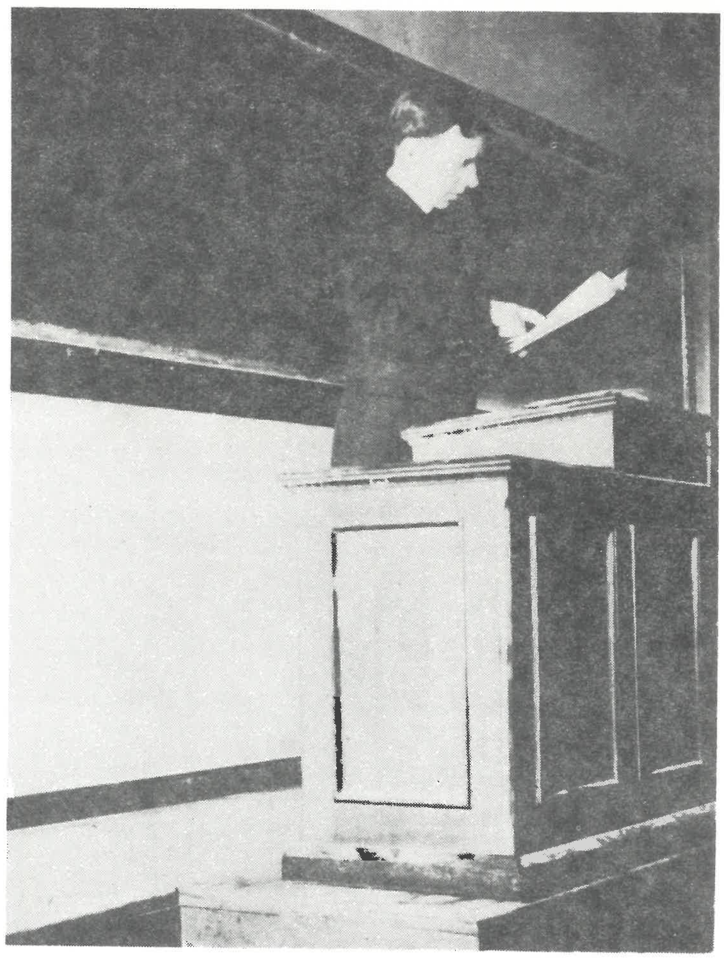
# QUEEN'S MATHEMATICAL COMMUNICATOR

January
1985



Norman Miller 1923

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**Cover Picture.** Norman Miller lecturing at Queen's about 1923. The picture was taken by a student, Pep Leadly, of Queen's football fame, probably in the "New Arts Building", Kingston Hall. The photo is in the possession of Mrs. C.H.R. Campling, Norman Miller's daughter, and we are grateful to her for permission to use it. Mrs. Campling also provided the photos on page 9, taken about 1960.


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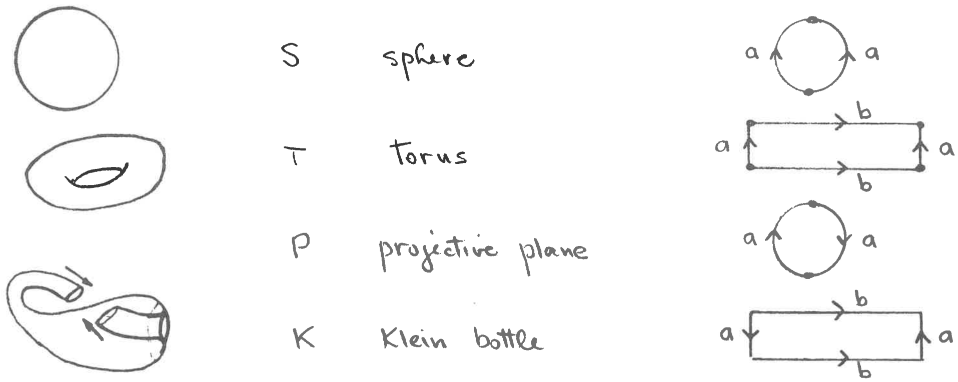
**The Chromatic Number of a Surface**
[The text of the first Coleman-Ellis Lecture,
by Peter Taylor, Oct. 23, 1984]

The famous four colour theorem says that for any map drawn on a sphere S (or a plane), the regions can be coloured with only 4 colours so that regions which share a common boundary line have different colours. The theorem was, in fact, only a conjecture for some 100 years. It was shown in 1890 that 5 colours were enough, and any example that was produced could be coloured with only 4, but the proof that 4 was enough could not be found. In 1976, Appel and Haken from Illinois, with the aid of a computer, showed that 4 colours sufficed. The computer was used to reduce the problem to a large but finite number of special cases, and to check these. It is still an open question whether a proof of this theorem exists whose details could be checked by one person in a reasonable length of time.

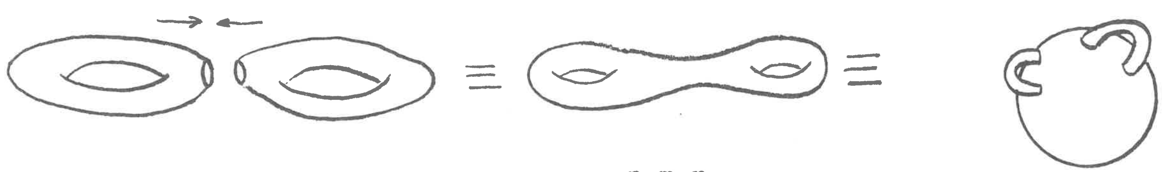
In this article I look at three other surfaces, the torus T , the projective plane P , and the Klein bottle K , all more complicated than the sphere, but for which the colouring problem is much simpler and has been solved for many years. These surfaces all belong to a class of objects known as compact 2-dimensional manifolds, all of which can be obtained by "pasting together" copies of S, T, P , and K . The colouring problem for all such surfaces was completely solved by 1967, except for the simplest one, the sphere S .

