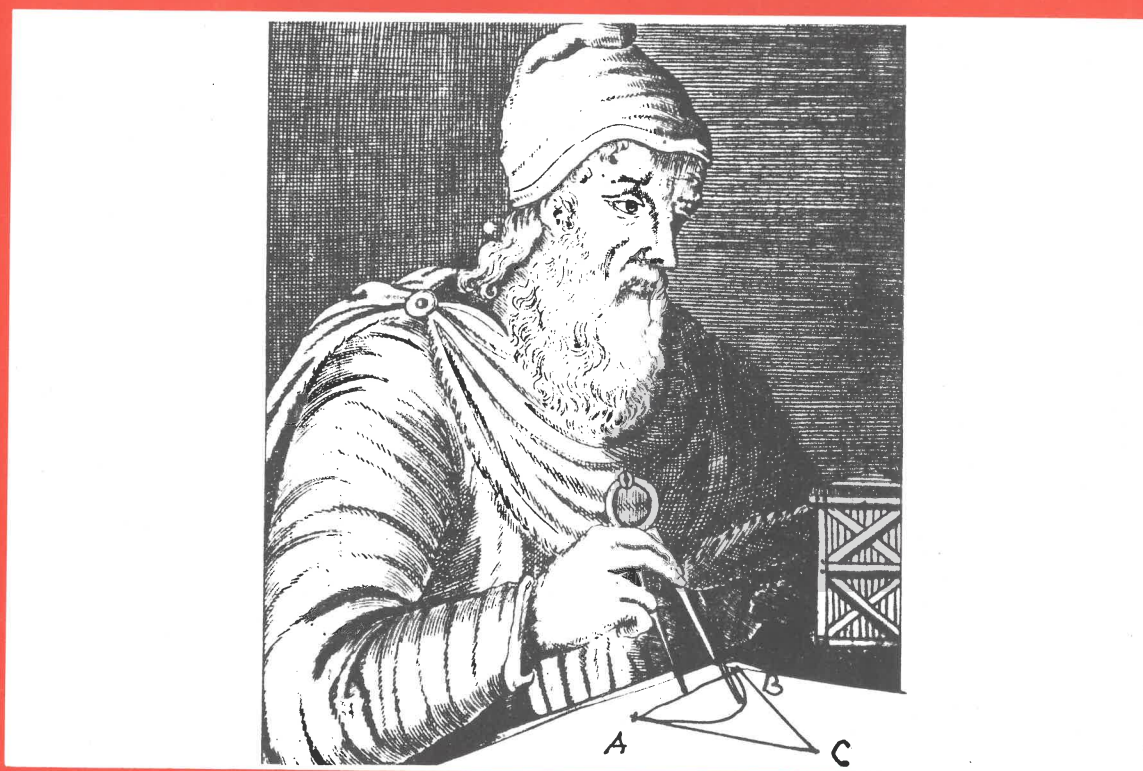


# QUEEN'S MATHEMATICAL COMMUNICATOR



SUMMER 1994



Archimedes Squaring A Parabolic Region

An aperiodical issued at Kingston, Ontario by the  
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Kingston, Ontario K7L 3N6

**QUEEN'S MATHEMATICAL COMMUNICATOR**  
**SUMMER 1994**

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**Address all correspondence, news, problems and solutions to:**

Queen's Mathematical Communicator  
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(Text of the Coleman-Ellis Lecture given January 13, 1994)

### Gem #1 — Squaring the Parabola

Of the three famous problems of antiquity — trisecting the angle, doubling the cube and squaring the circle — the last was shown to be impossible using standard Euclidean tools (unmarked straightedge and compasses) when Lindemann established the non-algebraic nature of  $\pi$  in 1882. (Given a unit length, the only numbers that are constructible using an unmarked straightedge and compasses are *algebraic*; that is, numbers that are roots of equations of the form

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x + a_0 = 0,$$

where the  $a_i$  are integers (not all zero), and  $n$  is a natural number. Squaring the circle would require that  $\sqrt{\pi}$  be constructible and therefore that  $\pi$  itself be constructible, contrary to Lindemann's result.)

Although squaring the circle is impossible, there are regions of the plane with curvilinear boundaries which *can* be squared. The region enclosed by the parabola and one of its chords is such an example. Around 240 BC, Archimedes related the area of this parabolic region to the area of the triangle formed by the chord and the tangents to the parabola at the end-points of the chord. With this information, a square with equal area can be obtained by Euclidean means. Archimedes' proof shows the high mathematical sophistication of the Alexandrian period of Greek mathematics.

