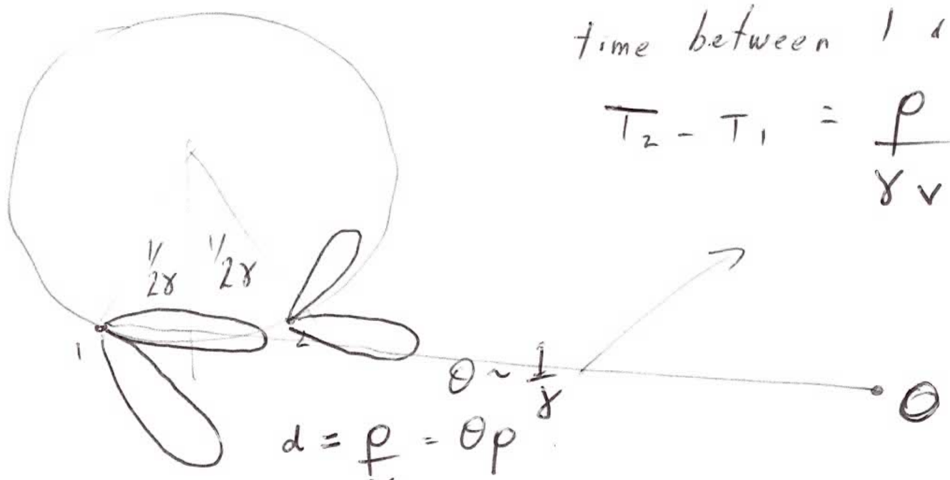
particle in extreme relativistic motion

radiation dominated by acceleration $\perp$ to velocity




$d = \frac{\rho}{\gamma} = \theta\rho$

time between 1 & 2

$$T_2 - T_1 = \frac{\rho}{\gamma v}$$

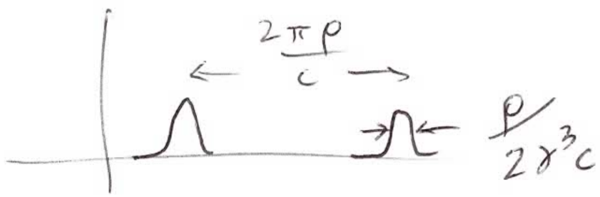
$\rho$ radius of curvature

$t_2 - t_1$ @ observer

$$= \underbrace{\frac{\rho}{\gamma v}}_{\text{moving particle}} - \underbrace{\frac{\rho}{\gamma c}}_{\text{moving EM radiation}}$$

$$t_2 - t_1 = \frac{\rho}{c\gamma}\left(\frac{c}{v} - 1\right)$$

$$= \frac{\rho}{2\gamma^3 c}$$




pulse train $\rightarrow$ Fourier

appreciable frequency components up to

$$\omega_c \simeq \frac{c}{\rho}\gamma^3 \leftarrow \text{critical frequency} \quad \left(\omega_c = \frac{3}{2}\frac{c}{\rho}\gamma^3\right)$$

$\omega_0 \simeq \frac{c}{\rho}$

fundamental frequency

e.g. Cornell light source synchrotron

10 GeV $\gamma_{max} \simeq 2 \times 10^4$

$\omega_0 = 3 \times 10^6\ s^{-1} \rightarrow \omega_c = 2.4 \times 10^{19}\ s^{-1}$ or 16 keV X-rays

$$\frac{dP(t)}{d\Omega} = |\vec{A}(t)|^2$$

P(t) - power in terms of observer's clock

where $$\vec{A}(t) = \left(\frac{1}{\mu_0 c}\right)^{1/2} \left(R\,\vec{E}_a\right)_{ret}$$

$\vec{A}$ is <u>not</u> the vector potential

$\vec{E}$ is the $\vec{E}$ field (at observer, generated @ retarded time)

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} |\vec{A}(t)|^2\, dt$$

is total energy radiated per unit solid angle

$$\tilde{\vec{A}}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cancel{|}\vec{A}(t)\cancel{|^2}\, e^{i\omega t}\, dt$$

$$\vec{A}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\vec{A}}(\omega)\, e^{-i\omega t}\, d\omega$$

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} |\tilde{\vec{A}}(\omega)|^2\, d\omega$$

Parseval's theorem

$$= 2\int_0^{\infty} |\tilde{\vec{A}}(\omega)|^2\, d\omega$$

if $\vec{A}(t)$ is real $\quad \tilde{\vec{A}}(-\omega) = \tilde{\vec{A}}^*(\omega)$

$$\frac{d^2 I}{d\omega\, d\Omega} = 2\,|\tilde{\vec{A}}(\omega)|^2$$

intensity per solid angle per unit frequency

$$\vec{A}(t) = \left(\frac{1}{\mu_0 c}\right)^{1/2} \left(\frac{e}{4\pi\epsilon_0 c}\right) \left[\frac{\hat{n} \times [(\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}]}{(1 - \vec{\beta}\cdot\hat{n})^3}\right]_{ret}$$

