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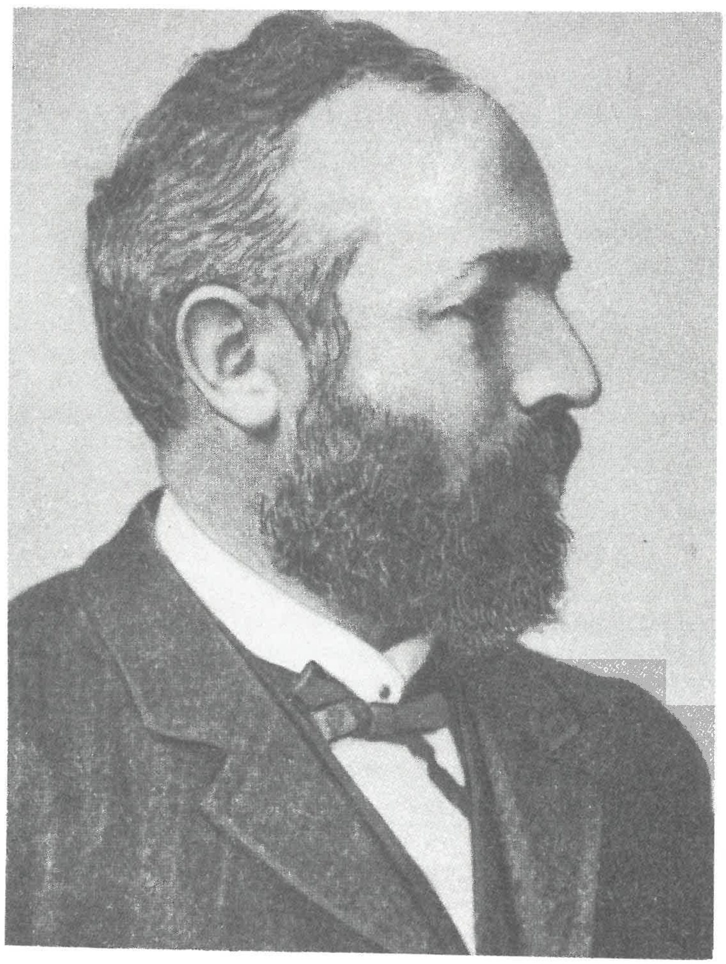

The Essence of Mathematics Lies in its Freedom
Georg Cantor

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Set Theory: Some of its Early History, Paradoxes, and Philosophy

Ole A. Nielsen

This essay is based on a Coleman-Ellis lecture given by the author in March, 1996.

Set theory is a part of mathematics but is different from the other parts of mathematics. It is a relatively recent theory, with its origins being only a little over one hundred years old. The foundations of mathematics are rooted in set theory and logic and yet the knowledge undergraduates have of set theory is typically acquired unsystematically in other courses as it is needed there or picked up in extra-curricular reading. All of this makes set theory a good topic for an essay on mathematics suitable for non-specialists.

Early History

Georg Cantor (1845–1918) was a student in Berlin in the late 1860's, where he studied number theory under such well-known professors as Ernst Kummer (1810–1893), Leopold Kronecker (1823–1891), and Karl Weierstrass (1815–1897). After he completed his studies he obtained a position at the university in Halle (in central Germany, near Leipzig). There he was somewhat isolated although Eduard Heine (1821–1881, of the Heine-Borel theorem) was there. Heine was an analyst and he persuaded Cantor to turn his attention from number theory to analysis. At that time the major problems in analysis revolved around integration and Fourier series. Fourier series were important because of their connections with heat conduction problems and integration was important partly for its own sake and partly because the formulae for the coefficients of a Fourier series involved integration. Remember that at this time (the early 1870's) the Riemann integral was not completely understood and the Lebesgue integral would not be introduced for another 30 years.

