$$P = \left[\frac{e^2}{6\pi\epsilon_0 c^3 m^2}\right]\left(\frac{d\vec{p}}{d\tau}\cdot\frac{d\vec{p}}{d\tau} - \beta^2\left(\frac{dp}{d\tau}\right)^2\right)$$

using $E = \gamma m c^2$, $\vec{p} = \gamma m \vec{v}$

$$\left[\frac{e^2}{6\pi\epsilon_0 c}\gamma^6\right]\left[(\dot{\vec{\beta}})^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2\right]$$

Liénard result

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examples linac

$$P = \left[\frac{e^2}{6\pi\epsilon_0 c^3 m^2}\right]\left[\underbrace{\frac{d\vec{p}}{d\tau}\cdot\frac{d\vec{p}}{d\tau}}_{=\left(\frac{dp}{d\tau}\right)^2} - \beta^2\left(\frac{dp}{d\tau}\right)^2\right]$$

$$= \left[\frac{e^2}{6\pi\epsilon_0 c^3 m^2}\right]\left(\frac{dp}{d\tau}\right)^2(1-\beta^2)$$

$$1-\beta^2 = \frac{1}{\gamma^2} = \left(\frac{d\tau}{dt}\right)^2$$

$$P = \left[\frac{e^2}{6\pi\epsilon_0 c^3 m^2}\right]\left(\frac{dp}{dt}\right)^2 = \left[\frac{e^2}{6\pi\epsilon_0 c^3 m^2}\right]\left(\frac{dE}{dx}\right)^2$$

$$\frac{P}{(dE/dt)} = \left[\frac{e^2}{6\pi\epsilon_0 c^3 m^2}\right]\frac{1}{v}\frac{dE}{dx}$$

take $v = c$

$$\frac{e^2}{4\pi\epsilon_0(m_e c^2)} = 2.8\times10^{-15}\ \text{m}$$

$$[\ \ ] = \frac{4}{6}(2.8\times10^{-15})\frac{1}{mc}$$

$$\frac{P}{(dE/dt)} = \frac{1.88\times10^{-15}\ \text{m}}{0.511\ \text{MeV}}\left(\frac{dE}{dx}\right)$$

SLAC $\Delta E = 50$ GeV

$\Delta x = 3$ km

for SLAC

$$\frac{P}{(dE/dt)} = 6.1\times10^{-14}$$

tiny amount of radiated power loss in linac circular accelerators

$\vec{p}$ changes direction but $|\vec{p}|/_{\text{revolution}}$ small

$$\vec{p} = |\vec{p}|\,(\cos\omega t\ \hat{x} \pm \sin\omega t\ \hat{y})$$

$$\frac{d\vec{p}}{d\tau} = \gamma\omega\,|\vec{p}|\,(-\sin\omega t\ \hat{x} \pm \cos\omega t\ \hat{y})$$

$$\frac{d\vec{p}}{d\tau}\cdot\frac{d\vec{p}}{d\tau} = \gamma^2\omega^2|\vec{p}|^2 \gg \frac{1}{c}\frac{dE}{d\tau}$$

$$P = \left[\frac{e^2}{6\pi\epsilon_0 c^3 m^2}\right]\left[\gamma^2\omega^2|\vec{p}|^2\right] = \left[\frac{e^2}{6\pi\epsilon_0 c^3 m^2}\right]\left[\gamma^2\frac{v^2}{R^2}\gamma^2 m^2 v^2\right]$$

use $\omega = \dfrac{v}{R}$

$$\boxed{P = \frac{e^2}{6\pi\epsilon_0}\frac{c\,\beta^4\gamma^4}{R^2}}$$

$$\Delta E \text{ per revolution} = P\,\Delta t = P\left(\frac{2\pi R}{\beta c}\right)$$

$$= \frac{e^2}{6\pi\epsilon_0}\frac{2\pi\beta^3\gamma^4}{R} = \frac{e^2}{3\epsilon_0}\frac{\beta^3\gamma^4}{R}$$

4th power of E

-1 power of R

$$= \frac{4}{6}(m_e c^2)\,2\pi\frac{\beta^3\gamma^4}{R}\,r_e$$ ← classical electron radius

LEP electrons 100 GeV on 100 GeV (104.5 GeV to be precise)

$R = 4.3$ km (circumference $= 27$ km) $\quad \underline{\gamma = 204{,}500}$

$\Delta E \simeq 2.4$ GeV per revolution <u>substantial</u> !

what about 1 TeV in the same ring (electrons)? 10^4 more <u>radiated power</u>

or 24 TeV $\quad$ ∴ ILC or μ collider ring

7 TeV protons on the other hand in the same ring

LHC $\quad \gamma = 7{,}500$ and $\Delta E = 4.4$ keV per revolution

no problem!

