$$\Box A^\mu = \mu_0 J^\mu \qquad \partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta$$

$\left(\text{Lorenz gauge: } \partial_\alpha A^\alpha = 0\right)$

sol$^{\underline{n}}$ $$\Phi(\vec{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\rho(\vec{x}', t_r)}{|\vec{x}-\vec{x}'|}$$

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{J}(\vec{x}', t_r)}{|\vec{x}-\vec{x}'|}$$

where $t_r = t - \dfrac{|\vec{x}-\vec{x}'|}{c}$ is the retarded time

$$A^\mu = \left(\frac{\Phi}{c}, \vec{A}\right) \qquad J^\mu = \left(c\rho, \vec{J}\right)$$

Covariant Wave Equation and Green's Functions

$$\Box_x D(x,x') = \delta^{(4)}(x-x') = \delta(x^0 - x^{0'})\,\delta(\vec{x}-\vec{x}')$$

x - spacetime coordinate of observer

x' - coordinate of source

$D(x,x')$ response @ x to idealized source (δ-fn) @ x'

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recall :

$L\, u(x) = f(x)$

↑ differential operator; $u(x)$: something; $f(x)$: source term

$$\boxed{\int L\, G(x,s) f(s)\, ds} = \int \delta(x-s) f(s)\, ds = f(x) = \boxed{L\, u(x)}$$

$L\, G(x,x) = \delta(x-s)$ L acts on x alone (not on s variable being integrated over)

what is $u(x)$? $L\, u(x) = L \int G(x,s) f(s)\, ds$

$u(x) = \int G(x,s) f(s)\, ds$

$$\nabla^2 \Phi(\vec{x}) = -\frac{\rho(\vec{x})}{\epsilon_0} \qquad \nabla^2 G(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}')$$

$$\Phi(\vec{x}) = \int d^3x' \, G(\vec{x},\vec{x}')\,\rho(\vec{x}')\left(\frac{1}{4\pi\epsilon_0}\right)$$

$$\nabla^2 G(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}') \qquad \therefore\; G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|}$$

$$\Box A^\mu = \mu_0 J^\mu \qquad \Box_x D(x,x') = \delta^{(4)}(x-x')$$

then $$\boxed{A^\mu(x) = \mu_0 \int D(x,x')\, J^\mu(x')\, d^4x'}$$

boundary surfaces – assume none (fields fall off at infinity)

$D(x, x')$ depends only on $x - x'$

$z^\alpha \equiv x^\alpha - x^{\alpha'}$ (and since always talking about 4-vectors, drop the index)

$$z = x - x'$$

$$\Box_z D(z) = \delta^4(z)$$

Fourier analysis

$$D(z) = \frac{1}{(2\pi)^4} \int d^4k \, e^{-ik\cdot z} \tilde{D}(k)$$

$\tilde{D}(k)$ – Fourier transform of Green function

$$\boxed{\delta^{(4)}(z) = \frac{1}{(2\pi)^4} \int d^4k \, e^{-ik\cdot z}}$$

standard Fourier transform delta function

$$\boxed{\tilde{D}(k) = \int d^4z \, e^{ik\cdot z} D(z)}$$

$$\Box_z D_z = \left(\frac{1}{2\pi}\right)^4 \left[\int d^4k \, \tilde{D}(k) e^{-ik\cdot z} (-k^2)\right]$$

$$\Box_z D_z = \delta^{(4)}(z) = \left(\frac{1}{2\pi}\right)^4 \int d^4k \, e^{-ik\cdot z}$$

$$\therefore \tilde{D}(k) = -\frac{1}{k^2}$$

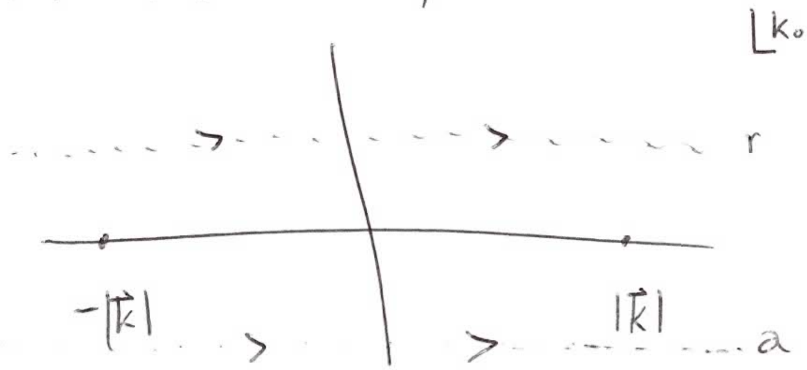
$$\therefore D(z) = -\frac{1}{(2\pi)^4} \int d^4k \, \frac{e^{-ik\cdot z}}{(k\cdot k)}$$