Cantor did not turn to Fourier series themselves but, instead, to trigonometric series. A trigonometric series is an expression of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta.$$

Such an expression need not converge for all values of θ but is periodic with period 2π. So in considering such a series it is only necessary to consider $\theta's$ in the interval $[0, 2\pi)$. The first problem Cantor tackled was the following: if the series converges to zero for all values of θ must all of its coefficients be zero, i.e., must $a_n = 0$ and $b_n = 0$ for all values of n? He determined that the answer is yes. The next question he considered was whether the same conclusion holds if the series converges to zero for all $\theta \neq \theta_0$ for some θ_0. The answer is again yes. He then generalized the question as follows: what can can be said if the series converges to zero for all $\theta \notin S$ for some subset S of $[0, 2\pi)$? Not surprisingly, this turns out to involve the structure of the set S.

Cantor soon turned away from the study of trigonometric series and devoted his energies to the investigation of the structure of subsets of the real line $\mathbb{R}$. This marked the first time that the subsets of $\mathbb{R}$ were studied systematically and was the beginning of set theory. Cantor introduced many of the ideas of elementary topology which are familiar to most of today's undergraduates.

Paradoxes

Paradoxes entered the realm of set theory as early as last century. Certainly by the mid-1890's at least three paradoxes were known: two due to Cantor dealing with the set of all sets and with the set of all cardinal numbers and one due to Cesaro Bureli-Forti (1861–1931) dealing with the set of all ordinal numbers. These paradoxes were not particularly worrisome in that they involved technical issues well inside set theory and it was felt that, by working towards a better understanding of the foundations of set theory, these paradoxes would be understood and thus resolved.

Russell's paradox was discovered by Bertrand Russell (1872–1970) in about 1904.

This was more of a paradox in logic than in set theory and so was more fundamental and was thus viewed more seriously by logicians and by many mathematicians. This paradox goes as follows: for each set x it must be the case that either $x \in x$ or $x \notin x$, so consider the set

$$r = \{x : x \notin x\}$$

consisting of all those sets x for which $x \notin x$. In particular, the set r itself must satisfy either $r \in r$ or else $r \notin r$, and this leads to a contradiction. For if $r \in r$ then, by the definition of r, it follows that $r \notin r$; and, on the other hand, if $r \notin r$ then, again by the definition of r, it follows that $r \in r$. But this is clearly a contradiction and is known as **Russell's paradox**. There are at least three ways of accommodating Russell's paradox: by "layers", by not allowing impredicative definitions, or by distinguishing between classes and sets. (The last of these is the one currently used by most working mathematicians.)

We can think of sets as being arranged in layers, with the sets in one layer being defined in terms of those in lower layers. Alternatively, we can think of sets as being built up in time, with the sets built at any one time necessarily being defined in terms of those built at earlier times. This approach is due to Russell himself and is formalized in his theory of types and ramified types. This approach makes the question $x \in x$ meaningless for any set x since any element of a set must lie in a lower level than the set itself. This approach turns out to be unwieldy and not well-suited to the needs of mathematicians and is today regarded more as a historical curiosity than a viable mathematical theory.

The definition of Russell's set r given above is said to be *impredicative* since, in building r, it is necessary to check whether $x \in x$ for each set x, including the set r itself which is only just being built. This is a vicious circle of sorts and Henri Poincaré (1854–1913) argued that one should not allow such definitions in mathematics. But such definitions are useful in mathematics and to not allow them would be too high a price to pay for ridding mathematics of Russell's paradox. Mathematicians today do use impredicative definitions and they are not regarded as being troublesome.

The third way to rid mathematics of Russell's paradox is, in effect, to say that some collections of objects are too large to be called sets and are, instead, called *classes.* According to this approach, only sets are small enough to be elements of sets or classes. So in an expression of the form $x \in y$ the object x must be a set but y may be a set or a class. Russell's set turns out not to be a set but to be a class and so the question of whether $r \in r$ is meaningless. The precise way in which one distinguishes between classes and sets is rather technical although in practise it is quite easy for the working mathematician to maintain the distinction.