Angular Distribution of Radiation from Accelerated Charge

$$\left(\vec{S}\cdot\hat{n}\right)_{ret} = \frac{1}{\mu_0}\left(\vec{E}_a \times \vec{B}_a\right)_{ret}$$

$\vec{B} = (\hat{n}\times\vec{E})/c$

$\therefore \vec{E}\times\vec{B} = \frac{|\vec{E}|^2}{c}\hat{n}$

$$= \frac{1}{\mu_0 c}\left|\vec{E}_a\right|^2$$

$$\left(\vec{S}\cdot\hat{n}\right)_{ret} = \frac{1}{\mu_0 c}\left\{\left(\frac{1}{4\pi\epsilon_0}\right)\frac{e}{c}\cancel{\left[\frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{R}\right]_{ret}} \left[\frac{\hat{n}\times\{(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\}}{(1-\vec{\beta}\cdot\hat{n})^3 R}\right]_{ret}\right\}^2$$

$\hat{n}$ determined at retarded time

$\frac{dE}{dA\,dt}$ energy per unit area per unit time detected at some $\vec{x}, t$ from radiation emitted by accel. charge at $t' = t - \frac{R}{c}$

E radiated between $t' = T_1$ to $t' = T_2$

$$\frac{dE}{dA} = \int_{T_1}^{T_2} (\vec{S}\cdot\hat{n})\frac{dt}{dt'}\,dt'$$

$$\frac{dt}{dt'} = 1 + \frac{1}{c}\frac{d}{dt'}R(t')$$

$$R = |\vec{x} - \vec{r}(t')|$$

$$\frac{dR}{dt'} = -\frac{(\vec{x}-\vec{r}(t'))\cdot\frac{d\vec{r}}{dt}}{|\vec{x}-\vec{r}(t')|} = -\hat{n}\cdot\vec{\beta}$$

$$\frac{dP(t')}{d\Omega} = R^2(\vec{S}\cdot\hat{n})\frac{dt}{dt'} = R^2(\vec{S}\cdot\hat{n})(1-\vec{\beta}\cdot\hat{n})$$

$$\boxed{\therefore \frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c}\left(\frac{1}{4\pi\epsilon_0}\right)\frac{\left|\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]\right|^2}{(1-\vec{\beta}\cdot\hat{n})^5}}$$

Example linear motion $\vec{\beta} \parallel \dot{\vec{\beta}}$ $\qquad \vec{\beta}\cdot\hat{n} = \beta\cos\theta$

θ angle between observation and motion

$$\hat{n}\times\left[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\right]$$
$$= \hat{n}\times(\hat{n}\times\dot{\vec{\beta}}) = \hat{n}(\hat{n}\cdot\dot{\vec{\beta}}) - \dot{\vec{\beta}}$$

$\cancel{\dot{\vec{\beta}}}$

• Observer

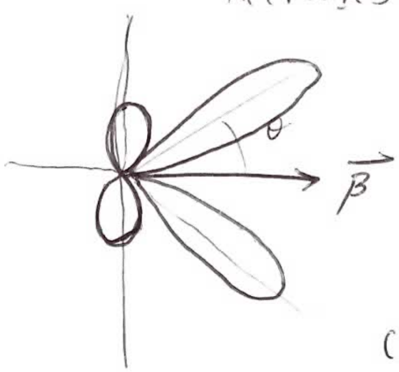
$\vec{\beta}, \dot{\vec{\beta}}$ direction

retarded time position

$$\boxed{\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c}\left(\frac{1}{4\pi\epsilon_0}\right)\frac{\dot{\beta}^2\sin^2\theta}{(1-\beta\cos\theta)^5}}$$

for $\beta \ll 1$, Larmor result $\sin^2\theta$ angular distribution

for $\beta \to 1$ angular distribution is tipped forward and increases in magnitude



Solve for θ_{max} i.e. angle with max $\frac{dP(t')}{d\Omega}$

by setting derivative w.r.t. θ or $\cos\theta$ $= 0$

$$\cos\theta_{max} = \frac{1}{3\beta}\left[(1+15\beta^2)^{1/2} - 1\right]$$

as $\beta \to 0$ $\quad \cos\theta_{max} = 0 \quad \therefore \theta_{max} = \pi/2$

as $\beta \to 1$ $\quad \boxed{\theta_{max} = \frac{1}{2\gamma}}$ relativistic particles θ_{max} very small!