note 4-vector scalar product

$$D(z) = -\frac{1}{(2\pi)^4} \int d^3k \, e^{i\vec{k}\cdot\vec{z}} \int_{-\infty}^{\infty} dk_0 \frac{e^{-ik_0 z_0}}{[k_0^2 - |\vec{k}|^2]}$$

use complex analysis

treat k_0 as complex



$$\int_{-\infty}^{\infty} dk_0 \, \frac{e^{-ik_0 z_0}}{k_0^2 - |\vec{k}|^2} = \oint dk_0 \, \frac{e^{-ik_0 z_0}}{k_0^2 - |\vec{k}|^2}$$

if we can close contour where integrand vanishes

$$= -2\pi i \, \mathrm{Res}\left(\frac{e^{-ik_0 z_0}}{k_0^2 - |\vec{k}|^2} \right)$$

$f(z) = \frac{g(z)}{h(z)}$

$\mathrm{Res}(f, c) = \frac{g(c)}{h'(c)}$

$e^{-ik_0 z_0}$ increases in upper half-plane $\rightarrow$ close contour on the lower half plane

for $z_0 > 0$, contour r

$$= -\frac{2\pi}{|\vec{k}|} \sin(|\vec{k}| z_0) \qquad = -2\pi i \left(\frac{e^{i|\vec{k}| z_0}}{-2|\vec{k}|} + \frac{e^{-i|\vec{k}| z_0}}{2|\vec{k}|} \right)$$

for $z_0 < 0$, integral vanishes (close contour in upper-half plane encloses no singularities)

$$D_r(z) = \frac{\Theta(z_0)}{(2\pi)^3} \int d^3k \, e^{i\vec{k}\cdot\vec{z}} \, \frac{\sin(|\vec{k}| z_0)}{|\vec{k}|}$$

Θ: Heaviside step function

$R = |\vec{z}|$

$R = |\vec{x} - \vec{x}'|$

$d^3k = 2\pi |\vec{k}|^2 dk \, d(\cos\theta)$

$\vec{k}\cdot\vec{z} = kR\cos\theta$

$$= \frac{\Theta(z_0)}{2\pi^2 R} \int_0^{\infty} d|\vec{k}| \, \sin(|\vec{k}| R) \sin(|\vec{k}| z_0)$$

$$= \frac{\Theta(z_0)}{8\pi^2 R} \int_{-\infty}^{\infty} d|\vec{k}| \left[e^{+i(z_0 - R)|\vec{k}|} - e^{+i(z_0 + R)|\vec{k}|} \right]$$

integrals are δ-fns; second $= 0$ because $z_0 > 0$, $R > 0$; first: δ-fn

$$\boxed{D_r(x - x') = \frac{\Theta(x_0 - x_0')}{4\pi R} \, \delta(x_0 - x_0' - R)}$$

source time $x_0' = ct_0'$

observation " $x_0 = ct_0$

t_0' earlier than t_0

→ retarded Green's function $D_r(x-x')$

similarly for $z_0 < 0$, take contour a

t_0' later than t_0

and $\boxed{D_a(x,x') = \dfrac{\theta[-(x_0-x_0')]}{4\pi R}\,\delta(x_0-x_0'+R)}$

advanced Green's fn

note: $\delta[(x-x')^2] = \delta[(x_0-x_0')^2 - (\vec{x}-\vec{x}')^2]$

4-vector scalar product

$= \delta[(x_0-x_0'-R)(x_0-x_0'+R)]$

$= \dfrac{1}{2R}\left[\delta(x_0-x_0'-R) + \delta(x_0-x_0'+R)\right]$

$\delta(g(x)) = \sum_i \dfrac{\delta(x-x_i^0)}{|g'(x_i^0)|}$

$\delta(x^2-a^2) = \dfrac{1}{2|a|}[\delta(x+a)+\delta(x-a)]$

and θ Heaviside selects then

Lorentz invariant

$D_r(x-x') = \dfrac{1}{2\pi}\,\theta(x_0-x_0')\,\delta[(x-x')^2]$

$D_a(x-x') = \dfrac{1}{2\pi}\,\theta((x_0'-x_0))\,\delta[(x-x')^2]$

θ doesn't look it ... but is with δ-fns.