A Little Philosophy of Mathematics

This discussion of ways of eliminating Russell's paradox from mathematics seems to suggest that one has a choice and that the method is not dictated by mathematics itself. This seems a little strange at first and is very different from what is done in any of the experimental sciences (where nature itself is the ultimate arbitrator). But this is not the least bit strange in mathematics once one realizes that the rules of mathematics are not dictated by nature but are made by mathematicians. At the risk of overly simplifying the situation, there are two philosophical approaches — **formalism** and **Platonism** — to mathematics and to the question of what it is that mathematicians do. Formalism asserts that mathematics is nothing more or less than a game in which mathematicians put marks on pieces of paper in accordance with certain rules which are made by mathematicians themselves and which may be changed by mathematicians. This is an austere view which some mathematicians such as Paul Cohen (b. 1934) subscribe to. Platonism (in so far as set theory goes) asserts that there really are such things as sets independent of our mental constructs and that the task of the set-theorist is to map out this set-theoretic landscape much as the early explorers mapped out the seas and the continents. Kurt Gödel (1906-1978), for example, was a Platonist. So the answer to the question of whether a mathematician *invents* or *discovers* a new theorem when she proves it for the first time depends on one's philosophical persuasions: the answer is invents to a formalist and discovers for a Platonist.

Logical versus Combinatiorial Sets

Not all mathematicians were dismayed by Russell's paradox. There is, for instance, evidence that both Cantor and Ernst Zermelo (1871-1956) independently of one another and of Russell discovered Russell's paradox a year before Russell but were not at all upset by it. At this time (*circa* 1904) there were two quite different ways of viewing sets or, more properly, of answering the question "what is a set"? Mathematicians and logicians who give one of the two possible answers would be disconcerted by Russell's paradox whereas those who give the other answer would not be.

Most logicians (including Russell himself) and some mathematicians had a **logical** conception of a set which allowed them to regard any collection of objects as being a set. Those who held this view would regard Russell's set r as a legitimate set and would be disconcerted by it. On the other hand, some mathematicians (and this included Cantor and Zermelo) had a **combinatorial** conception of a set according to which only those collections which satisfied a certain criterion were to be regarded as being sets. The criterion in play here is that a collection of objects constitutes a set if it can be **well-ordered**, meaning that its elements can be listed in the following sort of order:

$$0, 1, \ldots, \infty, \infty + 1, \ldots, 2\infty,$$

$$2\infty + 1, \ldots, \infty^2, \ldots, \infty^\infty,$$

$$\infty^\infty + 1, \ldots.$$

There seems to be no way to well-order Russell's set r and so those who regarded sets as combinatorial objects would not regard r as a set. At the same time, there seems to be no way to well-order the real line $\mathbb{R}$ or even the unit interval $[0, 1]$ and so they also had difficulty regarding $\mathbb{R}$ and $[0, 1]$ as sets. Nevertheless, Cantor was so firmly committed to the combinatorial view of sets that he sought well-orderings of the real line and of the unit interval off and on for much of his working life.

Well-orderings and the Axiom of Choice

In 1904 Zermelo published a research paper in which he proved that every set could be well-ordered. His proof made use of a principle which had never before been explicitly articulated but which had been used implicitly on many occasions by many other mathematicians. This principle is now known as the **Axiom of Choice**. One way of stating this principle is to say that if we are given a collection of non-empty sets then it is possible to select one element out of each of the sets. Putting it more precisely, if $\mathcal{A}$ is a non-empty set all of whose elements are themselves non-empty sets then there is a function f whose domain is $\mathcal{A}$ and which has the property that $f(A) \in A$ for each $A \in \mathcal{A}$. (Such an f is known as a *choice function* for $\mathcal{A}$.)