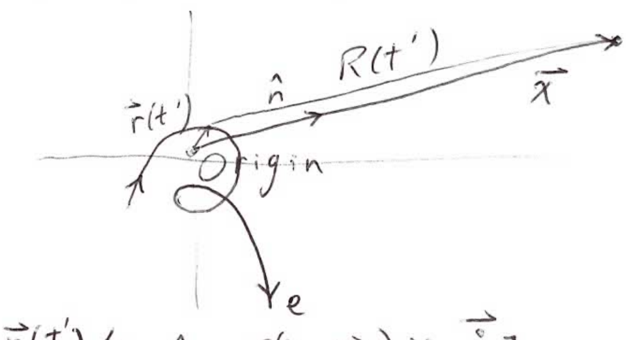
$$= \left(\frac{e^2}{\mu_0 c^3}\right)^{1/2} \left(\frac{1}{4\pi\epsilon_0}\right) \left[\qquad\qquad\right]_{ret}$$

$$\tilde{\vec{A}}(\omega) = \left(\frac{e^2}{2\pi\mu_0 c^3}\right)^{1/2} \left(\frac{1}{4\pi\epsilon_0}\right) \int_{-\infty}^{\infty} e^{i\omega t} \left[\qquad\qquad\right]_{ret} dt$$

$$\tilde{\vec{A}}(\omega) = \left(\frac{e^2}{2\pi\mu_0 c^3}\right)^{1/2}\left(\frac{1}{4\pi\epsilon_0}\right)\int_{-\infty}^{\infty} e^{i\omega\left(t' + R(t')/c\right)} \frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^{2}}\, dt'$$

observer is assumed far away from spacetime event where acceleration occurs ; unit vector $\hat{n} \approx$ const. in time

$$R(t') \simeq x - \hat{n}\cdot\vec{r}(t')$$




$$\tilde{\vec{A}}(\omega) = \left(\quad\right)^{1/2}\left(\frac{1}{4\pi\epsilon_0}\right)\int_{-\infty}^{\infty} e^{i\omega\left(t' - \hat{n}\cdot\vec{r}(t')/c\right)} \frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^{2}}\, dt'$$

$$\frac{d^2 I}{d\omega\, d\Omega} = \left(\frac{e^2}{\pi\mu_0 c^3}\right)\left(\frac{1}{4\pi\epsilon_0}\right)^2 \left|\int_{-\infty}^{\infty} e^{i\omega\left(t' - \hat{n}\cdot\vec{r}(t')/c\right)} \frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^{2}}\, dt'\right|^2$$

$$\boxed{\frac{d^2 I}{d\omega\, d\Omega} = \frac{e^2}{4\pi^2 c}\left(\frac{1}{4\pi\epsilon_0}\right)\left|\qquad\qquad\right|^2}$$

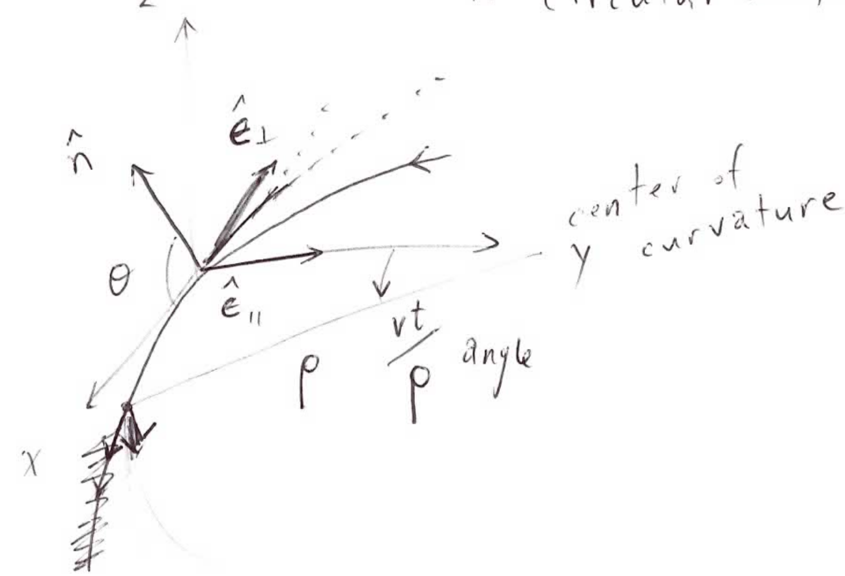
from Jackson $\displaystyle \frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^{2}} = \frac{d}{dt}\left[\frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{1-\vec{\beta}\cdot\hat{n}}\right]$

integration by parts

$$\frac{d^2 I}{d\omega\, d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left(\frac{1}{4\pi\epsilon_0}\right)\left|\int_{-\infty}^{\infty} \hat{n}\times(\hat{n}\times\vec{\beta})\, e^{i\omega\left(t - \hat{n}\cdot\vec{r}(t)/c\right)}\, dt\right|^2$$