**The Basic Surfaces.** The surfaces we will be drawing maps on are displayed below. S and T can be realized in 3-space, but P and K cannot. In the "picture" of K I have drawn, the two open ends must meet without passing through the surface of the bottle. This is impossible in 3-space but can be done in 4-space. I haven't drawn a "picture" of P 'cause I don't really know how. In any case, to draw maps on these surfaces we want to "spread them out flat" and to do that we make one or two cuts. The diagrams on the right are the planar representations we get. Two edges with the same name are the same edge and the arrows indicate how the edges are put together to reconstruct the surface. Thus these representations define the surface (in particular they tell you what P is). To get the surface you take your needle and thread and sew up the cuts in the indicated orientation. You can actually do that for S and T , but for P and K you get stuck near the end of your job by the impossibility of pulling the surface through itself.

A word about K . If I join the "a" edges I get what's called a Möbius band. The Klein bottle is then obtained from the Möbius band by joining its two edges, actually by joining the two halves of what has now become a single edge.

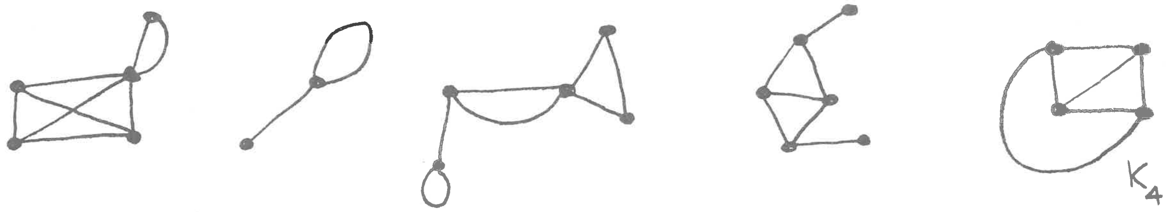



**General Surfaces.** Two surfaces M and N can be pasted together to form a new surface MN by cutting small discs out of each and pasting the edges together. The process is illustrated below for two tori, to get the surface TT which is also written $T^2$, and is called the double torus or "pretzel". It can also be regarded as a "sphere with 2 handles"; we regard two surfaces as equivalent (homeomorphic) if one can be continuously deformed into the other. Thus $TS = T$ (a tire with a weak spot is still a tire) and indeed S acts as an identity element for the operation. It is also true that the operation is associative and commutative, so the surfaces that can be built out of the basic ones are all of the form $T^nP^mK^r$ for $n, m, r \geq 0$; by convention $M^0 = S$. It is a famous and difficult theorem that every compact 2-manifold is of this form.



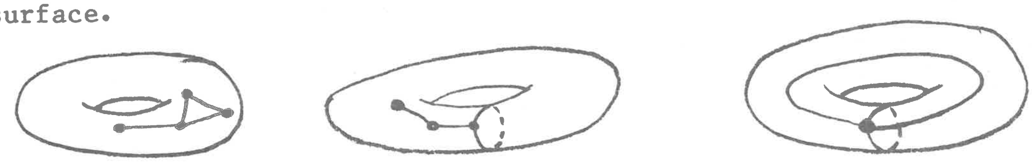
Actually not all of the surfaces $T^nP^mK^r$ are (homeomorphically) different. It turns out that $P^2 = K$ and $P^3 = TP$. Using this we see that every surface can be written in the form $T^n$ or $T^nP$ or $T^nK$ for some $n \geq 0$. We take these representations as canonical forms. The surface $T^n$ can be visualized as a sphere with n handles. The surfaces $T^nP$ and $T^nK$ cannot be realized in 3-space.

**Graphs.** Actually we're not going to colour maps. We're going to colour the vertices of graphs. A graph, roughly speaking is a connected complex of vertices (dots) and edges (lines). I allow multiple edges and loops, but I will call a graph **simple** if it has neither of these. In the examples below the two graphs on the right are simple.




An important family of graphs are the **complete graphs** $K_n$. $K_n$ has $n$ vertices and one edge joining each vertex pair. Thus $K_3$ is a triangle. A graph is said to be **drawn** on a surface M if the vertices and edges can be arranged so no two edges cross. Thus $K_4$ can be drawn on S (above right) but $K_5$ cannot (try it!). Note the edges don't have to be straight lines. Henceforth whenever I talk of a graph on surface M, I will assume it is drawn on M.

A graph, drawn on a surface M, is said to **decompose** M if when we cut along the edges the pieces we get can all be flattened. Thus only the graph on the right below decomposes T. Indeed the graph on the right produces a single piece, called a **face**, which is, in fact, the flat representation of the torus we drew earlier. Similarly, the flat representations for S, P, and K came from graphs which decomposed the surface.



One more definition: the **degree** of a vertex is the number of edges which are incident to it.

**The Characteristic.** If I have a graph which decomposes a surface M, I let V, E, and F denote the number of vertices, edges and faces (the pieces I get if I cut along the edges). If I add more vertices or edges I will change these numbers, but it's easy to see I will never change the quantity $V - E + F$ (try it!), which in fact depends only on the surface and is called its characteristic $\chi$. Precisely, the **characteristic** $\chi(M)$ of the surface M is the value $V - E + F$ for all graphs drawn on M which decompose it.

We can compute $\chi$ for our basic surfaces from the planar representations. The results are

| | V | E | F | $\chi$ |
|---|---|---|---|---|
| S | 2 | 1 | 1 | 2 |
| T | 1 | 2 | 1 | 0 |
| P | 1 | 1 | 1 | 1 |
| K | 1 | 2 | 1 | 0 |

The characteristics of the surfaces $T^nP^mK^r$ can be computed with the rule: start with 2 (for S), and subtract 1 for every copy of P and 2 for every copy of T or K. Thus $\chi(T^nP^mK^r) = 2 - 2n - m - 2r$ and for the canonical forms, $\chi(T^n) = 2 - 2n$, $\chi(T^nP) = 1 - 2n$, and $\chi(T^nK) = -2n$. Thus, except for the basic surfaces above, every surface has $\chi < 0$; there is one surface $(T^nP)$ of every odd negative characteristic and two surfaces ($T^{n+1}$ and $T^nK$) of every even negative characteristic.