It may be helpful to review a few facts about a parabola at this point. First, the parabola is the locus of a point P moving so that its distance from a fixed point F (the focus) is equal to its (perpendicular) distance from a fixed line, d, (the directrix). In Figure 1, the point P on the parabola lies on the perpendicular to d through the point H and on the right bisector of the segment HF. The right bisector of HF is a tangent to the parabola at the point P.

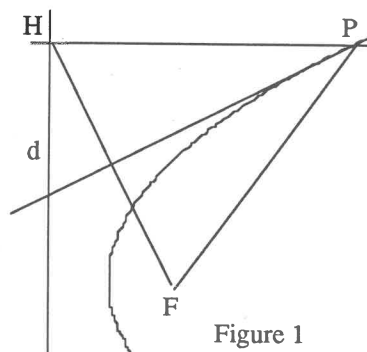


Figure 1

Figure 2, below, shows two right bisectors CA and CB (corresponding to the points H and K, respectively, on the directrix) both of which are tangents to the parabola (not shown) at the points A and B on the parabola. These two right bisectors of FH and FK meet at C which must therefore be the circumcentre of triangle HFK.

Figure 3 shows a family of parallel chords of the parabola along with the line connecting the mid-points of the chords. This line is known as the *diameter* of the parabola for the particular family of parallel chords with that given slope. The diameter is always parallel to the axis of symmetry of the parabola. The tangent to the parabola at the point where the diameter meets the parabola is parallel to the family of chords associated with the diameter. (Think of the tangent as the limiting position of the family of chords.) It is also true that every line parallel to the axis of the parabola is a diameter for a set of parallel chords of some particular slope.

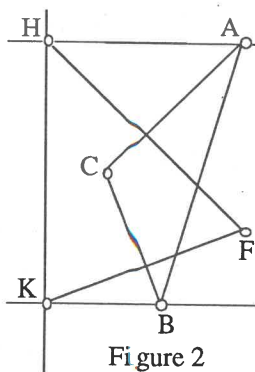


Figure 2

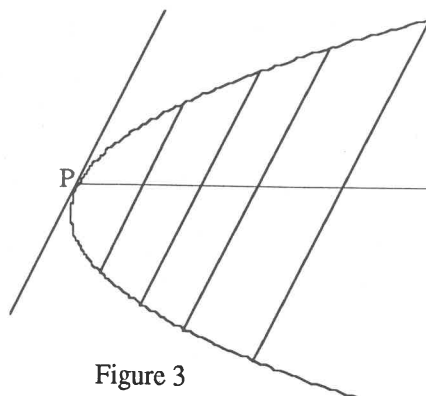


Figure 3

Figure 4 shows the region bounded by the parabola and one of its chords and the triangle ABC formed by the chord and the tangents to the parabola drawn at the extremities of the chord. Such a triangle is known as an Archimedes triangle and the chord is referred to as the base of the triangle.

The development below illustrates how Archimedes showed that this parabolic region itself, as well as its complement relative to the Archimedes triangle, can be tiled with triangles in such a way that to each triangle in the parabolic region there is a corresponding triangle in the complement with half the area.

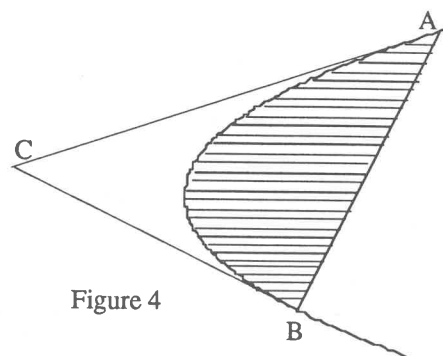


Figure 4

In Figure 5, since C is the circumcentre of triangle HKF, the perpendicular to the directrix HK through C must also bisect HK — here, at the point T. The line TC is a mid-line of the trapezoid HABK and hence TC bisects AB at U. Therefore, CU is a median of triangle ABC and we have the result that, in general, *the median to the base of an Archimedes triangle is parallel to the axis of the parabola*. This result will be used again. Also, since U is the mid-point of AB and TU is parallel to the axis of the parabola, TU is a diameter for the set of chords parallel to AB.

The point W at which the diameter TU meets the parabola is the intersection of TU and the perpendicular bisector of TF. This right bisector, VW, is tangent to the parabola at W and hence is parallel to the base AB of the Archimedes triangle CAB. The tangent VW meets sides CA and CB of the Archimedes triangle at  $A_1$  and  $B_1$  respectively.

