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SIR ISAAC NEWTON

In October 1987, Queen's hosted a celebration of the tercentary of the publication of the *Principia*. Symposium participants included: Stephen Smale (Field Medalist), Werner Israel (F.R.S., London), Sir Denys Wilkinson (F.R.S., London), and Stephen Weinberg (Nobel Laureate).

The seminars and lectures presented will appear in the Spring 1988 edition of the Queen's Quarterly, which may be obtained by writing the Campus Bookstore.

SOME PROBLEMS THAT CAN BE SOLVED USING THE PIGEONHOLE PRINCIPLE

BY D. DE CAEN

Prof. de Caen is one of the fortunate few Canadian mathematicians holding an N.S.E.R.C. University Research Fellowship. This award restricts his teaching to about half the usual load and frees him to pursue his research in discrete mathematics.

At a recent problem session for high school students, I presented the *Pigeonhole Principle* to them. As you may know, this principle asserts that no matter how we place $N + 1$ objects into N boxes, some box will receive at least two objects. It is rather surprising that this obvious statement has non-trivial consequences. In this note we shall present a number of examples. Most of these problems are well known, and I have not tried to trace the original sources.

To begin with, I would like to mention that the Pigeonhole Principle follows from a more general principle, the *Averaging Argument*. This asserts that given any numbers $x_1, x_2, \ldots, x_N$, then at least one of these numbers is at least as large as the average $(x_1+\ldots+x_N)/N$. Again, this is an obvious statement, but it has useful applications, even startling ones. Note that the Averaging Argument implies a more general form of the Pigeonhole Principle: if one arbitrarily places $A \cdot N + 1$ objects into N boxes, then some box receives at least $A + 1$ objects. Indeed, if we let x_i denote the number of objects in the i^{th} box, then the average $(x_1+\ldots+x_N)/N$ equals $A + 1/N$; thus some x_i is at least $A + 1/N$, and since x_i is an integer in this case, we see that x_i is at least $A + 1$.

I will now present a list of problems, each solvable by means of the Pigeonhole Principle or Averaging Argument. The classification I give into easy and hard is quite subjective, of course. I will then present some hints, solutions and comments. No peeking!

A. **Relatively Easy Problems**

1. Some two people in Ontario have the same number of hairs on their head. (This is true even excluding all the bald people!)

2. At any party, some pair of guests have the same number of acquaintances among those present.

3. Given a set of N integers, none of them divisible by N, then two of them have difference divisible by N.

4. Place the integers 1 to 10 at random around a circle. Show that there are three consecutive numbers whose sum is at least 17. Generalize!

B. **Intermediate Problems**

5. Two circular disks each have ten 0's and ten 1's in some arbitrary orders. Show that the disks can be superimposed so that at least ten positions have the same digit.

6. Among any six people, there is a set of three mutual acquaintances or a set of three mutual strangers. (Geometrical reformulation: Given six points in space in general position, if we arbitrarily color the fifteen segments joining pairs of points by two colors, say black and white, then there is necessarily a black triangle or a white triangle.)

7. Nine lattice points (i.e. points (x,y,z) with x, y and z integers) are taken arbitrarily in space. Then at least one of the segments joining pairs of points contains another lattice point.

8. Given a set S of $n + 1$ distinct integers chosen from $1,2,\ldots,2n$, show that S contains two distinct integers A and B such that A divides B.

C. **Difficult Problems**

9. Given any N integers, then some non-empty subset sums to an integer divisible by N.

10. Generalization of problem 6: there is, for every integer k, an integer $R(k)$ with the following property. Given at least $R(k)$ points in space in general position, if we randomly two-color the segments joining pairs of points, there will necessarily be k points with all $\frac{1}{2}k(k-1)$ segments between pairs of these k points having the same color.

11. Given any sequence of $N^2 + 1$ distinct integers, then there is an increasing subsequence of $N + 1$ terms or a decreasing subsequence of $N + 1$ terms. (Example: with $N = 3$ we are given the 10-sequence 7, 8, 9, 4, 5, 6, 1, 2, 3, 0. This contains the decreasing subsequence 7, 5, 2, 0, among others.)

