recap

$$A^\alpha(x) = \mu_0 \int d^4x' \, D_r(x, x') \, J^\alpha(x')$$

4-vectors
x observer
x' over all sources

$$D_r(x, x') = \frac{1}{2\pi} \Theta(x_0 - x_0') \, \delta\left[(x - x')^2\right]$$

$$J^\alpha(x') = ec \int d\tau \, u^\alpha(\tau) \, \delta^{(4)}[x' - r(\tau)]$$

point charge in arbitrary motion

$r^\alpha = (ct, \vec{r}(t))$

$u^\alpha = (\gamma c, \gamma \vec{v}(t))$

$$A^\alpha(x) = \frac{\mu_0}{2\pi} ec \int d\tau \, u^\alpha(\tau) \, \Theta(x_0 - r_0(\tau)) \, \delta\left[(x - r(\tau))^2\right]$$

$$\boxed{A^\alpha(x) = \left(\frac{\mu_0 c}{4\pi}\right) \frac{e \, u^\alpha(\tau)}{(x - r(\tau)) \cdot u} \Bigg|_{\tau = \tau_0}}$$

evaluated at τ_0 retarded time

Liénard-Wiechert potential

$$u \cdot (x - r(\tau_0)) = u_0(x_0 - r_0(\tau_0)) - \vec{u} \cdot (\vec{x} - \vec{r}(\tau_0))$$
$$= \gamma c R - \gamma \vec{v} \cdot \hat{n} R$$

$\vec{x} - \vec{r}(\tau_0) = R\hat{n}$
$x_0 - r_0(\tau_0) = R$

$$\therefore u \cdot (x - r(\tau_0)) = \gamma c R (1 - \vec{\beta} \cdot \hat{n})$$

find $\vec{E}$ and $\vec{B}$ or $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$

$$\partial^\alpha A^\beta = \frac{\mu_0}{2\pi} ec\, \partial^\alpha \int d\tau\, u^\beta(\tau)\, \Theta(x_0 - r_0(\tau))\, \delta\left[(x - r(\tau))^2\right]$$

∂^α acts on x_0 and x

$$\partial^0 \Theta(x_0 - r_0(\tau)) = \delta(x_0 - r_0(\tau))$$

but $\delta(x_0 - r_0(\tau))\,\delta\left[(x - r(\tau))^2\right]$ corresponds to $R=0$

moving charge particle on the observer

which we ignore

$$\therefore \partial^\alpha A^\beta = \frac{\mu_0}{2\pi} ec \int d\tau\, u^\beta(\tau)\, \Theta(x_0 - r_0(\tau))\, \partial^\alpha \delta\left[(x - r(\tau))^2\right]$$

$\partial^\alpha \delta(f)$ where $f = (x-r)^2$

$$= \partial^\alpha f \frac{d}{df}\delta(f) = \partial^\alpha f \frac{d\tau}{df}\frac{d}{d\tau}\delta(f)$$

$$\partial^\alpha f = 2(x-r)^\alpha \qquad \frac{df}{d\tau} = -2u\cdot(x-r)$$

So $$\partial^\alpha A^\beta = -\frac{\mu_0}{2\pi} ec \int d\tau\, \Theta(x_0 - r_0)\, \frac{u^\beta (x-r)^\alpha}{u\cdot(x-r)} \frac{d}{d\tau}\left[\delta(x-r)^2\right]$$

integration by parts

$$= \frac{\mu_0}{2\pi} ec \int d\tau\, \Theta(x_0 - r_0)\, \delta\left[(x-r)^2\right] \frac{d}{d\tau}\left[\frac{(x-r)^\alpha u^\beta}{u\cdot(x-r)}\right]$$

use $$\delta\left[(x-r)^2\right] = \frac{\delta(\tau - \tau_0)}{2u\cdot(x-r)}$$ as before

$$\therefore \quad \partial^\alpha A^\beta = \frac{\mu_0 e c}{4\pi} \frac{1}{u\cdot(x-r)} \frac{d}{d\tau}\left[\frac{(x-r)^\alpha u^\beta}{u\cdot(x-r)}\right]$$

$\tau = \tau_0$ retarded time

$$F^{\alpha\beta} = \frac{\mu_0 e c}{4\pi} \frac{1}{u\cdot(x-r)} \frac{d}{d\tau}\left[\frac{(x-r)^\alpha u^\beta - (x-r)^\beta u^\alpha}{u\cdot(x-r)}\right]_{\tau=\tau_0}$$

$$(x-r)^\alpha = \left(x_0 - r_0(\tau), \vec{x}-\vec{r}\right)$$

$$= \left(R, R\hat{n}\right)$$

$$u^\alpha = \left(\gamma c, \gamma c\vec{\beta}\right)$$

$$\frac{du^\alpha}{d\tau} = \left(c\frac{d\gamma}{d\tau}, c\left(\frac{d\gamma}{d\tau}\vec{\beta} + \gamma\frac{d\vec{\beta}}{d\tau}\right)\right)$$

acceleration $\dot{\vec{\beta}} = \frac{d\vec{\beta}}{dt}$

$$\frac{d\gamma}{d\tau} = \frac{dt}{d\tau}\frac{d\gamma}{dt} = \gamma\frac{d}{dt}\left(\frac{1}{\sqrt{1-\beta^2}}\right) = \gamma\frac{\vec{\beta}\cdot\dot{\vec{\beta}}}{(1-\beta^2)^{3/2}} = \gamma^4\vec{\beta}\cdot\dot{\vec{\beta}}$$

$$\frac{d}{d\tau}\left(u\cdot(x-r)\right) = (x-r)^\alpha\frac{du_\alpha}{d\tau} - \frac{dr^\alpha}{d\tau}u_\alpha$$