**Chromatic Number.** Colouring is often discussed in terms of regions of a map, but it's simpler, and equivalent, to colour the vertices of a graph. Indeed I associate a graph to any map on a surface by putting a vertex in each region and an edge between any two vertices whose regions share a boundary. The edges can always be drawn so that the graph is drawn on the surface.

A **colouring** of a graph is an assignment of colours, one to each vertex, so that adjacent vertices (joined by at least one edge) have different colours. The **chromatic number Chr(M)** of a surface $M$ is the least number of colours required to colour all graphs that can be drawn on $M$.

It is worth observing that if $K_n$ can be drawn on $M$ then $\mathrm{Chr}(M) \geq n$. Thus we can see that the chromatic number of $S$ (and indeed of all surfaces) is at least 4. The 4-colour theorem states that $\mathrm{Chr}(S) = 4$.

**Heawood's Theorem.** The main technique for finding an upper bound for the chromatic number of a surface is to show that a simple graph on the surface must have a vertex of reasonably small degree. The fact that the graph is simple and must be drawn on the surface means you can't have a great many edges at every vertex. For example, it is impossible to draw a graph on $S$ which has every vertex of degree $\geq 6$ (proved below). It is possible to have a graph on $S$ with every vertex of degree $\geq 5$, but it takes a while to find one (Hint at end).

Heawood (1890) proved a general theorem of this type for all surfaces in terms of the characteristic of the surface. We let the Heawood number $H$ of a surface with characteristic $\chi$ be

$$\frac{7+\sqrt{49-24\chi}}{2} .$$

What a strange number! We will see in the proof of the next theorem how Heawood found it.

**Theorem 1.** (Heawood 1890). A simple graph $G$ which decomposes a surface of characteristic $\chi$ must have a vertex of degree $\leq 5$ if $\chi > 0$, and $H-1$ if $\chi \leq 0$.

Proof. Suppose we have such a graph. We can assume, by adding edges if necessary, that all faces are triangles (bounded by 3 edges). We use the fact that $G$ is simple here. [These additional edges if anything only increase the degree of each vertex.] Now since all regions are triangles, $2E = 3F$ (count edges by counting faces; every edge will be counted twice). Let $D$ be the average degree of all vertices. Then $2E = DV$ (count edges by counting vertices; every edge again counted twice). Plug the above two equations into $V - E + F = \chi$ to eliminate $E$ and $F$. We get

$$D = 6 - 6\chi/V .$$

Our objective is to show that $D$ is small (certainly there is always a vertex with degree $\leq D$, the average degree). Well if $\chi > 0$, $D < 6$, so there must be a vertex of degree $\leq 5$. That does the first case above. If $\chi \leq 0$, then $-\chi/V$ is positive and the way to get $D$ small is to have $V$ big. Well if $V \leq H$ then there are at most $H$ vertices and all have degree $\leq H-1$, and we are done. Otherwise $V > H$ which means $-6\chi/V \leq -6\chi/H$ and so $D \leq 6 - 6\chi/H$. If this were equal to $H-1$ we would be done. Indeed Heawood simply chose $H$ to make this true! The equation $6 - 6\chi/H = H-1$ is quadratic in $H$ with root given by the above formula. QED

**Two Basic Colouring Theorems.** Both due to Heawood 1890.

**Theorem 2.** If every simple graph on $M$ has a vertex of degree $\leq n$ then $\mathrm{Chr}(M) \leq n+1$.

**Theorem 3.** If every simple graph on $M$ has a vertex of degree $\leq n$ and in addition $K_{n+1}$ can't be drawn on $M$, then $\mathrm{Chr}(M) \leq n$.

The proof of both theorems is by induction on the number of vertices. First Theorem 2. We must show every graph on I can be coloured with $n+1$ colours. Suppose true for all graphs with $\leq V$ vertices. Take a graph on $M$ with $V+1$ vertices. Remove extra edges and loops if necessary to make it simple. Now it has a vertex of degree $\leq n$. Remove it and all incident edges. The remaining graph has $V$ vertices, so can be $n+1$ coloured. Now put the last vertex back. Can we colour it? Yes, since it has at most $n$ neighbours and we have $n+1$ colours to choose from. QED

Theorem 3 is similar but a bit more subtle. To show that $n$ colours are enough we suppose true for all graphs with $\leq V$ vertices, take a graph with $V+1$, simplify it, and produce a vertex $x_0$ of degree $\leq n$, as before. Now this vertex might have degree $\leq n-1$, in which case the removal trick used for Theorem 2 will allow us to colour the graph with $n$ colours. Otherwise it has degree $n$. In this case its $n$ neighbours cannot all be joined to one another by edges or we would have a copy of $K_{n+1}$ on $M$. So some pair $x_1$ and $x_2$ are not joined. Take these three vertices $x_0$, $x_1$ and $x_2$ and two edges $x_0x_1$ and $x_0x_2$, and collapse them into one supernode. The new graph has $V-1$ vertices so can be coloured with $n$ colours. Suppose the supernode has been coloured red. Now can we colour $x_0$, $x_1$ and $x_2$ ? Well colour $x_1$ and $x_2$ both red (they are not joined, and certainly won't be joined to a red vertex or the supernode would have been so joined). So the $n$ neighbours of $x_0$ use at most $n-1$ colours (two of them are red) and there must be a colour left for $x_0$. QED

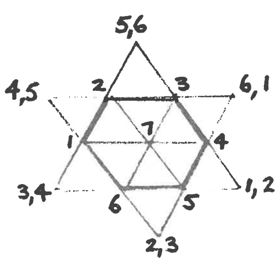
Historically, Theorem 3 first appeared as a special argument for $M = S$ (with $n = 5$). It provided the first proof that 5 colours suffice for the sphere.