12. Given m r-element sets $A_1,A_2,\ldots,A_m$ and m s-element sets $B_1,B_2,\ldots,B_m$. We assume that A_i and B_j are disjoint if and only if $i = j$; that is $A_i \cap B_i = 0$ for all i and $A_i \cap B_j \neq 0$ whenever $i \neq j$. Show that m is less than or equal to the binomial coefficient $\binom{r+s}{r} = \frac{(r+s)!}{r!s!}$.

We now proceed to outline some solutions. I hope that the references given will encourage a few readers to look further into this area of mathematics.

1. Here we need some outside information, namely that nobody has more than 100,000 hairs on their head, and Ontario has over six million inhabitants. Given this, it follows that there is a set of over 60 Ontarians with precisely the same number (67,311 perhaps) of hairs on their heads. This exercise serves to point out that the Averaging Argument is an *existential* argument, giving no method of finding an explicit set of equal-haired people. It would be rather pointless to try and do so; people shed hair all the time, and so the "solution sets" vary continually.

2. If N people are present at the party, then each guest has an "acquaintance number" of between 0 and $N-1$ inclusive. Since the acquaintance numbers 0, $N-1$ cannot both occur, then by the Pigeonhole Principle some two guests have the same number of acquaintances.

3. Hint: consider residues modulo N.

4. If $1,2,\ldots,N$ are placed around a circle, then the average sum of r consecutive integers is $\frac{1}{N} \cdot r \cdot (1+2+\ldots+N) = \frac{r(N+1)}{2}$. Thus, when $r = 3$ and $N = 10$, some three consecutive numbers sum to at least 16.5.

5. Hint: Consider one disk to be fixed. For each of the twenty ways of placing the second disk on top of the first, let x_i be the number of matches for the i^{th} placement. Show that $x_1 + x_2 + \ldots + x_{20} = 200$.

6. See the comments on problem 10.

7. There are eight possible odd-even patterns for a lattice point, for example (even, odd, even). Thus two of our nine points, say P and Q, have the same odd-even pattern. The midpoint $\frac{1}{2}(P+Q)$ is a lattice point.

8. The following elegant proof is due, I believe, to Esther Klein, many years ago. Every integer can be written as a power of two times an odd number, called the <u>odd part</u> of that integer. Among the numbers $1,2,\ldots,2n$, only n odd parts are available. Thus S contains integers A and B having the same odd part. But then either A or B divides the other.

9. Hint: If $x_1,x_2,\ldots,x_N$ are given, then consider $x_1, x_1+x_2, x_1+x_2+x_3, \ldots, x_1+x_2+\ldots+x_N$. If say x_1+x_2 and $x_1+x_2+x_3+x_4$ have the same residue modulo N, then the difference x_3+x_4 is divisible by N.

There is an interesting open problem generalizing this exercise. We consider k-tuples $\underline{a} = (a_1,a_2,\ldots,a_k)$ of arbitrary integers. Is it true that given any list of $k(N-1)+1$ k-tuples of integers, there must be a nonempty subset that sums to a k-tuple with all of its k coordinates divisible by N? A partial answer to this question is given in the paper of Alon, Friedland and Kalai, Journal of Combinatorial Theory B (1984), pp.79-91.

10. The assertion of this problem is known as Ramsey's theorem and was first proved by Frank Ramsey in 1930. In the past half-century an enormous body of work has evolved from this basic result. See the book "Ramsey Theory" by Graham, Rothschild and Spencer (Wiley 1980) for a detailed exposition of some of these developments. I might add that all proofs known of Ramsey's theorem use the Pigeonhole Principle in an essential way; and Ramsey's theorem itself may be viewed as a "higher dimensional" analogue of the Pigeonhole Principle. The Ramsey numbers $R(k)$ are in general unknown.

