Charged Particle Energy Loss

- typical treatment non-relativistic (for α particles)

charge ze, mass M, $E = \gamma M c^2$, $P = \gamma \beta c M$

collides with electron, charge $-e$, mass m

- ignore binding energy of electron in atom

Rutherford scattering formula

$$\frac{d\sigma}{d\Omega} = \left(\frac{ze^2}{8\pi\epsilon_0 p v}\right)^2 \operatorname{cosec}^4\left(\frac{\theta}{2}\right),$$

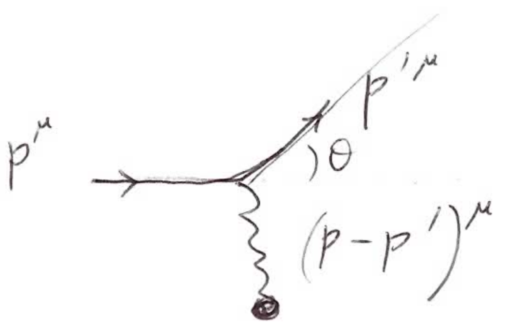
where $p = \gamma\beta c m$ momentum of electron in rest frame of heavy particle

$$\operatorname{cosec}^4\frac{\theta}{2} = \frac{1}{\sin^4(\theta/2)}$$

in Lorentz-invariant form Q is 4-momentum transfer

$$Q^2 \equiv -(p-p')^2$$

↑ to make quantity positive
→ it is the <u>momentum</u> transfer squared



heavy (in rest frame of heavy)

$$p^\mu = (\gamma m c, \gamma m c\beta, 0, 0)$$

$$p'^\mu = (\gamma m c, \gamma m c\beta\cos\theta, \gamma m c\beta\sin\theta, 0)$$

$$(p-p')^2 = -\left\{\left[\gamma m c\beta(1-\cos\theta)\right]^2 + \left[\gamma m c\beta\sin\theta\right]^2\right\}$$

$$Q^2 = -(p-p')^2 = (\gamma m c\beta)^2(\cos^2\theta + 1 - 2\cos\theta + \sin^2\theta)$$

$$Q^2 = p^2(2-2\cos\theta) = 2p^2(1-\cos\theta) = 4p^2\sin^2(\theta/2)$$

$$\frac{d\sigma}{dQ^2} = \int_0^{2\pi} \frac{d\sigma}{d\Omega} \sin\theta\, d\theta\, d\phi \quad 2\pi \frac{d\sigma}{d\Omega} \frac{\sin\theta\, d\theta}{dQ^2}$$

$$\frac{dQ^2}{d\theta} = 4p^2 \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = 2p^2\sin\theta$$

$$\therefore \frac{\sin\theta\, d\theta}{dQ^2} = \frac{1}{2p^2}$$

$$\therefore \frac{d\sigma}{dQ^2} = 2\pi\left(\frac{ze^2}{8\pi\epsilon_0 pv}\right)^2 \frac{1}{\sin^4\theta/2}\frac{1}{2p^2}$$

$$p^4\sin^4(\theta/2) = \frac{Q^4}{16}$$

$$\therefore \boxed{\frac{d\sigma}{dQ^2} = 4\pi\left(\frac{ze^2}{4\pi\epsilon_0 \beta c\, Q^2}\right)^2}$$

Q^2 in the rest frame of electron $\quad p^\mu = (mc, \vec{0})$

$p'^\mu = (\gamma mc, \gamma m\vec{v})$

$$Q^2 = -(p-p')^2$$
$$= -\left((\gamma-1)^2 - \gamma^2\beta^2\right)(m^2c^2)$$
$$= -(\gamma^2+1-2\gamma-\gamma^2\beta^2)m^2c^2 = -(\gamma^2(1-\beta^2)+1-2\gamma)m^2c^2$$
$$= (2\gamma-2)m^2c^2 = 2(\gamma-1)mc^2\, m = 2mT$$

where T is kinetic energy gained by electron

$T = (\gamma-1)mc^2$

= k.e. loss by heavy particle

$$\therefore \frac{d\sigma}{dT} = \frac{4\cancel{2}\pi z^2e^4}{(4\pi\epsilon_0)^2\beta^2c^2(4m^2T^2)} \; 2m \Rightarrow \boxed{\frac{d\sigma}{dT} = \frac{2\pi z^2e^4}{(4\pi\epsilon_0)^2\beta^2 mc^2 T^2}}$$

$\frac{d\sigma}{dT}$ for $T_{min} < T < T_{max}$

$T_{min} \geq \hbar\langle\omega\rangle$ where $\hbar\langle\omega\rangle$ is mean effective atomic binding energy