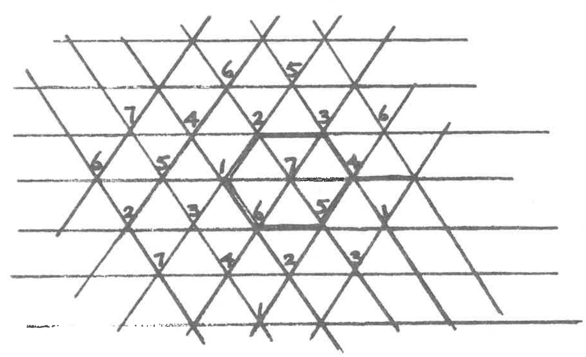
These three theorems of Heawood serve to give us upper bounds on $\mathrm{Chr}(M)$. Lower bounds can be obtained, as I have mentioned, by discovering which complete graphs can be drawn on M: if $K_n$ can be drawn on $M$ then $\mathrm{Chr}(M) \geq n$. Can we get these upper and lower bounds to match, thus determining the chromatic number? Let us look at the basic surfaces.

**The Sphere S.** The largest complete graph on S is $K_4$ . So $Chr(S) \geq 4$ . Since $K_6$ can't be drawn on S , Theorems 1 and 3 (with n = 5) give us $Chr(S) \leq 5$ . The upper and lower bounds are not the same and, as I have said, new and computationally long techniques were needed to settle the matter.

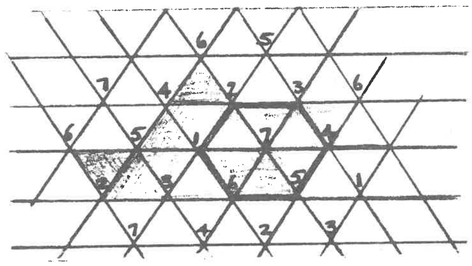
**The Projective Plane P.** Here, $\chi = 1$ , so Theorems 1 and 2 together give us $Chr(P) \leq 6$ . It follows from this that $K_7$ could never be drawn on P (it would need 7 colours!). Can $K_6$ be drawn? Try it as a good exercise. The answer is yes and here is the picture. Thus $\chi(P) = 6$ .

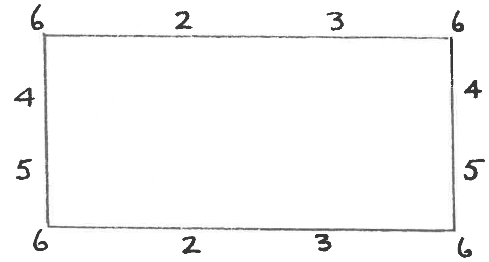
**The Torus T.** Here we have $\chi = 0, H = 7$ , so Theorems 1 and 2 give $Chr(T) \leq 7$ . Can $K_7$ be drawn on T ? Let's try Observe that for $K_7$, $V = 7$, $E = \binom{7}{2} = 21$ and since $V - E + F = 0$ (assume it decomposes T), $F = 14$ . Since $3F = 2E$ , we conclude all faces are triangles. So the graph must look in all directions like an infinite triangular grid. Let's take a vertex, and call it 7, and designate its neighbours, which form a hexagon, as 1-6. As we move over the grid outside this hexagon we will encounter these same vertices again (we are walking round the torus) but where will the various vertices be? Using the fact that every vertex must be joined to every other, we get the two possibilities shown below (left) for each of the points of the star. If we make one choice, say 4, for the 1, 2 edge then the rest are determined, as 5, 6, 1, 2, 3 in clockwise order. Let's do that. [It will turn out, and I'll leave this to you, that the other choice, 5, would give a completely equivalent analysis from this point on.] Then it's not hard to argue that all the remaining vertices are uniquely determined. I have filled in a number of these in the grid below (right).




Now, this infinite grid, should it represent a graph on the torus, must have 14 different triangles, and any line we draw enclosing one copy of each will represent the surface. It's not hard to see that the grid contains a repeating pattern with, in fact, 14 triangles, and there are many ways to draw such a line. I have drawn one below which shows that the surface is indeed a torus. [Look at the planar representation at the beginning.] We have a copy of $K_7$ on T . So $Chr(T) = 7$ .






**The Klein Bottle** K. Again we have $\chi = 0$, so as for the torus, $\mathrm{Chr}(K) \leq 7$, and any drawing of $K_7$ on $K$ must have triangular faces. But when we numbered such a grid as we have just done, we found our hand forced at every move (except once) and the surface we got was a torus and not a Klein bottle. [Had we chosen the 5, we would also have gotten a torus.] We conclude $K_7$ cannot be drawn on the Klein bottle! This observation was made by Franklin only in 1934; Heawood in 1890 knew that $K_7$ could be drawn on $T$, but did not know it could not be drawn on $K$.

Now $K_6$ is easy to draw on $K$ (try it; use the above drawing on $P$), so by Theorem 3, $\mathrm{Chr}(K) = 6$. This is the second application of Theorem 3, once for $S$ and once for $K$.

**Surfaces of negative characteristic.** Theorems 1 and 2 give us $\mathrm{Chr}(M) \leq [H]$ for all surfaces with $\chi \leq 0$. Can $K_{[H]}$ be drawn on any surface $M$ with $\chi < 0$? The answer is yes and was given bit by bit, in a series of special cases between 1952 and 1967 by Ringel and Youngs except for three instances which were settled in 1967 by Meyer, a professor of French Literature. The techniques rely on symmetry properties (as you might guess from the triangular grid) and make use of group theory.