Spectrum from Relativistic charged particle in circular motion

$\rho$ radius of curvature




observer in xz-plane

$$\hat{n} = \cos\theta\,\hat{x} + \sin\theta\,\hat{z}$$

$$\hat{\epsilon}_{\parallel} = \hat{y} \qquad \hat{\epsilon}_{\perp} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{z}$$

$$\vec{\beta} = \beta\left[\cos\left(\frac{vt}{\rho}\right)\hat{x} + \sin\left(\frac{vt}{\rho}\right)\hat{y}\right]$$

$$\vec{r}(t) = \rho\left[\sin\frac{vt}{\rho}\,\hat{x} - \cos\left(\frac{vt}{\rho}\right)\hat{y}\right]$$

$$\boxed{\hat{n}\times(\hat{n}\times\vec{\beta}) = \beta\left[-\hat{\epsilon}_{\parallel}\sin\left(\frac{vt}{\rho}\right) + \hat{\epsilon}_{\perp}\cos\left(\frac{vt}{\rho}\right)\sin\theta\right]}$$

$$\omega\left(t - \frac{\hat{n}\cdot\vec{r}(t)}{c}\right) = \omega\left[t - \frac{\rho}{c}\sin\left(\frac{vt}{\rho}\right)\cos\theta\right]$$

for small angle, short times, $\beta \to 1$ $\quad \beta \simeq 1 - \frac{1}{2\gamma^2}$

$$t - \frac{\rho}{c}\cos\theta\sin\left(\frac{vt}{\rho}\right) \simeq t - \frac{\rho}{c}\left(1 - \frac{\theta^2}{2}\right)\left(\frac{vt}{\rho} - \frac{1}{6}\left(\frac{vt}{\rho}\right)^3\right)$$

$$= t\left[1 - \left(1 - \frac{\theta^2}{2}\right)\left(\frac{v}{c} - \frac{1}{6}\frac{v^3 t^2}{c\rho^2}\right)\right] \qquad \beta \to 1$$

$$= t\left[1 - \left(1 - \frac{\theta^2}{2}\right)\left(1 - \frac{1}{2\gamma^2} - \frac{c^2t^2}{6\rho^2}\right)\right]$$

$$= t\left[\frac{\theta^2}{2} + \frac{1}{2\gamma^2} + \frac{c^2t^2}{6\rho^2}\right]$$

$$\frac{d^2 I}{d\omega\, d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left(\frac{1}{4\pi\epsilon_0}\right)\left|\int_{-\infty}^{\infty} dt' \left(\beta\theta\,\hat{\epsilon}_\perp - \beta\frac{vt'}{\rho}\,\hat{\epsilon}_\parallel\right) e^{i\omega(t' - \hat{n}\cdot\frac{\vec{x}(t')}{c})}\right|$$

$$= (\quad)(\quad)\left|\int_{-\infty}^{\infty} dt' \left(\beta\theta\,\hat{\epsilon}_\perp - \beta\frac{vt'}{\rho}\,\hat{\epsilon}_\parallel\right) e^{i\omega t'\left[\frac{\theta^2}{2} + \frac{1}{2\gamma^2} + \frac{c^2 t'^2}{6\rho^2}\right]}\right|^2$$

$\omega_0 = \frac{c}{\rho}$, $v \simeq c$, introduce $\xi = \frac{\omega\rho}{3c}\left(\frac{1}{\gamma^2} + \theta^2\right)^{3/2} = \frac{\omega}{3\omega_0}\left(\frac{1}{\gamma^2} + \theta^2\right)^{3/2}$

introduce $x \equiv \omega_0 t' / \left(\frac{1}{\gamma^2} + \theta^2\right)^{1/2}$

$$\frac{d^2 I}{d\omega\, d\Omega} = (\quad)(\quad)\frac{1}{\omega_0^2}\left|\left(\frac{1}{\gamma^2} + \theta^2\right)^{1/2}\theta\, I_\perp\,\hat{\epsilon}_\perp + \left(\frac{1}{\gamma^2} + \theta^2\right) I_\parallel\,\hat{\epsilon}_\parallel\right|^2$$