peak intensity $\propto \gamma^8$!

$$\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c}\left(\frac{1}{4\pi\epsilon_0}\right)\frac{\dot{\beta}^2\,\theta^2}{\left(1-\left(1-\frac{1}{2\gamma^2}\right)\left(1-\frac{\theta^2}{2}\right)\right)^5}$$

$\cancel{\gamma\left(1+\frac{\theta^2}{4\gamma^2}\right)}$

$\theta \ll 1$

$\sin\theta \simeq \theta$

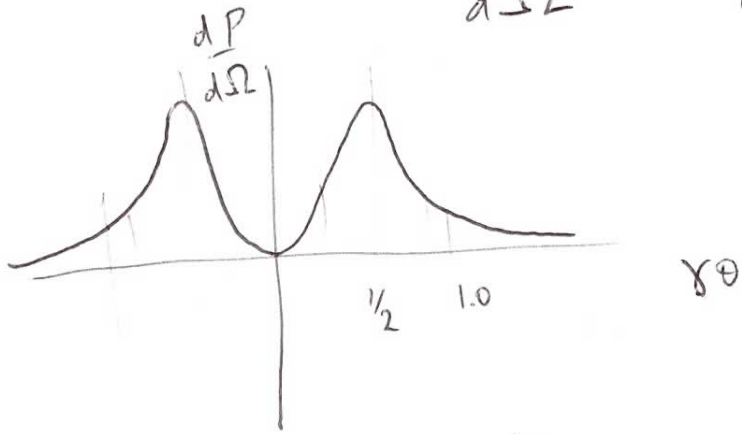
$\cos\theta \simeq 1 - \frac{\theta^2}{2}$

$\cancel{\gamma \simeq 1 - \frac{1}{2}\beta^2}$

$\gamma^2 = (1-\beta^2)^{-1}$

$\gamma^{-2} = 1-\beta^2$

for small angles $$\frac{dP(t')}{d\Omega} \simeq \frac{8}{\pi}\left(\frac{1}{4\pi\epsilon_0}\right)\frac{e^2\dot{\beta}^2}{c}\gamma^8\frac{(\gamma\theta)^2}{(1+\gamma^2\theta^2)^5}$$



peak @ $\gamma\theta = 1/2$

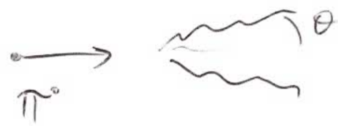
$\therefore \theta_{max} = \frac{1}{2\gamma}$

$1/2$ power @ $\gamma\theta = 0.23$ and 0.91

$$\langle\theta^2\rangle^{1/2} = \frac{1}{\gamma} = \frac{mc^2}{E}$$

e.g. $\pi^0 \to \gamma\gamma$

π^0 rest frame



$E_\pi = 2E_\gamma$

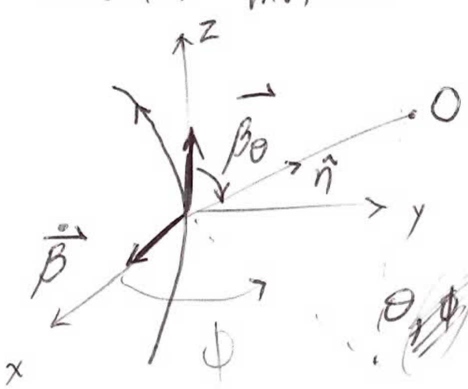
$p_\pi = 2p_\gamma\cos\theta$

$$\frac{cp_\pi}{E_\pi} = \beta_\pi = \frac{\cancel{2}cp_\gamma\cos\theta}{\cancel{2E_\gamma}}$$

for $\gamma \gg 1$, $$\beta \simeq 1 - \frac{1}{2\gamma^2} = \cos\theta = 1 - \frac{\theta^2}{2}$$

$$\Rightarrow \theta \simeq \frac{1}{\gamma}$$

Circular motion in xz plane



$$\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c}\left(\frac{1}{4\pi\epsilon_0}\right)\frac{\dot{\beta}^2}{(1-\beta\cos\theta)^3}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right]$$

θ, ϕ angles between $\vec{\beta}$ and $\hat{n}$

forward peaked b/c $1-\beta\cos\theta$ in denominator

can find P by $\int d\Omega$ or from $$P = \frac{e^2}{6\pi\epsilon_0 c}\gamma^6\left[(\dot{\vec{\beta}})^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2\right]$$

$$P = \frac{e^2}{6\pi\epsilon_0}\frac{|\dot{\vec{\beta}}|^2\gamma^4}{c}$$