of charge and current

must correspond to decrease in energy in EM field within V

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \rightarrow \text{express } \vec{J} \text{ as} \ldots$$

$$\int_V \left[\frac{\vec{E}}{\mu_0}\cdot(\vec{\nabla}\times\vec{B}) - \frac{\vec{E}}{\mu_0 c^2}\cdot\frac{\partial\vec{E}}{\partial t}\right] d^3x$$

$$\vec{\nabla}\cdot(\vec{E}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{B})$$ use Faraday's Law $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

$$\int_V \vec{J}\cdot\vec{E}\, d^3x = -\int_V \left[\vec{\nabla}\cdot\frac{(\vec{E}\times\vec{B})}{\mu_0} + \frac{\vec{B}}{\mu_0}\cdot\frac{\partial\vec{B}}{\partial t} + \epsilon_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}\right] d^3x$$

$$\boxed{\text{total energy density of EM field} \quad u = \frac{1}{2}\left(\vec{E}\cdot\epsilon_0\vec{E} + \vec{B}\cdot\frac{\vec{B}}{\mu_0}\right)}$$

$$-\int_V \vec{J}\cdot\vec{E}\, d^3x = \int_V \left[\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\left(\vec{E}\times\frac{\vec{B}}{\mu_0}\right)\right]$$

$$\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} = -\vec{J}\cdot\vec{E}$$ with $\vec{S} \equiv \vec{E}\times\frac{\vec{B}}{\mu_0}$

$\frac{\partial u}{\partial t}$: rate of change energy in the fields

$\vec{\nabla}\cdot\vec{S}$: energy flowing out boundary surfaces

$-\vec{J}\cdot\vec{E}$: work done by the fields on sources within volume

Poynting vector $\frac{\text{energy}}{\text{area}\cdot\text{time}} \rightarrow$ energy flux

(energy conservation)

momentum conservation Poynting vector

$$\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

$$\frac{d\vec{P}_{mech}}{dt} = \int_V \left(\rho\vec{E} + \vec{J}\times\vec{B}\right) d^3x \qquad \rho = \epsilon_0 \vec{\nabla}\cdot\vec{E}$$

$$\vec{J} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \epsilon_0 \frac{\partial\vec{E}}{\partial t}$$

$$\epsilon_0\left[\vec{E}(\vec{\nabla}\cdot\vec{E}) + \vec{B}\times\frac{\partial\vec{E}}{\partial t} - c^2\vec{B}\times(\vec{\nabla}\times\vec{B})\right]$$

write $\vec{B}\times\frac{\partial\vec{E}}{\partial t} = -\frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right) + \vec{E}\times\frac{\partial\vec{B}}{\partial t}$

add $c^2\vec{B}(\underbrace{\vec{\nabla}\cdot\vec{B}}_{=0})$ to square bracket

$$\rho\vec{E} + \vec{J}\times\vec{B} = \epsilon_0\Big[\vec{E}(\vec{\nabla}\cdot\vec{E}) + c^2\vec{B}(\vec{\nabla}\cdot\vec{B})$$
$$- \vec{E}\times\underbrace{(\vec{\nabla}\times\vec{E})}_{\text{Faraday}} - c^2\vec{B}\times(\vec{\nabla}\times\vec{B})\Big] - \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

$$\frac{d\vec{P}_{mech}}{dt} + \frac{d}{dt}\boxed{\int_V \epsilon_0(\vec{E}\wedge\vec{B})\,d^3x}^{\vec{P}_{field}} =$$

$$\epsilon_0\int_V\left[\vec{E}(\vec{\nabla}\cdot\vec{E}) - \vec{E}\times(\vec{\nabla}\times\vec{E}) + c^2\vec{B}(\vec{\nabla}\cdot\vec{B}) - c^2\vec{B}\times(\vec{\nabla}\times\vec{B})\right]d^3x$$

$$\vec{P}_{field} = \frac{1}{c^2}\frac{1}{\mu_0}(\vec{E}\times\vec{B}) \equiv \vec{g}$$

momentum density

$$\left[\vec{E}(\vec{\nabla}\cdot\vec{E}) - \vec{E}\times(\vec{\nabla}\times\vec{E})\right]_i = \sum_j \frac{\partial}{\partial x_j}\left(E_iE_j - \frac{1}{2}\vec{E}\cdot\vec{E}\,\delta_{ij}\right)$$

then define Maxwell stress tensor

$$T_{ij}^{(M)} = \epsilon_0 \left[E_i E_j + c^2 B_i B_j - \frac{1}{2}\left(\vec{E}\cdot\vec{E} + c^2 \vec{B}\cdot\vec{B}\right)\delta_{ij} \right]$$

$$\frac{d}{dt}\left(\vec{P}_{mech} + \vec{g}\right)_i = \sum_j \int_V \frac{\partial}{\partial x_j} T_{ij} \, d^3x$$

divergence theorem

$$= \oint_S \sum_j T_{ij} n_j \, da$$

outward normal vector

j^{th} component of momentum flow per unit area across the S $\frac{\text{into}}{\text{out of}}$ volume