where $I_\perp = \int_{-\infty}^{\infty} dt\, e^{i(3\xi t + \xi t^3)/2}$

$I_\parallel = \int_{-\infty}^{\infty} dt\, t e^{i(3\xi t + \xi t^3)/2}$

these are Airy integrals which are modified Bessel fns of order 1/3 and 2/3

$$\frac{\pi}{(3a)^{1/3}} Ai\left[\frac{x}{(3a)^{1/3}}\right] = \int_0^{\infty} dt\, \cos(xt + at^3) = \frac{1}{2}\int_{-\infty}^{\infty} dt\, e^{i(xt + at^3)}$$

$$I_\perp = \frac{2\pi}{(3\xi/2)^{1/3}} Ai\left[\left(\frac{3\xi}{2}\right)^{2/3}\right] \; ; \; I_\parallel = \frac{2\pi}{i(3\xi/2)^{2/3}} Ai'\left[\left(\frac{3\xi}{2}\right)^{2/3}\right]$$

$$Ai\left[(3\xi/2)^{2/3}\right] = \frac{1}{\pi}\left[\frac{(3\xi/2)^{2/3}}{3}\right]^{1/2} K_{1/3}(\xi)$$

$$Ai'\left[(3\xi/2)^{2/3}\right] = \frac{1}{\pi}\frac{(3\xi/2)^{2/3}}{\sqrt{3}} K_{2/3}(\xi)$$

$$K_\nu(\xi) = \begin{cases} \dfrac{\Gamma(\nu)}{2}\left(\dfrac{2}{\xi}\right)^{\nu} & \xi \ll 1 \\ \left(\dfrac{\pi}{2\xi}\right)^{1/2} e^{-\xi} & \xi \gg 1 \end{cases}$$ modified Bessel functions

$$\frac{d^2 I}{d\omega\,\Omega} = \left(\frac{e^2}{4\pi^2 c}\right)\left(\frac{1}{4\pi\epsilon_0}\right)\left(\frac{\omega}{\omega_0}\right)^2 \left| \left(\frac{1}{\gamma^2}+\theta^2\right)^{1/2}\theta\,\frac{2}{\sqrt{3}}K_{1/3}(\xi)\,\hat{\epsilon}_\perp + \left(\frac{1}{\gamma^2}+\theta^2\right)\frac{2}{i\sqrt{3}}K_{2/3}(\xi)\,\hat{\epsilon}_\parallel \right|^2$$

$$= \left(\frac{e^2}{3\pi^2 c}\right)\left(\frac{1}{4\pi\epsilon_0}\right)\left(\frac{\omega}{\omega_0}\right)^2\left(\frac{1}{\gamma^2}+\theta^2\right)^2\left[K^2_{2/3}(\xi) + \frac{\theta^2}{\left(\frac{1}{\gamma^2}+\theta^2\right)}K^2_{1/3}(\xi)\right]$$

polarized in plane of orbit

polarized $\perp$ to plane

"We now proceed to examine this somewhat complex result"

$K_\nu(\xi)$ is small for $\xi \gg 1$ $\qquad \xi = \dfrac{\omega\rho}{3c}\left(\dfrac{1}{\gamma^2}+\theta^2\right)^{3/2}$

for $\xi = 1/2$, $\theta = 0$ $\qquad \omega_c = \dfrac{3}{2}\gamma^3\dfrac{c}{\rho} = \dfrac{3}{2}\gamma^3\omega_0$

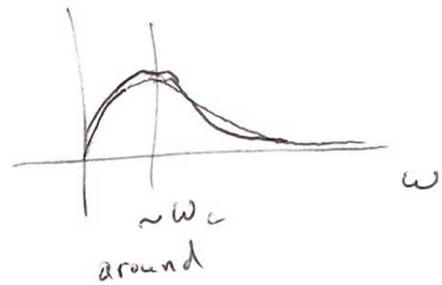
@ large $\theta$, negligible radiation
@ large $\omega > \omega_c$, " "

for $\omega \ll \omega_c$

$$\left.\frac{d^2 I}{d\omega\, d\Omega}\right|_{\theta=0} = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{e^2}{c}\left[\frac{\Gamma(2/3)}{\pi}\right]^2\left(\frac{3}{4}\right)^{1/3}\left(\frac{\omega}{\omega_0}\right)^{2/3}$$