Whether one believes the first of these formulations of the axiom of choice depends ultimately on one's view of infinity. It is obviously true for finite collections but, psychologically, there is a big difference between making a finite number of choices and making an infinite number. The second formulation really says that set theory contains a large number of functions and one could argue with this on the grounds that it might force set theory to contain unnecessarily many functions. But one should resist the temptation to identify the psychological and the technical formulation of the axiom of choice; the technical one is the important one and it is precisely this identification which has led to and continues to lead to misunderstandings and misstatements of the axiom of choice.

There is an example due to Russell which will help illustrate the axiom of choice very nicely. If someone has an infinite number of pairs of shoes there is no difficulty in selecting one shoe from each pair: just select the left one. But if someone has an infinite number of pairs of socks there is no obvious choice function or rule for selecting one sock from each pair.

Zermelo's proof that every set could be well-ordered was attacked by most but not all of the leading mathematicians of the day. They objected strenuously to his use of the axiom of choice as a method of proof. They said that the axiom of choice was not a legitimate technique in proving mathematical statements. Their objections were based largely on the identification of the psychological and technical formulations of the axiom of choice and on the fact that the axiom simply asserts that a choice function exits and gives no constructive way of finding one. In his defense Zermelo pointed out (quite correctly) that these same mathematicians had already themselves implicitly used the axiom of choice in their own research. For example, the equivalence of continuity and sequential continuity (which is proved in Math 220) depends on the axiom of choice.

It was soon realized that the axiom of choice was equivalent to the assertion that every set can be well-ordered.

In 1908 Zermelo published a second paper on the axiom of choice and well-orderings. In this paper he set out to rewrite his earlier paper very carefully, making all of his assumptions and all of his arguments crystal clear. His object in doing so was to make his derivation of well-orderings from the axiom of choice absolutely clear and unassailable. In this paper he presented a list of axioms which he said were the axioms of set theory (one of these axioms was, of course, the axiom of choice) and he deduced the existence of a well-ordering on any given set from these axioms. This marked the first attempt to axiomatize set theory and turned out to play a significant role in the subsequent development of set theory.

The axiom of choice is, as its name implies, an axiom. But is it true? Could it, for instance, be deduced from the other axioms of set theory or could it be that the other axioms will allow there to exist a set which cannot be well-ordered? This soon became a very important problem in mathematics since it was realized that there are a large number of important statements which mathematicians would like to be true but which can only be proven with the aid of axiom of choice. There are, in fact, so many such statements that without the axiom of choice mathematics would be very different from what it is today. These differences would not affect just pure mathematics but would extend into applied mathematics as well. In fact, in the absence of the axiom of choice even continuity of functions and high-school calculus would be very different from what they are today. The axiom of choice plays such an important role in all of mathematics that virtually all mathematicians today assume that it is true.

But this does not answer the question of whether the axiom of choice really is true. In spite of the best efforts of numerous mathematicians virtually no progress was made in answering this question until the late 1930's. At that time Gödel constructed a certain model of set theory now known as the **constructible universe** and denoted by L. Gödel was able to prove that the axiom of choice held in this model. Now any provable statement about set theory will hold in any model of set theory and so Gödel's theorem has, as a corollary, that it is impossible to prove that the axiom of choice is false. But that is a far cry from proving that it is true. In the early 1960's Cohen devised a method (now known as *forcing*) for producing models of set theory. This method turned out to be extremely powerful and gave a model of set theory in which the axiom of choice did not hold or, equivalently, a model of set theory which contained a set which cannot be well-ordered. (This set could be well-ordered if it were regarded as an element of a larger model of set theory, but that is not the point.) A corollary of this is that it is impossible to prove that the axiom of choice is true.