Since  $A_1A$  and  $A_1W$  are tangents to the parabola, triangle  $AWA_1$  is an Archimedes triangle and by the earlier result, if Z is the mid-point of AW, then the median  $A_1Z$  is parallel to the axis of the parabola. In particular,  $A_1Z$  is parallel to CW and hence  $A_1$  is the mid-point of CA.

Of course, the same observations can be made concerning the Archimedes triangle  $BWB_1$  with the corresponding result that  $B_1$  is the mid-point of CB. Also, since  $A_1B_1$  is parallel to AB, the point W on the parabola is the mid-point of CU.

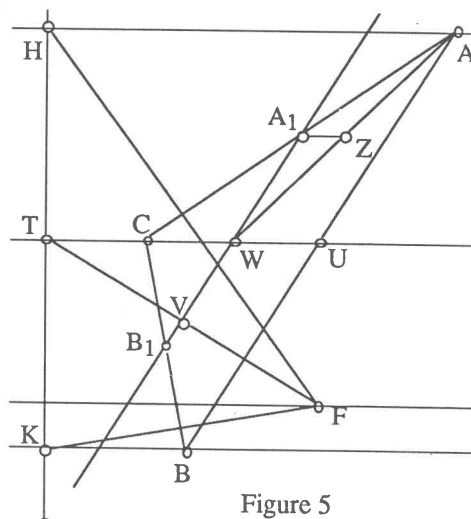


Figure 5

Figure 6 shows that the tangent  $A_1B_1$  and the chords WA and WB divide the triangle ABC into four sections: triangle  $A_1B_1C$  lying outside the parabola; triangle ABW lying entirely inside the parabola, and two residual Archimedes triangles  $AWA_1$  and  $BWB_1$  through which the parabola passes.

Triangles  $CWA_1$  and  $CUA$  are similar and have bases CW and CU (and hence altitudes from  $A_1$  and A respectively) in the ratio of 1:2. Therefore, triangles  $CWA_1$  and  $CUA$  have equal bases CW and CU and altitudes to those bases in the ratio 1:2. Similar observations hold for triangles  $CWB_1$  and  $CUB$ .

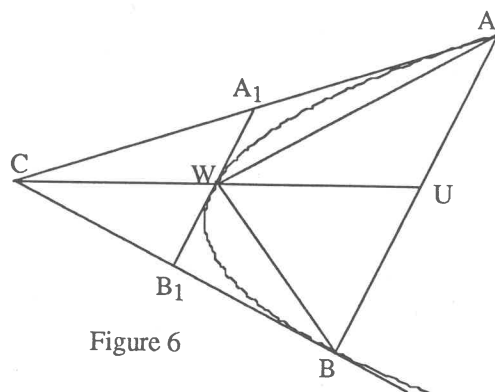


Figure 6

Therefore, the internal triangle ABW has twice the area of the external triangle  $A_1B_1C$ . In the same way, each residual triangle (like the two residual Archimedes triangles  $AWA_1$  and  $BWB_1$ ) gives rise to an internal triangle, an

external triangle and two residual Archimedes triangles. In every such Archimedes triangle, the internal triangle has twice the area of the external triangle.

By continuing in this fashion, the original Archimedes triangle ABC can be covered with internal and external triangles with the area of each internal triangle always twice that of the corresponding external triangle. Thus the area of the parabolic section is two-thirds of the area of the Archimedes triangle ABC. It is now a routine matter to construct a square equal in area to two-thirds of the area of the Archimedes triangle.

## Gem #2 — Wythoff's Game

Wythoff's game is a game similar to Nim for two players playing alternately. The game begins with two piles of counters; the number of counters in each pile is arbitrary. On any turn, a player must:

- (a) remove a non-zero number of counters from one pile (but only one pile), or
- (b) remove an equal number (non-zero) of counters from both piles.

The player who removes the last counter (or counters) is the winner.

There are certain safe combinations; that is, safe to leave on the table to ensure winning. For example, the unordered pair (1,2) is such a safe combination since your opponent's only options are to reduce (1,2) to one of (1,0), (1,1) or (2,0) none of which is a winning position and from each of which it is possible for you to win on your next turn.