The conclusion is that $\mathrm{Chr}(M) = [H]$ for all surfaces with $\chi < 0$. In fact this is true for all surfaces, regardless of $\chi$, except the Klein bottle for which $\mathrm{Chr}(K) = 6 = [H]-1$. Strange to have the Klein bottle as the one exception. Though I must say that the result, $\mathrm{Chr}(M) = [H]$ for $S$ and $P$, must be regarded as fortuitous because $H$, as can be seen from the proof of Theorem 1, really has no significance when $\chi > 0$. Or does it? That relates to my question of whether a reasonable proof that $\mathrm{Chr}(S) = 4$ will ever be found.

**References.** Mathematical Recreations and Essays, Ball and Coxeter, 12th Ed. (1974), U. of Toronto Press, contains a discussion of many of these results. A more advanced treatment is found in Map Colour Theorem, G. Ringel (1974), Springer-Verlag.

[Hint. The icosohedron is a graph on $S$ with every vertex of degree 5.]

**Norman Miller**
**1889–1984**

**Grace Miller**
**1894–1984**

Professor Norman Miller, who taught at Queen's for over forty years, was born in Aylmer, Ontario on November 1, 1889 and died on May 31, 1984 in his ninety-fifth year. He entered Queen's University in 1906, attended Teacher's College, taught school for two years and then, benefitting from a scholarship, pursued graduate study in mathematics at Harvard. It may well be that he was the first Canadian Mathematician to obtain a Harvard Ph.D., with a thesis entitled "Some Problems connected with the Linear Connectivity of Manifolds", 1916.

Immediately upon returning to Canada he enlisted in the army and was on active service in France at the conclusion of World War I. In 1918-19 he taught in the "Kahki" University in the U.K.

In 1919, on the invitation of Professor John Matheson, (Head of Mathematics and Dean of Arts and Science) he joined the Queen's Department of Mathematics which he served with extraordinary verve and enthusiasm until his retirement in 1959. Through his clear lectures, his encouragement of able students to take up the teaching of High School mathematics, his active participation in a succession of organizations of mathematics teachers and his co-authorship of high school text-books, Norman Miller made an enormous contribution to the teaching of mathematics in the Province of Ontario. He was undoubtedly one of the most beloved and respected teachers of mathematics in Canadian Universities.

Norman Miller's interest went beyond mathematics. He was an active United church man and, for twenty years, the Secretary of the Saturday Club. This is a group of professors from all disciplines at Queen's and the Royal Military College which meets about twelve times a year to discuss philosophical, social, or political issues. Miller's minutes of these meetings were always vivid and elegant. To the end of his life, even when suffering from Alzheimer's disease, Professor Miller projected an aura of gentleness, graciousness and civility.

Professor Miller's wife, Grace, who was an M.A. in mathematics at Queen's and a member of the faculty for a few years, predeceased him by three weeks. She played a leading role in the Girl Guides of Canada was a member of the School Board and made a very positive contribution to the quality of life in Kingston. The Millers are survived by four daughters.

A.J. Coleman

**Norman Miller**

I have received a letter about Norman and Grace Miller from Dr. **Jeanne** (leCaine) **Agnew**, Mathematics Professor at Oklahoma State University and Queen's Alumna (Arts '38). Her letter captures much of the spirit of the time, and I am grateful to her for permission to quote from it. PDT.

"I want to say a few words about my dear friend and counsellor, Dr. Norman Miller. He and his charming wife passed away within weeks of each other earlier this year.

I suppose the current student body and younger faculty, if they were aware of him at all, saw him as an old man, hard of hearing, wearing thick lensed glasses, yet walking to church every Sunday because it pleased him to spend that hour in the sanctuary even though he could not hear the sermon. Yet he is a man to whom the Mathematics Department at Queen's owes a great debt of gratitude for years of dedicated and creative service.

Scholar, Teacher, and Friend, he excelled at all three. Did you know he graduated from Harvard and was so well thought of there that he was able to open some doors for me? Did you know that he published several books among them a differential equations book that is beautifully concise and complete? Did you know that after his retirement he created a number of mathematical models from string? [They are still on display in Jeffery Hall and even used in the classroom. PDT]

As a teacher he was demanding but never sarcastic. You did not mind working when you knew your efforts would be appreciated. Among other things he taught me Complex Variables. When I went to Harvard they insisted that I retake the course - no course from Queen's would be the equivalent of a Harvard course! I made an A+ that semester (from G.D. Birkhoff), not through any merit of Birkhoff or myself but because I had such a solid foundation in the subject.

Like a great many other students in 1934, I came to Queen's a gauche midwesterner, feeling strange, scared, and out of place in the conservatism of Kingston. Yet I was awarded the same care and concern given to the better adjusted and more sophisticated students. The Millers had a custom of having students to tea on Sunday afternoon. Their four lovely daughters were then just old enough to help serve our cookies and tea while we listed to Gilbert and Sullivan records. What a treat to be accepted into a home!

Fortunately I have been able to keep a little contact with them over the years. I have enjoyed their continued interest in the University and their pride in their children, grandchildren, and great grandchildren. I have been pleased to find that that pride extends almost as much to their family of ex-students. I have never returned to Kingston without stopping by to see them, and even when hands shook so it was hard to hold the pot they insisted on sharing a cup of tea.

I had courses from many fine teachers at Queen's - Dean Matheson, Dr. Edgett, Dr. Gummer, Dr. Knox, Dr. MacIntosh. But at the top I place the Millers who to me represented Queen's at its best."