At first sight this seems to be a very peculiar: we can prove that we cannot prove that the axiom of choice is either true or false. This situation too will be viewed very differently by formalists and Platonists. To the formalist it is perfectly satisfactory and not the least disconcerting since, after all, mathematics is nothing but a formal game involving symbols on pieces of paper. But a Platonist, who believes that sets really and truly do exist, will also believe that the axiom of choice is either true or false and that one of the tasks of the set theorist is to decide whether it is true or false. Most Platonists would say that if we only had a better notion of "proof" this conundrum would disappear. In fact, they would say that if we only understood set theory better we would be able to write down one or more self-evident statements which everyone would agree should be axioms of set theory and that with these additional axioms we would be able to prove that the axiom of choice is either true or false. In the last thirty years considerable effort has gone into the search for such additional axioms. Much of this effort has been in the direction of so-called large cardinals but has not turned up a single widely-accepted candidate for such an additional axiom.

Continuum Hypothesis

For any two sets A and B we will write (i) $A \approx B$ to mean that there is a one-to-one function from A onto B and (ii) $A \prec B$ to mean that there is a one-to-one function from A into B but that there is no one-to-one function from A onto B. The relation $A \approx B$ is interpreted as saying that A and B have the same number of elements and $A \prec B$ as saying that A has fewer elements than B. Cantor proved that $\mathbb{N} \prec \mathbb{R}$ and today this is well-known to mathematics students (and is usually proven in Math 220).

Is there a subset X of $\mathbb{R}$ with the property that $\mathbb{N} \prec X$ and $X \prec \mathbb{R}$? Such a set would have more elements than the integers but fewer than the real line. Cantor formulated this question soon after he proved that $\mathbb{N} \prec \mathbb{R}$ and it became known as the **Continuum Hypothesis**. Cantor worked on this problem on and off for much of his active life. On a number of occasions he would write a letter to a colleague announcing a solution only to write another one a few days later retracting his claim. It is not difficult to show that if X is an infinite closed subset of $\mathbb{R}$ then either $X \approx \mathbb{N}$ or $X \approx \mathbb{R}$, and so the continuum hypothesis is correct for such sets.

Whether the continuum hypothesis is true or false is a vexing problem but, ultimately, not one which affects working mathematicians. Most mathematicians consequently have an open mind on this issue and rarely if ever worry about it. (This is very different from the axiom of choice.)

The resolution of the continuum hypothesis is identical to that of the axiom of choice. The continuum hypothesis holds in Gödel's constructible universe L and so it is impossible to prove that the hypothesis is false. And Cohen's method of forcing yields a model of set theory in which the continuum hypothesis is false and so it is also impossible to prove that it is true.

Gödel's Incompleteness Theorem

It has already been said that the axiom of choice and the continuum hypothesis are two statements about set theory which we can prove cannot be proven to be true or false. But putting it this way is being somewhat misleading if not outright inaccurate.

One difficulty with this statement is that it fails to specify the context in which mathematics is being done or to specify the meaning of "proof". Surely this statement would change if mathematicians were to change what they regard as an acceptable proof. In making this statement it is assumed that mathematics is formulated as a so-called first-order logical theory. In particular, if mathematics were to be regarded as a second-order theory then the notion of a proof would change and it may well be possible to prove that, say, the axiom of choice is true. But that would be changing the rules of the game of mathematics too drastically to suit most mathematicians. The distinction between first- and second-order theories was not made until about 1917 and the idea that mathematics should be first-order did not become widely accepted until the late 1920's. Some of the set theory developed prior to 1920 is second-order and there are today, in fact, some logicians who advocate a return to second-order mathematics on the grounds that it is much more natural and intuitive.