11. This result was first stated and proved by Erdös and Szekeres in 1935. I do not know who first came up with the following beautiful proof. For any integer x of the given sequence, let $\ell(x)$ be the length of a longest increasing subsequence starting at x. (In the example given earlier, $\ell(4) = 3$ and $\ell(2) = 2$.) Now if there is an x with $\ell(x) \geq N + 1$, we are done. In the opposite case, the function ℓ takes on at most N different values. Thus, by the Pigeonhole Principle, some $N + 1$ numbers have equal ℓ-value. It is not hard to see that these $N + 1$ numbers form a decreasing subsequence.

12. This problem is one of my favorite examples of how the Averaging Argument can be used to solve a difficult combinatorial problem. It may indeed require considerable imagination to see what the "objects" and "boxes" should be in a particular problem. It would take us too far afield to discuss the solution of this problem. I encourage the reader instead to look up the proof given as problem 13.32 in the book of L. Lovász "Combinatorial Problems and Exercises" (North-Holland 1979). This excellent text has many problems that are solvable using the Averaging Argument and other basic combinatorial principles.

In closing, a historical note. The number theorist Dirichlet was perhaps the first to use the Pigeonhole Principle in a non-trivial manner. There are many examples of the application of combinatorial principles to problems in Number Theory. See section 4.6 of Niven and Zuckerman "An Introduction to the Theory of Numbers" (Fourth edition, Wiley, 1980) for a good selection of such problems.

SOFTWARE FOR SOLVING DIFFERENTIAL EQUATIONS

BY N. RICE AND J. VERNER

The June 1986 issue described Microcomputers for Queen's Engineering. That article outlined the program which encourages each Engineering student entering first year to acquire an IBM-compatible micro-computer. A major aspect of this program is the development of software both for complementing traditional teaching techniques and for providing tools which may be used in professional activity.

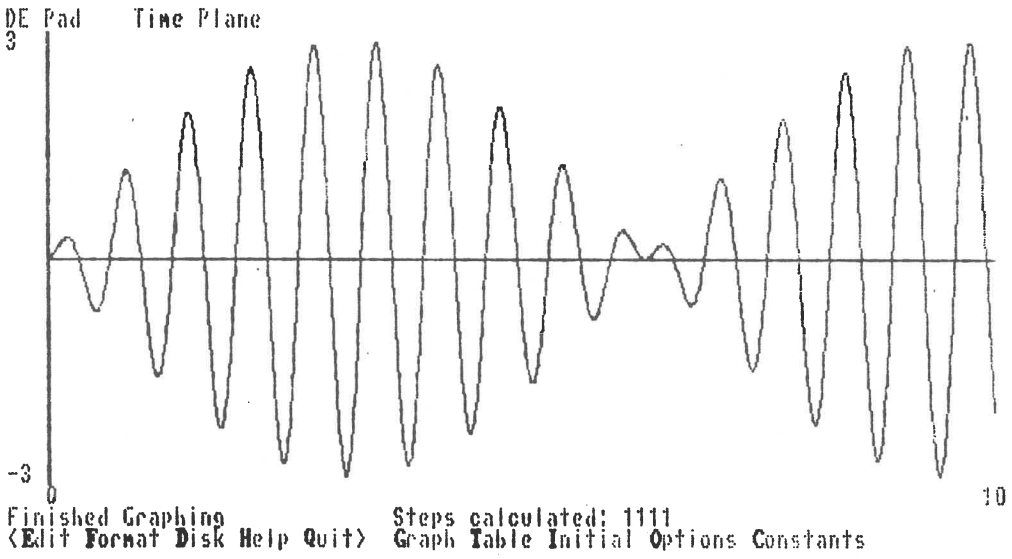
That issue and a sequel described Calculus Pad® and Matrixpad®. These are computing software tools for processing functions and matrices respectively. Members of the Department have continued the development of these and other tools. This fall a new package, DE Pad™ has been distributed to students for the first time.

DE Pad is designed for processing initial value problems in ordinary differential equations. Each second year engineering student takes a course in this subject. DE Pad builds on the basic design of Calculus Pad with similar but more flexible data entry, processing and graphical display.