Clearly, (1,k) for  $k \geq 3$  and (2,m) for  $m \geq 2$  are unsafe since they can be reduced to (1,2) (or (2,1) which is the same thing.) The next safe combination is (3,5). In seeking other safe combinations, it is necessary to eliminate combinations of the form (k,k+1) and (m,m+2) since these can be reduced to (1,2) and (3,5) respectively. Further investigation yields the following sequence of safe pairs:

$$(1,2), (3,5), (4,7), (6,10), (8,13), (9,15), \dots$$

The sequence is clear. The nth safe pair has components differing by n and the smaller natural number in the pair is the smallest natural number not already used in previous pairs.

To restate the problem, the set of safe pairs must satisfy two conditions:

- 1) the nth safe pair has components differing by n
- and 2) every natural number must appear once and only once as a component.

The attempt to find a general formula for a safe pair begins by finding a set of pairs satisfying the first condition stated above. Suppose x and y are positive irrational numbers and  $y = x + 1$ . Then x and y can be expressed as

$$y = k + 1 + f \quad \text{and} \quad x = k + f$$

where k is a non-negative integer and  $0 < f < 1$ . (In this representation k and k+1 can be thought of as the integer parts of x and y.) More formally,  $[y] = k + 1$  and  $[x] = k$ , where  $[x]$  means the greatest integer less than or equal to x. Since we're dealing with positive x and y, we can treat  $[x]$  as the integer part of x. Also if n is a natural number, then

$$ny = n(k + 1) + nf \quad \text{and} \quad nx = nk + nf$$

Since nf can be expressed as  $t + \epsilon$  where t is a non-negative integer and  $0 \leq \epsilon < 1$ , it follows that

$$ny = n(k + 1) + t + \epsilon \quad \text{and} \quad nx = nk + t + \epsilon.$$

$$\begin{aligned} \text{Consequently, } [ny] - [nx] &= n(k + 1) + t - (nk + t) \\ &= n \end{aligned}$$

That is  $[ny]$  and  $[nx]$  differ by n and hence the set of pairs  $\{([nx], [ny])\}$  satisfies the first condition required of the safe pairs in the solution of Wythoff's game.

The other condition demands that all natural numbers be used as components of the safe pairs and that no natural number be used more than once.

To satisfy this second condition and complete the solution of Wythoff's game, the result of a problem posed by Sam Beatty in the 1926, Vol. 33 edition of the American Mathematical Monthly will be used. This problem gives a surprising result — often referred to as "Beatty's Theorem" in problem solving circles. The statement given below of this theorem is equivalent to, but slightly different from that presented in Vol. 33 of the American Mathematical Monthly (Beatty's appearance in a Coleman-Ellis Lecture is appropriate; he was one of John Coleman's teachers.)

### Beatty's Theorem

Let  $x$  and  $y$  be positive irrational numbers such that  $x^{-1} + y^{-1} = 1$ . Then every natural number appears exactly once in the set  $\{[nx], [ny]\}$ , that is, every natural number appears exactly once among the numbers

$$[x], [2x], [3x], \dots, [y], [2y], [3y], \dots$$

An example or two might help emphasize this amazing result. Suppose that  $x = \sqrt{5} \approx 2.2360678$ . Then since

$\frac{1}{x} + \frac{1}{y} = 1$ ,  $\frac{1}{y} = 1 - \frac{1}{x} = \frac{x-1}{x} = \frac{\sqrt{5}-1}{\sqrt{5}}$  and hence  $y = \frac{\sqrt{5}}{\sqrt{5}-1} \approx 1.809017$ . The theorem says that the multiples of  $x$  and  $y$  have integer parts which comprise every natural number exactly once. The table below illustrates the theorem for these values of  $x$  and  $y$ .

<u>Multiples of x</u>	<u>Multiples of y</u>	<u>The integer parts of x and y</u>	
$x = 2.236068$	$y = 1.809017$	2	1
$2x = 4.472136$	$2y = 3.618034$	4	3
$3x = 6.708204$	$3y = 5.427051$	6	5
$4x = 8.944272$	$4y = 7.236068$	8	7
$5x = 11.180340$	$5y = 9.045085$	11	9
$6x = 13.416408$	$6y = 10.854102$	13	10
$7x = 15.652476$	$7y = 12.663119$	15	12
$8x = 17.888544$	$8y = 14.472136$	17	14
$9x = 20.124612$	$9y = 16.281153$	20	16
$10x = 22.360680$	$10y = 18.090170$	22	18
$11x = 24.596748$	$11y = 19.899187$	24	19
$12x = 26.832816$	$12y = 21.708204$	26	21
$13x = 29.068884$	$13y = 23.517221$	29	23
$14x = 31.304952$	$14y = 25.326238$	31	25
$15x = 33.541020$	$15y = 27.135255$	33	27

For a second example, suppose that  $x = 1 + e \approx 3.7182812$ . Solving for  $y$  gives  $y = \frac{1+e}{e}$ , or equivalently,  $y \approx 1.367879$ . The table below illustrates the theorem for these values of  $x$  and  $y$ .