A second difficulty with the statement in the first paragraph of this section is related to Gödel's incompleteness theorem. First recall from elementary logic that we only want to work with consistent logical systems since every statement is provable in an inconsistent system. Gödel's incompleteness theorem asserts basically that any first-order formulation of mathematics cannot be proven to be consistent by the methods of that theory. (Presumably any such formulation can be proven to be consistent by using a more powerful theory or by using a second-order theory, but that would defeat the purpose of a consistency proof: in proving that a theory is consistent it is necessary to use a theory which is no more likely to be inconsistent and hence no stronger.) In particular, if we define set theory as the theory whose axioms are those formulated by Zermelo in 1908 minus the axiom of choice together with one more axiom proposed independently by Abraham Fraenkel (1891–1956) and Thoralf Skolem (1887–1963) in about 1920 then we cannot prove that set theory is consistent. If we denote this set theory by ZF and, for any theory T, denote the statement that T is consistent by Con(T), then Gödel's incompleteness theorem asserts that within ZF it is not possible to prove Con(ZF).

Recall that we said that since the axiom of choice and the continuum hypothesis hold in Gödel's model L, then we cannot prove that either of these assertions is false. But this isn't quite the correct conclusion owing to the fact that we do not have a proof of the consistency of set theory: the correct conclusion should be that if set theory is consistent then it will remain consistent if the axiom of choice and the continuum hypothesis are added as additional axioms. In symbols, then, the correct conclusion to be drawn from the fact that the axiom of choice and the continuum hypothesis hold in Gödel's model L is that

$$\mathrm{Con(ZF)} \;\Rightarrow\; \mathrm{Con(ZF + AC + CH)}.$$

Cohen's method of forcing yielded a model of set theory in which neither the axiom of choice nor the continuum hypothesis held and this was earlier said to imply that we could not prove that either of these assertions is false. A more correct formulation of the conclusion of this is the assertion that

$$\mathrm{Con(ZF)} \;\Rightarrow\; \mathrm{Con(ZF} + \neg\mathrm{AC} + \neg\mathrm{CH)},$$

where $\neg$ is the usual logical symbol for negation.

So if the set theory ZF is consistent then adjoining to it either the axiom of choice or a suitable negation of it will yield another consistent set theory. This is usually stated by saying that the axiom of choice is independent of ZF. In a similar sense the continuum hypothesis is independent of ZF.

Disclaimer and References

I am not a set theorist, a historian, or a philosopher but I nevertheless agreed to give the Coleman-Ellis Colloquium Lecture on which this essay is based and write the essay itself because I find the subject matter fascinating and because I know that many undergraduate mathematics students would agree with me.

Here are a few references on set theory and logic I have found to be useful and readable and which I would urge the interested reader to consult:

- Joseph Warren DAUBEN, *Georg Cantor: his mathematics and philosophy of the infinite*, Harvard University Press, 1979.

- Shaughan LEVINE, *Understanding the Infinite*, Harvard University Press, 1994.
- Gregory H. MOORE, *Zermelo's Axiom of Choice: its origins, development, and influence*, Springer–Verlag, 1982.
- Stewart SHAPIRO, *Foundations without Foundationalism: a case for second-order logic*, Clarendon Press, 1991.

Teaching and Learning and Mathematics as a Social Science

Morris Orzech

This article is about teaching...and about learning, and about some of the departmental hubbub surrounding them. It is unlikely that you graduated from Queen's (particularly as a Math and Stats student) without a sense that providing a good education, and attending to the welfare of our students, were important concerns in this department. This hasen't changed, but in the past few years there have been institutional changes at Queen's and in this department which have altered the way in which the concern about teaching and about students manifests itself. And a good thing too, because they have helped us cope with our budgetary sandstorms and to approach our teaching with the kind of energy, curiosity, innovation and sense of mission that we bring to our research.

What institutional changes? For one, the establishment of the Queen's Instructional Development Centre (IDC) in 1992 following a recommendation by a committee chaired by Biology professor Dr. David Turpin. Another change came after Dr. Turpin became Dean of Arts and Science, when the Faculty of Arts and Science initiated funding for major instructional development initiatives, with the IDC playing a key role in evaluating the projects. Our department has been fortunate (and appreciative) that the Faculty has seen fit to fund our proposals in each of the three years of the competition (and our appreciation is heightened by having the new Dean decide to budget for instructional change (albeit at a reduced level) despite the current financial climate). Throughout our work on these projects we have benefitted from the expertise and support of the IDC.