$A^\alpha_{in}(x)$ are solution to source-free equation $A^\alpha_{out}(x)$

$\Box A^\alpha = 0$

incident potential @ $t=-\infty$

$$A^\alpha(x) = A^\alpha_{in}(x) + \mu_0 \int d^4x'\, D_r(x,x')\, J^\alpha(x')$$

@ $t=+\infty$ outgoing potential $= A^\alpha_{out}(x) + \mu_0 \int d^4x'\, D_a(x,x')\, J^\alpha(x')$

radiation fields difference between out and in

$$A^\alpha_{rad}(x) = A^\alpha_{out} - A^\alpha_{in} = \mu_0 \int d^4x' \, D(x, x') \, J^\alpha(x')$$

$$D(z) = D_r(z) - D_a(z)$$

---

how to use: $\rho(\vec{x}, t) = e \, \delta^{(3)}[\vec{x} - \vec{r}(t)]$

$\vec{J}(\vec{x}, t) = e \, \vec{v}(t) \, \delta^{(3)}[\vec{x} - \vec{r}(t)]$

Radiation from point charge in motion

point charge e with position $\vec{r}(t)$, $\vec{v}(t) = \dfrac{d\vec{r}(t)}{dt}$

4-vector

$$J^\alpha(x^\alpha) = ec \int d\tau \, U^\alpha(\tau) \, \delta^{(4)}[x - r(\tau)]$$

$r^\alpha = (ct, \vec{r}(t))$ $\quad U^\alpha = (\gamma c, \gamma \vec{v})$

make 4-vector

e.g. $\rho_{(x^\alpha)} = e \int dt' \, \delta(t - t') \, \delta^{(3)}[\vec{x} - \vec{r}(t')]$

$r^\alpha = (ct', \vec{r})$ $\quad \delta(t - t') = c \, \delta(x^0 - r^0)$

$$\rho = ec \int dt' \, \delta^{(4)}(x - r)$$

$$= ec \int \frac{dt'}{d\tau} d\tau \, \delta^{(4)}(x - r)$$

$u^0 = \left(c\dfrac{dt}{d\tau}, \dfrac{d\vec{r}}{d\tau}\right)$

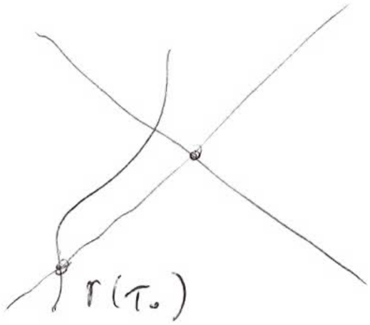
$$J^0 = c\rho = ec \int d\tau \, u^0 \, \delta^{(4)}(x - r)$$

$$A^\alpha(x^\alpha) = \mu_0 \int d^4x' \, \frac{1}{2\pi} \theta(x_0 - x_0') \, \delta[(x - x')^2] \, ec \int d\tau \, u^\alpha(\tau) \, \delta^4[x' - r(\tau)]$$

$$= \frac{\mu_0}{2\pi} ec \int d\tau \, u^\alpha(\tau) \, \theta(x_0 - r_0(\tau)) \, \delta[(x - r(\tau))^2]$$

light cone condition $(x - r(\tau_0))^2 = 0$

imposed by δ-fn



defines τ_0 - retarded time

$$\delta\left[(x - r(\tau))^2\right] = \frac{\delta(\tau - \tau_0)}{\left| df/d\tau \right|_{\tau=\tau_0}}$$

$$f = (x - r(\tau))^2$$

$$\frac{df}{d\tau} = -2\,(x - r(\tau))_\alpha \frac{dr^\alpha}{d\tau}$$

$$\therefore A^\alpha = \left(\frac{\mu_0 c}{4\pi}\right) \frac{e\, u^\alpha(\tau)}{(x - r(\tau)) \cdot u}\Bigg|_{\tau=\tau_0}$$

manifestly 4-vector

Liénard-Wiechert potential

$$u \cdot (x - r(\tau_0)) = u_0 (x_0 - r_0(\tau_0)) - \vec{u}\cdot(\vec{x} - \vec{r}(\tau_0))$$

$$= \gamma_c R - \gamma \vec{v} \cdot \hat{n} R$$

where $\vec{x} - \vec{r} = R\hat{n}$ and $x_0 - r_0(\tau_0) = R$

$$\therefore u \cdot (x - r) = \gamma_c R(1 - \vec{\beta} \cdot \hat{n})$$

$$\Phi(\vec{x}, t) = \left(\frac{1}{4\pi\epsilon_0}\right) \frac{e}{(1 - \vec{\beta}\cdot\hat{n})R}\Bigg|_{ret}$$

evaluate at retarded time τ_0 given by $x_0 - R = r_0(\tau_0)$

$$\vec{A}(\vec{x}, t) = \left(\frac{\mu_0 c}{4\pi}\right) \frac{e\,\vec{\beta}}{(1 - \vec{\beta}\cdot\hat{n})R}\Bigg|_{ret}$$