T_{max} from kinematics in rest frame of heavy particle

$\leftarrow e^-$ initial

$\rightarrow e^-$ final $(\overset{E'}{\gamma mc^2}, \overset{p'}{\gamma m\beta c})$

boost to lab frame

$$E = \gamma(E' + \beta c p') = \gamma^2(mc^2 + mc^2\beta^2)$$

T_{max} in lab frame is $E - mc^2 = (\gamma^2 - 1 + \beta^2\gamma^2)mc^2$

$$= 2\beta^2\gamma^2 mc^2$$

$\frac{dE}{dx}$ energy loss be unit length through

N atoms/volume

Z electrons/atom

$$= NZ \int_{\epsilon}^{T_{max}} T \frac{d\sigma}{dT} \, dT$$

$\epsilon >> \hbar\langle\omega\rangle$

$$\boxed{\frac{dE}{dx}(T>\epsilon) = \frac{2\pi NZ z^2 e^4}{(4\pi\epsilon_0)^2 mc^2\beta^2} \ln\left(\frac{2\gamma^2\beta^2 mc^2}{\epsilon}\right)}$$

energy loss / unit distance from collisions with energy transfer $> \epsilon$

when including QM spin correction (of electron) $\ln(\,) \rightarrow \ln(\,) - \beta^2$

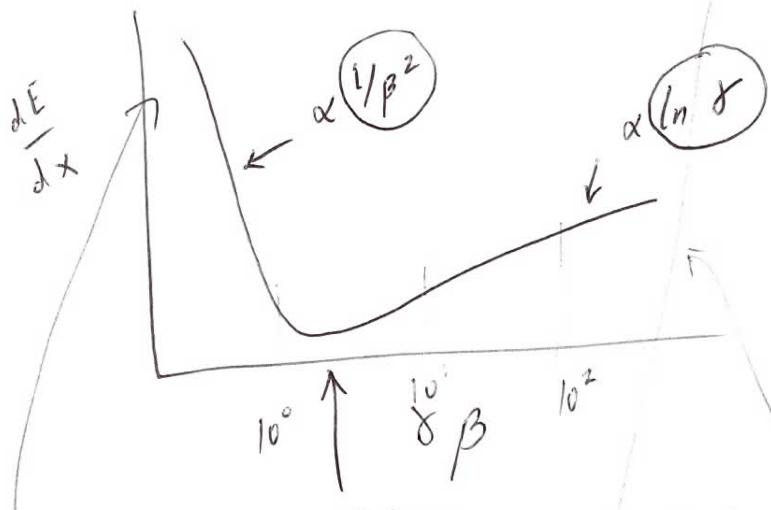
For energy loss with small energy transfer $< \epsilon$, must be QM calculation (Bethe)

$$\frac{dE}{dx}(T < \epsilon) = \frac{2\pi NZ}{(4\pi\epsilon_0)^2} \frac{z^2 e^4}{mc^2\beta^2}\left\{\ln\left[B_q^{\,2}(\epsilon)\right] - \beta^2\right\}$$

where $B_q(\epsilon) = \frac{\gamma v (2m\epsilon)^{1/2}}{\hbar\langle\omega\rangle}$

total $\frac{dE}{dx} = \frac{4\pi N Z z^2 e^4}{(4\pi\epsilon_0)^2 mc^2\beta^2}\left[\ln\left(\frac{2\gamma^2\beta^2 mc^2}{\hbar\langle\omega\rangle}\right) - \beta^2\right]$

Fig. 13.1 in Jackson (well known to you, perhaps)



transverse $\vec{E}$ field observer sees field ↑ by γ

minimum ionizing $\gamma\beta \sim 3\text{–}4$

@ higher energies - density corrections (Fermi)

@ lower energies : shell corrections (atomic)

Bethe-Bloch formula has corrections, the most important being the density correction @ high energies

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Density Effect

- calculation so far assumed incident particle effect one electron in one atom at a time, then sum incoherently all the energy transfers
- in dense media, many atoms in between (especially for large γ)
- these atoms perturb field or dielectric polarization of material alters field

Density Effect cont'd

- impact parameter $\sim$ atomic dimension dividing calculation of free-particle w/o polarization and with polarization
- join two logarithms (as before); dividing b need not be specified with great precision

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Solve using Fourier transforms in space and time

$$F(\vec{x}, t) = \frac{1}{(2\pi)^2} \int d^3k \int d\omega \, F(\vec{k}, \omega) e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

transformed wave equations: using Gaussian units

$$\left[k^2 - \frac{\omega^2}{c^2} \epsilon(\omega)\right] \Phi(\vec{k}, \omega) = \frac{4\pi}{\epsilon(\omega)} \rho(\vec{k}, \omega)$$

$$\left[k^2 - \frac{\omega^2}{c^2} \epsilon(\omega)\right] \vec{A}(\vec{k}, \omega) = \frac{4\pi}{c} \vec{J}(\vec{k}, \omega)$$