Recap: symmetric stress tensor for EM field

$$\Theta^{\alpha\beta} = \frac{1}{\mu_0}\left[g^{\alpha\mu}F_{\mu\lambda}F^{\lambda\beta} + \frac{1}{4}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}\right]$$

now, take $\partial_\alpha \Theta^{\alpha\beta} = \frac{1}{\mu_0}\left[\partial^\mu\left(F_{\mu\lambda}F^{\lambda\beta}\right) + \frac{1}{4}\partial^\beta\left(F_{\mu\nu}F^{\mu\nu}\right)\right]$

$$= \frac{1}{\mu_0}\left[\underbrace{\left(\partial^\mu F_{\mu\lambda}\right)}_{=\mu_0 J_\lambda}F^{\lambda\beta} + F_{\mu\lambda}\partial^\mu F^{\lambda\beta} + \frac{1}{2}F_{\mu\nu}\left(\partial^\beta F^{\mu\nu}\right)\right]$$

$$\boxed{\partial_\alpha \Theta^{\alpha\beta} + F^{\beta\lambda}J_\lambda} = \frac{1}{2\mu_0}F_{\mu\lambda}\left(\partial^\mu F^{\lambda\beta} + \underbrace{\partial^\mu F^{\lambda\beta} + \partial^\beta F^{\mu\lambda}}_{\substack{-\partial^\lambda F^{\beta\mu} \\ = +\partial^\lambda F^{\mu\beta}}}\right)$$

use $\partial^\mu F^{\lambda\beta} + \partial^\beta F^{\mu\lambda} + \partial^\lambda F^{\beta\mu} = 0$

(follows from $\partial_\alpha \mathcal{F}^{\alpha\beta} = 0$)

$$= \frac{1}{2\mu_0}\underbrace{F_{\mu\lambda}}_{\text{antisymm.}}\underbrace{\left(\partial^\mu F^{\lambda\beta} + \partial^\lambda F^{\mu\beta}\right)}_{\text{symm.}} = 0$$

$$\therefore \boxed{\partial_\alpha \Theta^{\alpha\beta} = -F^{\beta\lambda}J_\lambda}$$

did not cover in class

recall potential energy $W_i = q_i \Phi(\vec{x}_i)$; consider Φ produced by $n-1$ charges q_j

$$W = \sum_{i=1}^{n} q_i \sum_{j<i}^{n-1} \frac{q_j}{|\vec{x}_i - \vec{x}_j|}\left(\frac{1}{4\pi\epsilon_0}\right)$$

then system of n charges

$$W = \frac{1}{2}\sum_i \sum_j \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|}\left(\frac{1}{4\pi\epsilon_0}\right)$$ where $i=j$ terms are omitted

for continuous charge distribution

$$W = \frac{1}{2}\iint \frac{\rho(\vec{x})\rho(\vec{x}')}{|\vec{x} - \vec{x}'|(4\pi\epsilon_0)} d^3x\, d^3x'$$

$$= \frac{1}{2}\int \rho(\vec{x})\Phi(\vec{x})\, d^3x \qquad \nabla^2\Phi = -\frac{\rho}{\epsilon_0}$$

$$= -\frac{1}{2}\int \epsilon_0 \Phi \nabla^2 \Phi\, d^3x$$

$$= \frac{1}{2}\int \epsilon_0 |\vec{\nabla}\Phi|^2 d^3x = \frac{1}{2}\epsilon_0 \int |\vec{E}|^2 d^3x$$

integration by parts with Φ and $\vec{\nabla}\Phi$ evaluates $\Phi \to 0$ at infinity

energy density $\frac{1}{2}\epsilon_0 |\vec{E}|^2$

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inductor potential energy

$$U = \frac{1}{2} L I^2 \qquad L = \frac{\mu_0 N^2 A}{l}$$

EMF from $\frac{di}{dt}$

$$U = \frac{1}{2}\mu_0 \frac{N^2 A}{l} I^2$$

$$\boxed{B = \mu \frac{N}{l} I} = \frac{1}{2}\frac{\mu_0 N^2 A}{l}\frac{B^2 l^2}{\mu_0^2 N^2}$$

$$= \frac{1}{2}\frac{B^2}{\mu_0} A l$$

capacitor $U = \frac{1}{2}CV^2$

$$E = \frac{V}{d}, \quad C = \frac{\epsilon_0 A}{d}$$

$$U = \frac{1}{2}\frac{\epsilon_0 A}{d} E^2 d^2$$

$$U = \underbrace{\frac{1}{2}\epsilon_0 E^2}(Ad)$$

energy density in $\vec{E}$ field in capacitor "energy derived" "Hamiltonian derived" Equations of "motion"

$$\partial_\alpha \Theta^{\alpha\beta} = -F^{\beta\lambda} J_\lambda$$

energy and momentum conservation

time-component: $\dfrac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} = -\vec{J}\cdot\vec{E}$

space components: $\dfrac{\partial g_i}{\partial t} - \sum_{j=1}^{3} \dfrac{\partial}{\partial x_j} T_{ij}^{(M)} = -\left[\rho E_i + (\vec{J}\times\vec{B})_i\right]$

EM field interacting with sources

can define 4-vector $f^\beta \equiv F^{\beta\lambda} J_\lambda = \left(\vec{J}\cdot\vec{E},\ \rho\vec{E} + (\vec{J}\times\vec{B})\right)$

Lorentz force density

recall Lorentz force on charge q moving with U_β: $q F^{\alpha\beta} u_\beta$