**Problem Solving Sessions for High School Students.**

A series of biweekly problem solving sessions for high school students is running this year in Jeffery Hall. The problems are exploratory in nature and, during the two-hour session, students work in small groups and make periodic progress reports to the class. Problems are elementary, but challenging, are drawn from areas of combinatorics, probability, geometry, number theory, graph theory and logic. There are about 40 student participants (some 4 students from each of 10 schools) and a few faithful teachers. Students come not only from Kingston, but from as far away as Prescott, Smith's Falls and Belleville. Problems are selected, and the session is run, by **Peter Taylor.**

At the first session in September the following problem was put forward. Two people, each with a well-shuffled deck of cards play the version of SNAP in which you only call "snap" if the two cards which turn up are identical, i.e. same denomination and same suit. What is the probability there will be no call of "snap" during one play of all 52 cards? Try to guess whether you expect this probability to be greater than or less than 1/2. Most people guess greater than. But the answer is less than! The odds are that there <u>will</u> be a "snap". In fact the probability of there being no snap is $1/e \approx 1/2.718$. This was the result the students had to find.

[Some of you will be wrinkling your brow. So you should. How could it be exactly $1/e$ ? Shouldn't the answer depend on the number of cards in the deck? Indeed it should. The exact answer for a deck of $n$ cards turns out to be $P_n = 1 - 1 + 1/2! - 1/3! + 1/4! - 1/5! + \dots \quad 1/n!$ . As $n$ gets large this converges to $1/e$ [Recall $e^x = 1 + x + x^2/2! + \dots$], in fact since the series is alternating with decreasing terms, the difference between $P_n$ and $1/e$ is less than $1/(n+1)!$ . In particular $P_{52}$ is equal to $1/e$ to 69 places of decimal.]

* * *

**Quotations**

From a talk given by **Norman Miller** to a group of Queen's graduates in March 1951. (From John Coleman)

"... to judge the value of a teacher's work, my yardstick would be, not what proportion of his pupils does he get through the final examination, but does he make his pupils like the subject he teaches."

"I sometimes think that of all professional persons, the teacher, when he comes to retirement, has the most enduring satisfactions as he passes in review the years of his work and the generations of students whom he has known and taught... Who, more than the teacher, has the right to say with Ulysses 'I am a part of all that I have met' ... surely the measure of a teacher is his ability to suffuse even routine work with the romance of discovery, of pushing into the unknown sea 'whose margin fades forever and forever when I move'."

Professor **John Ursell** spoke at the Conference on Ordinary and Partial Differential Equations on his new method for solving first order, ordinary DE's of degree 1. He reports that the method is quite easy and expects its early appearance in undergraduate texts.

**Tor Gulliksen**, University of Oslo, a former Ph.D. student of **Paulo Ribenboim**, has been awarded an honorary doctorate by the University of Stockholm.

Professor **Jim Whitley** has received a letter from **Sally** (Cockburn) **Oerlemans** (M.Sc. '84). Its enthusiasm prompts us to quote at length.

"After a month in Botswana we feel that we know enough of the country to form our FIRST impressions. We LOVE it here - there is hardly a single thing to complain about. The weather has been wonderful; although it's getting pretty hot around midday, the evenings are still very pleasant. We've been given a brand-new, spacious semi-detached house on the school compound, which is much nicer than anything we could afford in Canada. David Matthews and the rest of the staff are EXTREMELY congenial - it took us no time at all to feel totally assimilated into the Maru a Pula community. The Botswana are, as you said, a beautiful people. It does not take long to notice their gentleness, warmth and courtesy. People SMILE here so much more often than in North America, it seems; just a simple greeting of "Dumela, mma" or "Dumela, rra" elicits a friendly response from a complete stranger. In Canada, it would elicit only embarrassment, if anything. Our teaching jobs are exciting and challenging, although I feel that it would be of greater benefit to both the school AND me if I were given a full-time position, instead of my present part-time status. There are enough math courses currently distributed among science teachers, English teachers and David Matthews himself to justify hiring another full-time mathematics teacher. However, I'm lucky to be working at all this term.

I'm EXTREMELY impressed with the level of mathematics at Maru a Pula. There's a second form class that does trigonometry, advanced geometry, even matrix multiplication!! It's absolutely incredible! I help supervise that class, and answer individual questions, and I'm quite astounded by how bright they are, and how quickly they pick up just about any concept you feed them. More generally, the level of secondary school math seems higher here than in Canada. The "A-level" students study statistics that I didn't learn until my 3rd year at Queen's! I've no doubt that teaching the sixth form students here is comparable to teaching at a first-year university level.

We haven't learnt much Setswana beyond the usual greeting dialogue, yet, but we've enrolled in a language course at the Botswana Orientation Centre that starts next week. It seems to me imperative to learn Setswana in order to learn much about Botswana's culture and people.

We've seen so many slides and photos, and heard so many stories of other teachers' vacations in this area that we're positively itching for the Christmas holidays. We've bought a Chevrolet Nomad, a creature I'd never seen or heard of before arriving in Botswana. It wasn't too expensive, and it seems very practical for the country's numerous dirt roads. We saw one of your Honda 50's in David's yard; it looked like a fun machine, but not suitable for the elaborate safaris we've been planning!"

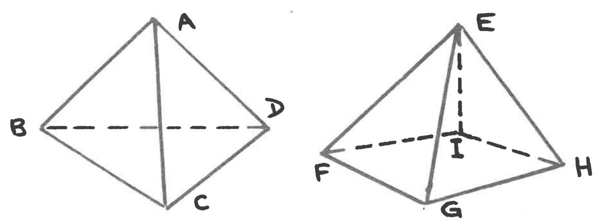
**Coleman-Ellis Lectureship Launched**

In honour of Emeritus Professors A.J. Coleman and H.W. Ellis, the Department of Mathematics and Statistics at Queen's University has established a special undergraduate lecture series which will expose some interesting ideas in mathematics and statistics in an elementary fashion. The lectures are held once a month on a Tuesday evening, and are followed by refreshments and informal discussion. The first two lectures were given this fall to enthusiastic audiences consisting of high school students and teachers, university undergraduates and graduates, and even a sprinkling of professors. The full program for 1984-85 is given below.