$$\rho(\vec{x}, t) = ze\,\delta(\vec{x} - \vec{v}t)$$

$$\vec{J}(\vec{x}, t) = \vec{v} \rho(\vec{x}, t)$$

$$\rho(\vec{k}, \omega) = \frac{ze}{2\pi} \delta(\omega - \vec{k}\cdot\vec{v})$$

$$\vec{J}(\vec{k}, \omega) = \vec{v} \rho(\vec{k}, \omega)$$

$$\therefore \Phi(\vec{k}, \omega) = \frac{2ze}{\epsilon(\omega)} \frac{\delta(\omega - \vec{k}\cdot\vec{v})}{k^2 - \frac{\omega^2}{c^2}\epsilon(\omega)}$$

$$\vec{A}(\vec{k}, \omega) = \epsilon(\omega) \frac{\vec{v}}{c} \Phi(\vec{k}, \omega)$$

$$\vec{E}(\vec{k},\omega) = i\left[\frac{\omega\,\epsilon(\omega)}{c}\frac{\vec{v}}{c} - \vec{k}\right]\Phi(\vec{k},\omega)$$

$$\vec{B}(\vec{k},\omega) = i\,\epsilon(\omega)\,\vec{k}\times\frac{\vec{v}}{c}\,\Phi(\vec{k},\omega)$$

energy loss to an electron @ impact parameter b

$$\underset{\substack{\uparrow \\ \text{energy}}}{\Delta E} = -e\int_{-\infty}^{\infty}\vec{v}\cdot\vec{E}\,dt = 2e\,\mathrm{Re}\int_0^{\infty} i\omega\,\vec{x}(\omega)\cdot\vec{E}^{*}(\omega)\,d\omega$$

$\vec{x}(\omega)$ is electron coordinate Fourier transform

$\vec{E}(\omega)$ Fourier transform of $\vec{E}$ field

$$\vec{E}(\omega) = \frac{1}{(2\pi)^{3/2}}\int d^3k\;\vec{E}(\vec{k},\omega)\,e^{ibk_2}$$

observed at $(0, b, 0)$ ← impact parameter

e.g. calculate $E_1(\omega)$ $\vec{E}$ field $\parallel$ to $\vec{v}$

$$E_1(\omega) = \frac{2ize}{\epsilon(\omega)\,(2\pi)^{3/2}}\int d^3k\,e^{ibk_2}\left[\frac{\omega\,\epsilon(\omega)}{c^2}v - k_1\right]\frac{\delta(\omega - vk_1)}{k^2 - \frac{\omega^2}{c^2}\epsilon(\omega)}$$

$$\hookrightarrow\; E_1(\omega) = \frac{-ize\,\omega}{v^2}\left(\frac{2}{\pi}\right)^{1/2}\left[\frac{1}{\epsilon(\omega)} - \beta^2\right]K_0(\lambda b)$$

← modified Bessel function

where $\lambda^2 = \dfrac{\omega^2}{v^2} - \dfrac{\omega^2}{c^2}\epsilon(\omega) = \dfrac{\omega^2}{v^2}\left[1 - \beta^2\epsilon(\omega)\right]$

similar $E_2(\omega) = \dfrac{ze}{v}\left(\dfrac{2}{\pi}\right)^{1/2}\dfrac{\lambda}{\epsilon(\omega)}K_1(\lambda b)$

$B_3(\omega) = \epsilon(\omega)\,\beta\,E_2(\omega)$

then Fermi density effect:

$$\left(\frac{dE}{dx}\right)_{b>a} = \frac{2}{\pi}\frac{(Ze)^2}{v^2} \mathrm{Re}\int_0^\infty i\omega \lambda^* a K_1(\lambda^* a) K_0(\lambda a)\left(\frac{1}{\epsilon(\omega)} - \beta^2\right) d\omega$$

take relativistic $\beta \simeq 1$

$d\omega$ important fequencies optical, $a \sim$ atomic dimensions

$|\lambda a| \sim (\omega a / c) \ll 1$, then Bessel fns $\rightarrow$ small argument limit

$$\left(\frac{dE}{dx}\right)_{b>a} \simeq \frac{2}{\pi}\left(\frac{Ze}{c^2}\right)^2 \mathrm{Re}\int_0^\infty i\omega\left(\frac{1}{\epsilon(\omega)} - 1\right)\left\{\ln\left(\frac{1.123c}{\omega a}\right) - \frac{1}{2}\ln[1-\epsilon(\omega)]\right\} d\omega$$

$$\left(\frac{dE}{dx}\right)_{b>a} = \frac{(Ze)^2 \omega_p^2}{c^2}\ln\left(\frac{1.123c}{a\omega_p}\right)$$

$\ln[1-\beta^2\epsilon(\omega)]$ if $\epsilon = 1$ gives classical

where $\omega_p^2 = \frac{4\pi N Z e^2}{m}$.

w/o density effect

$$\left(\frac{dE}{dx}\right)_{b>a} \simeq \frac{(Ze)^2 \omega_p^2}{c^2}\ln\left(\frac{1.123\gamma c}{a\langle\omega\rangle}\right)$$

the effect $\rightarrow$ asymptotic energy loss no longer depends on $\langle\omega\rangle$ details atomic structure

just depends on density of electrons