<u>Multiples of x</u>	<u>Multiples of y</u>	<u>The integer parts of x and y</u>	
$x = 3.718282$	$y = 1.367879$	3	1
$2x = 7.436564$	$2y = 2.735759$	7	2
$3x = 11.154845$	$3y = 4.103638$	11	4
$4x = 14.873127$	$4y = 5.471518$	14	5
$5x = 18.591409$	$5y = 6.839397$	18	6
$6x = 22.309691$	$6y = 8.207277$	22	8
$7x = 26.027973$	$7y = 9.575156$	26	9
$8x = 29.746255$	$8y = 10.943036$	29	10
$9x = 33.464536$	$9y = 12.310915$	33	12
$10x = 37.182818$	$10y = 13.678794$	37	13
$11x = 40.901100$	$11y = 15.046674$	40	15
$12x = 44.619382$	$12y = 16.414553$	44	16
$13x = 48.337664$	$13y = 17.782433$	48	17
$14x = 52.055946$	$14y = 19.150312$	52	19
$15x = 55.774227$	$15y = 20.518192$	55	20

Tables like the ones above are easily prepared on a spreadsheet.

The following proof of Beatty's problem was devised jointly by J. Hyslop in Glasgow and A. Ostrowski in Göttingen and appears on page 159 of the March, 1927, Vol. 34 edition of the American Mathematical Monthly.



For a given positive integer  $N$ , the number of members less than  $N$  of the sequences

$$x, 2x, 3x, 4x, \dots \quad \text{and} \quad y, 2y, 3y, 4y, \dots$$

are, respectively,  $\left[ \frac{N}{x} \right]$  and  $\left[ \frac{N}{y} \right]$ .

Now,  $\frac{N}{x} = \left[ \frac{N}{x} \right] + f_1$  where  $0 < f_1 < 1$ , and  $\frac{N}{y} = \left[ \frac{N}{y} \right] + f_2$  where  $0 < f_2 < 1$ . (Notice the end points 0 and 1 have been excluded since  $\frac{N}{x}$  and  $\frac{N}{y}$  cannot be rational and hence cannot be integral.)

Now, from the relationships above,

$$\begin{aligned} \left[ \frac{N}{x} \right] + \left[ \frac{N}{y} \right] &= \frac{N}{x} + \frac{N}{y} - (f_1 + f_2) \\ &= N \left( \frac{1}{x} + \frac{1}{y} \right) - (f_1 + f_2) \\ &= N - (f_1 + f_2), \text{ since } \frac{1}{x} + \frac{1}{y} = 1. \end{aligned}$$

But  $\left[ \frac{N}{x} \right]$ ,  $\left[ \frac{N}{y} \right]$  and  $N$  are all integers. Therefore  $f_1 + f_2$  must also be an integer, and must satisfy the condition  $0 < f_1 + f_2 < 2$ . Therefore,  $f_1 + f_2 = 1$ .

Thus the number of members of the set  $\{x, 2x, 3x, 4x, \dots, y, 2y, 3y, 4y, \dots\}$  less than  $N$  is  $N - 1$ . That is, there are no members less than 1, one member less than 2, two members less than 3, and so on. This means that there is exactly one member of the set between each pair of positive integers, and hence the integer parts  $[nx]$  and  $[ny]$  must be the natural numbers themselves.

So to summarize, if  $x$  and  $y$  are positive irrationals such that  $y = x + 1$ , then the  $n$ th ordered pair in the set  $W = \{([nx], [ny]) \mid n \text{ is a natural number}\}$  has components differing by  $n$ ; and if we further add the condition that  $x^{-1} + y^{-1} = 1$ , the set  $W$  consists of ordered pairs whose components are natural numbers, each natural number appearing once and only once. These are precisely the conditions for the safe pairs in Wythoff's game.

Solving  $y = x + 1$  and  $\frac{1}{x} + \frac{1}{y} = 1$ , gives  $x = \frac{1 + \sqrt{5}}{2} = \tau$  (the Golden Section!), and  $y = \frac{3 + \sqrt{5}}{2} = \tau^2$ . Hence the  $n$ th safe combination in Wythoff's game is  $([n\tau], [n\tau^2])$ .