October 23 - The Chromatic Number of a Surface
Peter Taylor
November 20 - A New Chapter in the Guiness Book of Records
Paulo Ribenboim
January 15 - Infinitesimals and Non-Standard Analysis
Norman Rice
February 12 - Morse Code, False Coins, Entropy, and Like Matters
Lorne Campbell
March 12 - Finite Geometry and a Golf Tournament Scheduling Problem
Norman Pullman

**PROBLEMS**

Problem 1. (An Educational Testing Service "Scholastic Aptitude Test" question.)




In the regular tetrahedron ABCD and pyramid EFGHI shown above, all faces except FGHI are equilateral triangles of equal size. If face ABC were glued congruently to face EHI , how many faces would the resulting solid have?

Problem 2. A fisherman is 1/3 of the way across a long, narrow, high railway trestle when he hears a train coming behind him at 60 m.p.h. (constant). He starts running instantly at his top (constant) speed, and can just save his life by running to either end of the bridge. How fast can he run?

Colin Blyth

Problem 3. Find the next term in this sequence.

$$(x+y)^3 = x^3 + y^3 + 3xy(x+y)(x^2+xy+y^2)^0$$

$$(x+y)^5 = x^5 + y^5 + 5xy(x+y)(x^2+xy+y^2)^1$$

$$(x+y)^7 = x^7 + y^7 + 7xy(x+y)(x^2+xy+y^2)^2 \quad .$$

Peter Taylor and Doug Dillon.

**Perfect Ellipsoids**

Let's call an ellipsoid in 3-dimensional space empty if it contains no integer points in its interior. Now blow such an ellipsoid up until it touches some integer points and call it $E$. Is there another empty ellipsoid, $E'$, distinct from $E$, which touches the same integer points as $E$? If there is no such ellipsoid call $E$ perfect; otherwise call $E$ imperfect. To be perfect an ellipsoid has to touch a lot of integer points. The purpose of this problem is to find out something about perfect ellipsoids and the number of integer points they much touch. An ellipsoid $E$ is the solution set of an equation of the form

$$f(x_1,x_2,x_3) = a_0 + \Sigma_{i=1}^{3} a_i x_i + \Sigma_{ij=1}^{3} a_{ij} x_i x_j = 0$$

where $a_{ij} = a_{ji}$ and the matrix $\{a_{ij}\}$ is positive definite; the interior of $E$ is the set where $f < 0$. Thus $E$ is empty if $f(z_1,z_2,z_3) \geq 0$ for all integers $z_1, z_2, z_3$; $E$ touches an integer point $(n_1,n_2,n_3)$ if $f(n_1,n_2,n_3) = 0$. Turning things around, this equality provides a linear equation which the coefficients, $a_0,a_1,a_2,a_3,a_{11},a_{12},\ldots,a_{33}$ must satisfy. Perfection is assured for $E$ if the integer points which $E$ touches provide a sufficient number of such equations to determine the coefficients of $f$. Since $f$ has 10 coefficients (determined up to a scalar multiple), $E$ must touch at least 9 integer points.

Of course we can look for perfect ellipsoids in any dimension. In one dimension an ellipsoid amounts to a pair of points; an empty ellipsoid is a pair of points which have no integer point lying in between. Any adjacent pair of integer points is an empty ellipsoid (A quadratic on $R$ has 3-coefficients). With a simple argument (involving the oddness and evenness of integers) it can be shown that there are no perfect ellipsoids in 2 and 3-dimensions. Can you find such an argument?

It is known that there are no perfect ellipsoids in $R^4$, but this fact was established using a complicated argument - it is conceivable, however, that a simple argument would work. Can you find one? In five dimensions the conjecture is - no perfect ellipsoids.

In 6 dimensions a perfect ellipsoid would have to touch at least 27 integer points (why?) and such an ellipsoid has been found! Consider the following problem: Find all of the integer solutions of the following pair of equations:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 - x_7^2 = 0$$

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 - 3x_7 + 2 = 0 .$$

If you can find all of these integer solutions (and prove that you have them all) you are well on your way to establishing the existence of perfect ellipsoids in 6-dimensional space!

**Bob Erdahl**

## The Convex Needle Problem

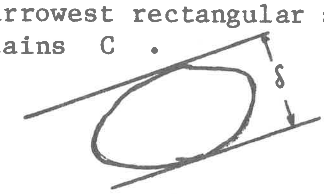
In the last issue we posed a convex version of the Kakeya-Besicovitch Needle Problem. It seemed like a nice little problem at the time, but it turned out to be non-trivial and of some historical interest. We chanced upon a reference to this very problem in Ball and Coxeter, p. 101, (Math. Recreations and Essays, 12th Ed. 1974; what does this book not contain!) to a 1921 paper in German. **Hans Kummer** agreed to read and understand this paper and distil its results for you. It involved him in other papers, equally German!, by Blaschke. I am most grateful for his efforts. It's really nice stuff, the sort of thing you don't get much of anymore. Here is Hans' report.

By an <u>oval</u> we mean a compact (bounded and closed) convex set in the plane. The following famous problem was posed early in this century by the Japanese mathematician Fujiwara:

**Find the oval of smallest area inside which a needle of unit length can be completely turned around** (without its endpoints ever leaving the oval!)

Fujiwara conjectured (as most of use would do after some thought!) that the minimum area is attained by the equilateral triangle of height 1, and therefore of side length $2/\sqrt{3}$ and of area $1/\sqrt{3}$. To prove the Fujiwara conjecture however is a non-trivial matter. A proof using some insights of W. Blaschke was designed in 1920 by the Hungarian mathematician Julius Pal [Math. Ann. 83, 311-319, 1921]. Here I outline for you Julius Pal's proof in some detail.