A basic understanding behind these projects was that the funding was seed money, and that the changes brought about would have to be self-sustaining. It was also a given that the changes should be aimed at benefits for the students, rather than efficiency without regard for its impact. A large part of our effort went into a major revamping of our first year calculus service courses, both in terms of content and delivery. The changes are described in the article by Grace Orzech and Joan Geramita following this one, but I would summarise them thus: We have reformed the content with the intended audience in mind, and we have redeployed the significant resources devoted to the course. We try to provide excellent support materials and instructors for the large lectures; we offer a large and varied amount of the help students seem to need; and we administer the course with a great deal of care, attention to detail, regard for individual student special circumstances and a serious effort at getting feedback and addressing problems. (Sounds good eh?)

Much of our work has been aimed at developing and assessing materials and methods to do some things we have not done before, and to do some traditional things better. One project focused on developing student skills at reading, evaluating and writing mathematical reports. There has a been a continuing effort to identify and integrate appropriate technology in our courses, with the educational aspects as the primary concern. (Software packages such as Maple, Matlab, Mathematica and Minitab are used in one or another of most of our first and second year courses.) If you were to drop in on student and faculty conversations in Jeffery Hall relating to their courses you would likely hear references to group projects, to course chats, to interactive notes, and labs for Math 110 and Math 120. You might get some sense of the changes brought about through the institutional support for instructional development and you would certainly get a sense that our instructional activities were not on automatic pilot. (One of the outgrowths of our work was a start last year of a Teaching and Learning Seminar, a forum for presenting and discussing ideas, experiences and materials about mathematics and statistics postsecondary education.)

Our projects have involved cooperation among about a dozen regular and adjunct faculty members, but the participation by faculty relates to only one part of what the "hubbub" to which I referred in my opening sentence is about. The other part involves our students. One of the most rewarding aspects of the instructional development work has been the opportunity to hire about a dozen of our students over the past three years to work with us on the development, implementation and assessment of our projects. I have always been cognizant of how lucky we are to have so many students who sport a variety of talents, enthusiasm, good sense and a positive attitude towards their work and their peers. Being able to bring some of these students on board as helpers and consultants provided us all with a concrete example of education as an experience shared by teacher and learner. (Naturally, this shared experience included a few significant disagreements.) University faculty experience this kind of interaction with graduate students, but we have too little opportunity to share with our undergraduate students the sense of community that for many teachers is a necessary underpinning for a good education.

Which brings me to the second part of my title, mathematics as a social science. I recently attended a workshop presentation by Dr. Christopher Knapper, director of the IDC, on fostering "deep learning," the kind of cognitive development which helps people sort out the less important details from significant principles and which leaves useful traces when a course is done. Dr. Knapper referred to various studies which tried to identify factors correlated to this "deep learning." Among the factors so identified were interaction, feedback and rapport between students and faculty and between students and students, and teaching methods based on interaction and student activities. I was gratified to note that these factors have been explicit principles in our thinking about what and how we teach. I suspect we "did the right thing" as much because of the guidance from the IDC as out of our own good instinct. But it made me think again about a certain phenomenon some of my colleagues and I have noticed in Jeffery Hall: students in different years (including graduate students) working and talking together in various parts of the building, stopping faculty members to ask questions or arriving as a group in our office to clarify something. I used to think of this easy and natural interaction as a bonus (for the students and for us) which exists atop of the good mathematics education we provide. Dr. Knapper's presentation made me realize that the interaction isn't just a bonus. It is an indicator that we have fostered an environment where the good material we offer can remain valuable to our students after their stay with us is done.