## References

### Gem #1 — Squaring the Parabola

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### Gem #2 — Wythoff's Game

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After completing his B.Sc. and M.A. in Mathematical Statistics at the University of Toronto, Hugh Allen taught secondary school mathematics for five years. In 1968 he went to the University of Waterloo as a lecturer in the Faculty of Mathematics while he completed his Ph.D. in Mathematics (Biometry). He has been teaching Mathematics Education at Queen's since 1971. He is the co-author of mathematics textbooks and has published numerous articles on mathematics education. At present he is working on a collection of "gems" and a book on hidden symmetries in geometric forms. He enjoys music, the outdoors and, particularly, his approaching retirement.

## MATHEMATICS AND POETRY

*Peter Taylor*

Some of you may know that my long-time (10 years) Math and Poetry co-teacher Bill Barnes died some 18 months ago of a heart attack. For the past few years, Bill had been gamely fighting a whole host of diabetes-related afflictions, and at the time of his death, having lost both legs to infections, he was confined to a wheelchair and reporting to KGH three times a week for dialysis. Bill was never one to find it easy being confined, and his sense of frustration and courageous acceptance was displayed in much of his poetry and enriched his teaching.

In fact the following poem of his is one of my favorites, for a whole host of reasons, and I used it in the first class I taught without him, along with a sonnet of Shakespeare ("That time of year thou mayest in me behold"). Last fall, in his wheelchair, he was often "cramped and awkward," and there were lots of cut edges to prick his flesh – he was always scraping the backs of his hands as he wheeled himself, just missing, through the heavy Watson Hall doors. And the last stanza I find wonderful – "the guts of friends are ropes paid out to let you down and set you free." Notice the ambiguity in the phrase, "let you down." The poem is about St. Paul's escape from Damascus.

### ESCAPE ARTISTS

(2 Cor. 11:33)

No flying carpet, this, no freakish bald  
giant coaxed from a lamp to execute  
the three commands, no magic rod to pluck  
you out of this new bind and whisk you away

Instead, an ordinary basket made of reeds  
(a kind of truck? a clothes-hamper?) stands near  
at hand and does the trick. The strands are woven,  
criss-crossed for strength, but there's no more than room  
enough for even your small body, cramped  
and awkward, as the others tie down the lid,  
the cut edges of reeds pricking your flesh.

But it works: under darkness, through the thin  
window, the guts of friends are ropes  
paid out to let you down and set you free.

W. J. Barnes

July 1986

In 1992, Bill wrote the Queen's Sesquicentennial Hymn and both the music and the lyrics are quite wonderful. Shortly before he died he received a 3M teaching award, the first person at Queen's ever to be so honoured.

I feel enormously privileged to have been able to "share a lectern" with him for so many years. I put the phrase in quotes because neither of us managed to stay very close to the lectern; during Bill's (physically) fragile times (even when emotionally fragile, he always managed to rise for the class) it was amusing to watch him grab his stick or crutches and lever himself over to the board to draw a picture ("let's play hangman") because the class was slow to see what was patently obvious!

Over the years we learned a lot from one another's methods and styles. He gave me a new set of standards for engaging a large class in dialogue - there is enough development work in that to keep me busy for the rest of my teaching life.



By the way: David Helwig and I have produced a book of Bill's poetry and music; it is published by Quarry press and will appear this fall. If you are interested in obtaining a copy, please let me know.

## WHAT CAN ONE ACHIEVE WITH A DEGREE IN MATHEMATICS OR STATISTICS?

### An Invitation To Our Former Graduates

A frequent question one hears from gifted first year students is: "I like Mathematics but what can I do later on with a degree in Math besides teaching?" To this question we usually give a vague answer such as: "There are many employers who go for Math graduates, since having been able to acquire a degree in Math or Stats shows that you are good at problem solving and logical thinking. Consequently many of our graduates have been able to create for themselves successful careers in industry and government."