$\hat{n}$ determined at retarded

$$= (x-r)\cdot\frac{du}{d\tau} - c^2$$

$$\boxed{\vec{B} = \frac{1}{c}\left(\hat{n}\times\vec{E}\right)_{ret}}$$

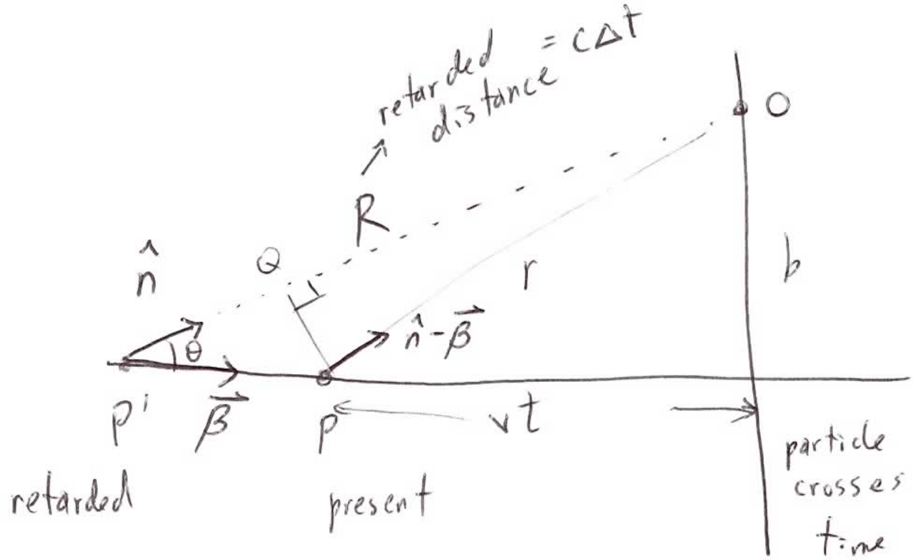
$$\vec{E}(\vec{r},t) = \frac{e}{4\pi\epsilon_0}\left[\left(\frac{\hat{n}-\vec{\beta}}{\gamma^2(1-\vec{\beta}\cdot\hat{n})^3R^2}\right)_{ret} + \frac{1}{c}\left(\frac{\hat{n}\times\{(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\}}{(1-\vec{\beta}\cdot\hat{n})^3 R}\right)_{ret}\right]$$

depends on velocity, goes as $\frac{1}{R^2}$
Coulomb electrostatic-like
near-field

depends on $\dot{\vec{\beta}}$
goes as $\frac{1}{R}$, $\frac{1}{c}$
radiation

Particle in uniform, constant $\vec{\beta}$ motion

$$\vec{E} = \frac{e}{4\pi\epsilon_0}\left(\frac{\hat{n}-\vec{\beta}}{\gamma^2(1-\vec{\beta}\cdot\hat{n})^3R^2}\right)_{ret}$$



$r^2 = b^2 + (vt)^2$

$$P'P = v\Delta t = \beta R$$

$$P'Q = \beta R\cos\theta = (\vec{\beta}\cdot\hat{n})R$$

$$[QO]^2 = \left[1-(\vec{\beta}\cdot\hat{n})R\right]^2 \quad \text{denominator}$$

$$= r^2 - (\beta R\sin\theta)^2 \qquad \sin\theta = \frac{b}{R}$$

$$= b^2 + (vt)^2 - \beta^2 b^2$$

$$= (1-\beta^2)b^2 + v^2t^2 = \frac{1}{\gamma^2}(b^2+\gamma^2v^2t^2)$$

$$\therefore E_y = \frac{e}{4\pi\epsilon_0}\left.\frac{\hat{n}\cdot\hat{y}}{(\gamma^2(1-\vec{\beta}\cdot\hat{n})^3R^2}\right|_{ret} = \frac{e}{4\pi\epsilon_0}\left.\frac{\hat{n}\cdot\hat{y}\;R\gamma}{\gamma^3(1-\vec{\beta}\cdot\hat{n})^3R^2}\right|_{ret}$$

$$= \frac{e}{4\pi\epsilon_0}\frac{\gamma\,R\sin\theta}{(b^2+\gamma^2v^2t^2)^{3/2}} = \frac{e}{4\pi\epsilon_0}\frac{\gamma b}{(b^2+\gamma^2v^2t^2)^{3/2}}$$

as before!

in x-direction

$$E_x = \frac{e}{4\pi\epsilon_0}\left(\frac{\hat{n}\cdot\hat{x} - \vec{\beta}}{\gamma^2(1-\vec{\beta}\cdot\hat{n})^3R^2}\right)_{ret} = \frac{e\gamma}{4\pi\epsilon_0 R}\frac{1}{(b^2+\gamma^2v^2t^2)^{3/2}}(R\cos\theta - \beta R)$$

$$= \frac{e\gamma v t}{(b^2+\gamma^2v^2t^2)^{3/2}}$$ as before

$$\vec{E} = \frac{e}{4\pi\epsilon_0}\frac{\gamma}{(b^2+\gamma^2v^2t^2)^{3/2}}(vt, b, 0)$$

points to P
not P'

Jackson notes: [for particle in uniform motion u^α is constant]

$$F^{\alpha\beta} = \frac{\mu_0 e c^2}{4\pi[u\cdot(x-r)]^3}\cdot\left[(x-r)^\alpha u^\beta - (x-r)^\beta u^\alpha\right]$$

Can write $\vec{E}$ from this too