for $\omega >> \omega_c$

$$\left.\frac{d^2 I}{d\omega\, d\Omega}\right|_{\theta=0} = \left(\frac{1}{4\pi\epsilon_0}\right) \frac{3}{4\pi} \frac{e^2}{c} \gamma^2 \frac{\omega}{\omega_c} e^{-\omega/\omega_c}$$




$$\cancel{\frac{dI}{d\omega} = \int d\Omega}$$

In low frequency range

$\omega << \omega_c$

$$\theta_c \simeq \left(\frac{3c}{\omega\rho}\right)^{1/3} = \frac{1}{\gamma}\left(\frac{2\omega_c}{\omega}\right)^{1/3}$$

low frequency emitted at wider angles than $\langle \theta^2 \rangle^{1/2} \sim \frac{1}{\gamma}$

at high frequencies

$$\theta_c \simeq \frac{1}{\gamma}\left(\frac{2\omega_c}{\omega}\right)^{1/2}$$

emitted much smaller angle than average

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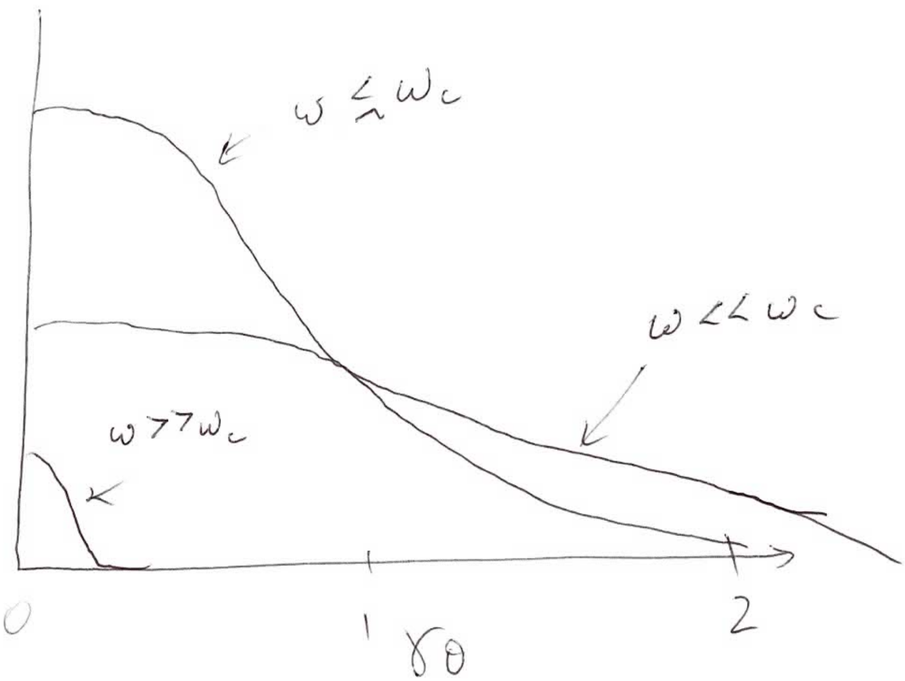
$$\frac{dI}{d\omega} = \int d\Omega \simeq 2\pi \int_{-\infty}^{\infty} \frac{d^2 I}{d\omega\, d\Omega}\, d\theta$$

$$\simeq 2\pi\, \theta_c \left.\frac{d^2 I}{d\omega\, d\Omega}\right|_{\theta=0} \sim \begin{cases} \dfrac{e^2}{c}\left(\dfrac{\omega}{\omega_0}\right)^{1/3} & \text{for } \omega << \omega_c \\ \simeq \sqrt{\dfrac{3\pi}{2}} \left(\dfrac{e^2}{c}\right) \gamma \left(\dfrac{\omega}{\omega_c}\right)^{1/2} e^{-\omega/\omega_c} & \text{for } \omega >> \omega_c \end{cases}$$

proper integration gives

$$\frac{dI}{d\omega} = \left(\frac{1}{4\pi\epsilon_0}\right)\sqrt{3}\, \frac{e^2}{c} \gamma \left(\frac{\omega}{\omega_c}\right) \int_{\omega/\omega_c}^{\infty} K_{5/3}(x)\, dx$$

don't worry about this




e.g. synchrotron radiation in astrophysics

- galaxies radio $\nu < 30$ GHz
- galactic plane synchrotron from electrons in the galactic magnetic field
- radio emission from Jupiter - electrons spiralling in magnetic field
- pulsar wind nebula (Crab) electrons up to $10^{13}$ eV from radio to X-ray $B \sim 10^{-4}$ gauss

Can calculate energy loss due to synchrotron, deduce electrons would have radiated away energy in less time than age of Crab $\therefore$ source of energetic electrons being injected (pulsar wind)