It occurred to us that it would be nice to be able to make the answer more specific by pointing to real persons who were able to use their degree in Mathematics or Statistics as a basis for their professional career. Therefore, dear Queen's graduate in Mathematics or Statistics, we invite you to introduce yourself to the *Mathematical Communicator* and describe to us your professional career and how your degree in Mathematics or Statistics helped you to get started. We would be delighted if we could fill the next issue of the *Communicator* with letters from you containing a description of your particular career and how your degree helped you to reach the position you now occupy! Please send your letter to:

*Queen's Mathematical Communicator*  
*Department of Mathematics and Statistics*  
*Queen's University*  
*Kingston, Ontario K7L 3N6*

## NEWS OF GRADUATES

Vijay Bhargava (Math. & Eng. 1970) has won the 1993 B.C. Science and Engineering Gold Medal in Engineering and Applied Sciences from the Science Council of British Columbia, for his research and innovative methods and techniques in improving wireless communication.

Ian B. MacNeill (M.A. 1963) is one of two Canadians to be elected Fellow of the Institute of Mathematical Statistics, the foremost international organization fostering the development and application of mathematical statistics and probability theory.

## A YEAR IN THE LIFE OF THE DEPARTMENT

*Leo Jonker*

If necessity is the mother of invention, we are in for exciting times in the Department of Mathematics and Statistics. While in the early 1980's the Department had as many as 48 tenured faculty members, that number is now down to 38! This year we are losing three more of our faculty members through retirement. Norman Pullman, Cedric Schubert and Madanlal Wasan are no doubt remembered by many of our readers for their fine teaching. Norm Pullman over the years built up a very fine research group in the area of Combinatorics and Discrete Mathematics. Recently, his contribution to the subject was recognized by the publication of a Festschrift in his honour

on the occasion of his 60th birthday. Cedric Schubert's specialty is analysis, particularly partial differential operators. He coordinated graduate studies within the Department for a number of years, and later became chairman of the science division in the School of Graduate Studies. Wasan, an expert in stochastic processes has published a number of books during his distinguished career. Along with Professor Edgett, Wasan has been extremely influential in creating the Statistics group in the Department. When he arrived at Queen's University he and Professor Edgett were our only Statisticians! Though all three retirees will undoubtedly remain involved in some of the Department's activities, we will miss their teaching. While the faculty reduction was taking place, especially over the last 6 years, the number of students taught has not gone down, and at the same time there has been a marked increase in the demands put on the Department to be engaged in original research, and in fundraising to offset the increasing unavailability of research support.

Fortunately, through the skillful work of Grace Orzech, who continues as chair of undergraduate studies, we have been able to rationalize our teaching effort so that in spite of the faculty cutbacks the average teaching load has actually gone down. Gone are the first year classes of 30 or 40 students that were the rule twenty years ago. One of the sections of mathematics 121 is now taught in Dunning Hall auditorium to 420 students at once! In fact in many of the big introductory courses at Queen's class size is limited only by the unavailability of good large lecture halls.

While freshmen are undoubtedly intimidated initially when they find themselves packed in a lecture hall with 420 others, the Department (with the help of Queen's Instructional Development Centre) has been remarkably successful in maintaining the quality of education and even improving it. To ensure this success, mounting a course such as Calculus 121 now requires a major organizational effort, involving a team of faculty and graduate students who provide help outside the classroom. We believe that as a result of the restructuring students now find it easier rather than harder to get help. As well, a lot has been done over the last few years, at Queen's and elsewhere, to rethink the undergraduate mathematics curriculum. Peter Taylor's problem oriented approach to Calculus has gained attention throughout North America and has been an inspiration to many leaders in the so-called Calculus Reform movement now sweeping the continent.

This summer, under the leadership of Grace and Morris Orzech many of our first year courses are undergoing further reorganization. Courses will continue to become more problem oriented and will engage the student more fully. We are rethinking the way we handle assignments and tests, and we are designing ways to help students use computers to learn and use mathematics.

Another goal in our efforts to rethink the curriculum is to see an increase in the number of students continuing in a major or medial program in mathematics or statistics. At the moment these numbers are far too small. While this ensures that students in their third and fourth years will experience the small and friendly classes they missed in their first, the situation decreases the Department's efficiency. Since the problem of attrition is common everywhere in Canada (and the USA) it has serious implications for the future of science and industry in Canada. To enhance our program in the upper years we have introduced new courses at the second and third year levels.

As further evidence that the Department is concerned to provide high quality instruction, I am proud to say that two of the three Golden Apples this year were awarded in Mathematics and Statistics. Both Ron Hirschorn and Wen Cebuhar were selected for this honour by the students of the Faculty of Applied Science. At the same time, Jon Davis and Ron Hirschorn, with the help of Mathematics and Engineering student Serge Mister, are building a sophisticated control theory laboratory which will be used to enrich the study programme for students in Mathematics and Engineering.