After initiating the program by typing DEPAD, the user types in a (system of) differential equation(s). A typical equation is

$$Y'' + 81\ Y = 27\ \cos(10\ T)\ , \quad Y(0) = Y'(0) = 0\ .$$

DE Pad then calculates an approximate a solution of this equation, and plots the result:



The program allows the user to choose which variables are to be plotted and over what range. The program also provides a variety of supplementary facilities: for instance, functions, constants and comments may be entered; constants and initial conditions can be directly changed to study corresponding changes in a solution; and values of the solution may be displayed in a table.

To compare solutions derived by conventional techniques, the user may enter exact solutions, plot them, and hence compare them with DE Pad's solution.

Two other modules complement the basic structure of DE Pad. For any function entered, an approximate Fourier series of up to 9 terms can be calculated. Such a Fourier series may be graphed or subsequently used in the definition of a differential system. The solution of a DE with a forcing function can be compared to that with the corresponding Fourier series.

For differential systems based on time dynamics (for example, motion of a satellite under forces from the earth and moon) the solutions can be simulated in either two or three dimensions. The position of each body is determined by current values of X and Y (and Z) components of the system, and plotted on the screen. The user may thus "watch" the motion of the system in "real time".

Work on improving and extending this program is continuing. Because the hardware environment is always changing, a continuing effort to maintain and improve all the software already developed is necessary. Other software is planned or already under development; for instance, two separate projects for computing tools for Statistics were initiated last summer. Look to future issues for reports on further development.

NEW PROBLEMS

Brian Manning (B.Sc. '75), who assumed the position of Head of Mathematics at Adam Scott Collegiate Vocational Institute in Peterborough in 1983, observes that "there are numerous examples where the graphs of $y = \sin x$ (and $y = \cos x$) occur in connection with natural phenomena"; he wonders whether there are any natural occurrences of the graph of $y = \tan x$ (or other trigonometric functions). He notes that such occurrences might at first be thought unlikely because the tangent function is discontinuous; but there are discontinuities in nature. Note that what is wanted is examples where the graph of the tangent function appears; there is, of course, no shortage of illustrations of natural applications of the function itself. Can anyone help?

Louis Levine, who teaches Mathematics at Humberside C. I. in Toronto, is interested in the sequence 1, 2, 3, 5, 8, 12, 18, 27, 41,... whose n^{th} term is $x_n = \left\lceil \frac{3}{2} x_{n-1} \right\rceil$ where $\lceil x \rceil$ means the smallest integer greater than or equal to x ; in other words, $x_n = \frac{3}{2} x_{n-1}$ if x_{n-1} is even, $x_n = \frac{3}{2} x_{n-1} + \frac{1}{2}$ if x_{n-1} is odd. He would like a formula for x_n .

Herb Shank, currently visiting at Queen's, passes along the following problem. Let $f(x)$ be a polynomial with integer coefficients and of degree d . Suppose that there are at least $4d$ integers n such that $f(n)$ is either ± 1 or a (positive or negative) prime. Show that $f(x)$ is irreducible, that is, show that there is no factorization $f(x) = g(x)h(x)$ where $g(x)$, $h(x)$ are polynomials with integer coefficients and of positive degree.

OLD PROBLEMS

Charlie Small, currently on leave at McMaster University, sends the following comments on "Amending the Constitution" (problem in Summer 1987 issue). "The answers are 1093; 112, 380, 434, 167. Number-theory afficionados will recognize 1093. For any odd prime p , it's easy to prove that $2^{p-1}-1$ is divisible by p (in fact Fermat's "little theorem" says p divides $a^{p-1}-1$ for any a not divisible by p). It's known that if $2^{p-1}-1$ is not divisible by p^2 then the "first case" of Fermat's Last Theorem is true for p , that is, $x^p + x^p = z^p$ has no solutions in

integers x, y, z not divisible by p. Now 1093 is famous as the smallest prime p for which $2^{p-1}-1$ is divisible by p^2; 3511 is the only other, up to 3×10^9. It's known for other reasons that the first case of F.L.T. does hold for $p = 1093$ and 3511; the "p^2 doesn't divide $2^{p-1}-1$" criterion is sufficient, but not necessary, for the truth of F.L.T.Case1."