With any oval $C$ we will associate three important numbers. The first is the <u>area</u> $\alpha$ of $C$. The second is the <u>width</u> $\delta$ defined as the width of the narrowest rectangular strip (with parallel sides) that completely contains $C$.





The third is the <u>inner radius</u> $\rho$ defined as the radius of the largest circle (called the inscribed circle) that can be drawn inside $C$. These three numbers obey the three inequalities

$$\alpha \geq \pi\rho^2 \tag{1}$$
$$\delta \geq 2\rho \tag{2}$$
$$\delta \leq 3\rho \tag{3}$$

Now (1) and (2) are obvious, but (3) is quite difficult and is due to W. Blaschke (1915). At the same time Blaschke proved the following.

<u>Theorem A</u>

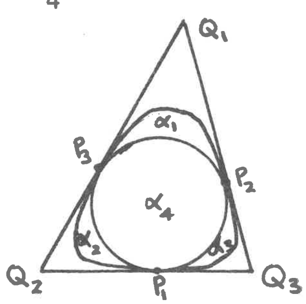
If $\delta > 2\rho$ then the inscribed circle is unique and it shares with the boundary of $C$ (at least) three points $P_1, P_2, P_3$ which form the vertices of an acute angled triangle (cf. Fig. 2). We will not discuss the proof of this.

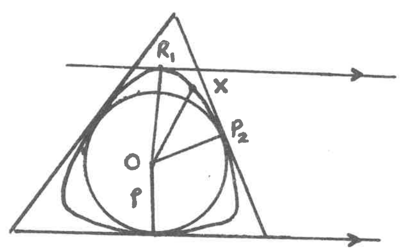
Now note that a needle of length $L$ can be turned around inside $C$ if and only if $\delta \geq L$. It follows that a $C$ of minimal area for a needle of length 1 must have $\delta = 1$. The following immediately establishes the conjecture of Fujiwara.

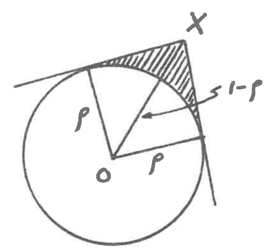
Theorem B

Suppose $C$ is an oval with $\delta = 1$. Then $\alpha \geq 1/\sqrt{3}$.

Outline of Proof. Define $r_0 = \sqrt{1/\pi\sqrt{3}} \simeq .43$. If $\rho \geq r_0$ then $\alpha \geq 1/\sqrt{3}$, by (1) and we are done. So henceforth we assume $\rho < r_0$. Since $\delta = 1$, $\rho < r_0 < \delta/2$ and Theorem A gives us three points $P_i$ at which the inscribed circle touches the boundary of $C$. Let $Q_1Q_2Q_3$ be the triangle formed from the tangents to the inscribed circle at the $P_i$. It can be argued that $C$ must lie inside this triangle (if it poked out we could move the circle slightly and make it bigger). Let $\alpha_i$ be the areas of the four parts of $C$ shown (below left). So $\alpha = \Sigma\alpha_i$, and $\alpha_4 = \pi\rho^2$. We now estimate $\alpha_1$.




First note (above centre) that $OR_1 \geq 1-\rho$ (since $\delta = 1$, the parallel lines are at least 1 unit apart). Since $OP_2 = \rho < 1-\rho$ (since $\rho<1/2$) there must be a point $X$ on the boundary of $C$ between $R_1$ and $P_2$ at which $OX = 1-\rho$ exactly. Now the hatched region (above right) has area (can you compute it?)

$$\beta_1 = \rho\sqrt{1-2\rho} - p^2 \arccos \frac{\rho}{1-\rho} .$$

and is contained in the $\alpha_1$ region, so $\beta_1 \leq \alpha_1$. A similar argument applies to $\alpha_2$ and $\alpha_3$. We get $\alpha = \Sigma\alpha_i \geq \pi\rho^2 + 3\rho\sqrt{1-2\rho} - 3\rho^2 \arccos \frac{\rho}{1-\rho}$.

Call the expression on the right $f(\rho)$. Then you can verify $f'(\rho) > 0$ on $(\frac{1}{3},\frac{1}{2})$ which implies that for $1/3 < \rho < r_0$,
$\alpha \geq f(\rho) > f(1/3) = 1/\sqrt{3}$
and we are done.

It is not too hard to argue, using the figures above, that the equilateral triangle is unique: if $\delta = 1$ and $\alpha = 1/\sqrt{3}$, then $C$ must be the equilateral triangle of height 1.

## A Financial Appeal

With the Department budget becoming increasingly hard-pressed we have been wondering over the past year whether we can really afford to continue to produce and send out 1700 copies of the Communicator every 6 months. One idea we had was to reduce costs by cutting circulation down to those who really wanted to receive it: indeed to those who were prepared to fill out and return a response which was included in the last issue (June 1984). We got some 200 responses back (from some 1100 alumni on our list), a percentage we consider quite good and we would like to thank all those who took the trouble (and cost) to respond. A number of these responses included cheques for $10-$20 as a contribution towards production costs and we were delighted to receive these.

Indeed, this gave us the idea to maintain full circulation for a few years, and try to subsidize the magazine with voluntary contributions. To cut circulation back drastically would mean losing touch with many alumni, some of whom are undoubtedly interested in the Department's activities, and are or may soon become, interested in our mathematical notes and articles.

We have decided to try this out. For the next few issues, each issue will contain a request for a voluntary contribution. It costs approximately one dollar to print and mail a single copy (labour costs not included) so periodic contributions of, say, $5 every two years or $10 every 4-5 years would certainly pay for issues received.

Along with your contribution, send along a piece of news, an opinion or a problem, and we'll put you in the next issue! All contributions gratefully received. Cheques should be made payable to The Communicator, Queen's University.