Lest I give the impression that teaching is all we do, let me also say a little about our research activity.

As in the past few years, we had a rich variety of long term visitors. Dr. V. Futorny (Ukraine), continued his work with the Lie Algebraists, Dr. Y. Ledyayev (Russia), works in Control Theory

and Differential Inequalities, Dr. R. Michler (Germany), in Algebraic Geometry, Dr. C. Lee-Shader (USA), in Lie Algebra, Dr. W. Oldford (Canada), in Statistics, Dr. Y. Pitteloud (Switzerland), in Algebraic Geometry, Dr. B. Shader, (USA), in Combinatorics, Dr. Y. S. Shin (Korea), in Algebraic Geometry, Dr. L. Yukalov (Russia), in Numerical Analysis, and Dr. V. Yukalov in Mathematical Physics. Sir M. Atiyah visited the Department for a few days in the spring, and was awarded an honorary degree by the University.

## OLD PROBLEM

**Solution to Taylor's Divisibility Problem**, received from Thomas Cook, Graduate Student, Department of Mathematics & Statistics, Queen's University. Also solved by Doug Dillon, Head of Mathematics, Athen's District High School.

**Problem:** The number 374,625 has a remarkable property. Not only is it divisible by 37, but if I permute the first three digits in any way, and permute the last three digits in the same way, the resulting number is also divisible by 37. For example, 734,265 is divisible by 37. (a) Find all 6-digit numbers with this property. (b) What is the corresponding problem for 8-digit numbers?

**Solution:** Let  $x : y$  be " $x$  modulo  $y$ ", noting that  $x : y = 0$ , is a shorthand for " $x$  is divisible by  $y$ ". Since  $1000 : 37 = 1$ , a number of the form  $a00b$ , with  $a$  and  $b$  representing digits in the number, will have the property that  $a00b : 37 = a + b$ . Thus, by expressing a general six digit number  $abcdef$  in the form  $100(a00d) + 10(b00e) + c00f$ , the following are equivalent:

$$abcdef : 37 = 0$$

$$(100(a + d) + 10(b + e) + 1(c + f)) : 37 = 0$$

and (since  $100 : 37 = 26$  and ":" comfortably distributes through addition and multiplication),

$$(26(a + d) + 10(b + e) + (c + f)) : 37 = 0 \quad (1)$$

Let  $\alpha = a + d$ ,  $\beta = b + e$  and  $\gamma = c + f$ , noting that the permutations of digits allowed do not change the values of  $\alpha$ ,  $\beta$  and  $\gamma$ . Substituting into (1) gives:

$$(26\alpha + 10\beta + \gamma) : 37 = 0 \quad (2)$$

First note that all triples with  $\alpha = \beta = \gamma$  satisfy (2). Conversely, since we may permute the positions of the pairs  $a$  and  $d$ , etc. and still have a number divisible by 37 (as the problem states), the roles of  $\alpha$ ,  $\beta$  and  $\gamma$  may be permuted in equation (2). Use this fact to obtain the following equation by permuting  $abcdef$  to  $bacedf$ :

$$(26\beta + 10\alpha + \gamma) : 37 = 0 \quad (3)$$

Subtract equation (3) from equation (2) to get:

$$(16\alpha - 16\beta) : 37 = 0 \quad (4)$$

Since 16 is relatively prime to 37, this gives

$$(\alpha - \beta) : 37 = 0 \quad (5)$$

And since  $\alpha$  and  $\beta$  are non-negative, but no greater than  $18 (= 9 + 9)$ , the only solution to (5) is  $\alpha = \beta$ . Similarly, by using the permutation  $abcdef$  to  $cbafed$ , it can be shown that  $\alpha = \gamma$ .

Thus a number  $abcdef$  satisfies the conditions of the problem iff  $(a + d) = (b + e) = (c + f)$ .  $\square$

For the eight digit analog, we use 101, since it is a prime dividing  $1111 (= 1000 + 100 + 10 + 1)$  in the same way that 37 divides  $100 + 10 + 1$  and  $10000 : 101 = 1$  in the same way that  $1000 : 37 = 1$ .

### NEW PROBLEM

*Peter Taylor*

I have to fly from  $A$  to  $B$ , at some distance due south, and back again. Assuming a fixed air speed, rank order the following options according to total time of transit.

- (a) No wind
- (b) A north wind of 5 knots
- (c) A west wind of 5 knots
- (d) An east wind of 7 knots.





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