Jim Whitley's problem (minimize the total area A enclosed by N regular polygons with pre-specified side numbers $n_1, n_2, \ldots, n_N$ and total perimeter L) was solved both by himself and by James Hodder, M.Sc 1981, using Lagrange multipliers. It would be nice to have a direct proof, without calculus, that the minimizing polygons satisfy $L_i/A_i = L/A$ for each $i = 1,2,\ldots,N$.

MATH CONTEST WINNERS HOSTED BY QUEEN'S AND RMC

Each year the American High School Mathematics Exam (affectionately known as AHSME) is written by thousands of students throughout North America. The top thirty Ontario students are invited to a day-long seminar at some Ontario university, and this year it was the turn of Queen's and RMC to do the honours. Most of the winners this year came from the Toronto or Ottawa area, but there were a couple from right here in Kingston (and one who drove down from Georgian Bay with his girlfriend).

The sessions began Sunday evening (April 24) with **Jim Verner** giving the students some hands-on challenges with Calculus-Pad, the calculus graphics software that has been developed in the Department (and has been featured in previous Communicator articles). On Monday morning **Leo Jonker** and **Malcolm Griffin** each gave lecture/problem/workshop sessions designed to engage thirty mathematically quick, knowledgeable, aggressive, lively boys (no girls, unfortunately).

After lunch the students were taken in an army bus to RMC (I thought it would be good for them to walk over to RMC, but wiser heads prevailed). There they had a lecture by Queen's-grad-turned-RMC-prof **Peter Buckholtz**, and got a tour of some of RMC's impressive research facilities. After a banquet in the Senior Officer's Mess at RMC and a few games of billiards, etc. in the Games Room, they headed off home.

Norman Rice (who gets the credit for all the local organizing actually done by **Marge Lambert**).

AIM PROJECT MODULES FOR CLUB AND CLASS

The Mathematical Association of America (MAA) provides, free of charge, excellent modules on Applications in Mathematics (AIM) suitable for high school courses, club projects, career counseling, or independent study. Each AIM module consists of a video cassette, resource books for the student and for the teacher, and a computer diskette.

Available are:

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The problems are taken right from industry. Each video opens with an on-site interview with a person from the industry, continues with some hints and a career discussion, and concludes with a solution and a re-examination of the problem.

For a brochure write the MAA, 1529 Eighteenth St., N.W., Washington, D.C. 20036.

NEWS

Among the honorary doctorates being granted by Queen's this year is one to **Jeanne Le Caine Agnew**, Emeritus Professor of Mathematics at the University of Oklahoma. Dr. Agnew has received awards as Outstanding Teacher, Outstanding Woman and Outstanding Educator at Oklahoma.

While at Queen's she was influenced expecially by her Master's supervisor, Prof. Norman Miller (Communicator, January 1985). Assisted by a Marty Memorial Scholarship, she completed a Ph.D. thesis in difference equations at Harvard (Radcliffe) under the direction of G. D. Birkhoff.

Later, her teaching led to her main research interest, number theory. Her graduate supervision and instruction in that field resulted in the book 'Explorations in Number Theory' (Brooks, Cole 1972).

Currently, she is an editor of the College Journal of Mathematics and is working with the Mathematical Association of America on the development of learning modules in applied mathematics (AIM, this Communicator).

A native of Port Arthur, Ontario, Dr. Agnew still returns annually to the family cottage on Lake Superior.

Three Queen's students won NSERC 1967 (travelling) graduate scholarships this year: **Virginia de Sa** (Eng. Mathematics, Applied Science Gold Medalist), **Scott Wilson** (Mathematics Gold Medalist, Putnam team) and **Krishna Rajagopal** (Physics Gold Medalist, Putnam team).

Valued at $18,000 per year for up to four years, only 47 of these prestigious scholarships are granted nationwide each year.

We have now received more than $5000 in contributions to the **G. L. Edgett Scholarship fund**. It is expected that the conditions for the scholarship will be settled during the summer and that the first scholarship will be awarded in 1989.

THANK YOU

Our thanks to the many people who have sent donations to help keep the Coomunicator coming. Our cost is approximately $1800.00 for each issue. If you also would like to help, please send your cheque to the address below, payable to the Communicator, Queen's University.

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