And I hope our interaction with our students can remain valuable to us as well as to them. Are you interested in knowing more about our instructional development work? Do you think you can offer help or advice? Do you have any questions relating to a young person you know who might be interested in studying with us? Do you just want to say hello? Don't hesitate to get in touch with me or with someone else in the department.

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Math 121

Joan Geramita and Grace Orzech

Calculus is one of the great inventions of the eighteenth century. Characterizing calculus as an "invention" is in keeping with the spirit of our largest undergraduate course, MATH 121. Ideally, the successful graduates of MATH 121 will have acquired intellectual equipment which they can apply in a variety of familiar and unfamiliar situations.

MATH 121 is taken by Arts and Science students who need a calculus course but are not contemplating more than a minor concentration in mathematics. In fact, most students in MATH 121 are not planning to take any other mathematics course at all. About ten years ago, Joan Geramita and Leo Jonker undertook to redesign the course to recognize the needs of its students by providing a strong sense of how calculus interfaces with other disciplines. Since then, MATH 121 has evolved from a less demanding version of our honours calculus course (MATH 120) to its current incarnation where the emphasis is on calculus as a useful tool in many disciplines.

A great deal of thought and effort have gone into the transformation of MATH 121. For about two decades there was a widespread consensus on what a first-year calculus course should look like, at least in North America. Most of our readers and we ourselves took such a course. Many of us were quite comfortable teaching a course much like the one we had as students and that some of us had been teaching for a long time. The value of the experience that supports such a tradition should not be underestimated. Nevertheless, there was also a growing concern at Queen's and elsewhere in the North American mathematical community that many of the students who took calculus as their only math course were not well served by the standard approach. For many of them the course they had boiled down to learning a set of procedures together with a selection of key words that provided clues about which procedure to use.

Learning procedures has a place in the redesigned course. Students take three technical mastery tests. As the name implies, these tests are meant to ensure that important techniques and facts have been mastered. This is a side of calculus that many students have seen in high school and that the excellent students who come to Queen's can learn on their own if they have good materials to help them. We provide a list of sample questions for each test. Before they take a technical mastery test, students know how the questions will be worded, which techniques will be tested, and how many questions of each type there will be. The grading on these tests is strict. Students are required to provide accurate answers. The three mastery tests account for 25% of the final mark.

The rest of the grade is earned through tests that require more independence and creativity. In the past, the midyear and final exams in MATH 121 tended to be long. Sometimes students found that they could not get to all the questions. In recent years, these exams have consisted of a smaller number of challenging problems. It may take a well-prepared student some time to decide on a strategy for doing a given problem. Weekly assignments help provide practice in the required level of thinking. Rather than handing in solutions to the assigned problems, students take biweekly quizzes which consist of problems that are closely related to those that have been assigned. This system encourages students to work with their peers and to make use of all the extra help provided for the course while ensuring that they have achieved their own understanding of the material.

The onset of hard times gave a final impetus to our decision that MATH 121 would be taught in large lecture theatres by a small team of like-minded professors. MATH 121 has gone from ten to three sections (for 700 or so students). Teaching and learning familiar material from a new perspective is exciting but anxiety provoking. Doing it in a large lecture setting is a real challenge.

The Faculty of Arts and Science has assisted the department by providing generous curriculum development funds. We have used these to prepare good written materials that the students can use to prepare for the technical mastery tests and the biweekly quizzes. They can also purchase a set of notes designed to make it easier to follow lectures in a large lecture theatre. Finally, there are many ways for students to get assistance with studying for MATH 121. Several graduate and undergraduate teaching assistants are assigned to the course. There is an electronic forum called m121chat where students can post questions and get responses from one another or from the instructors in the course. The course coordinator is available to help through email. Finally, the course is organized to encourage students to study together.

How is all this working? A perusal of questionnaires handed in by students over a two year period indicates considerable enthusiasm for technical mastery tests, guarded approval of homework quizzes, and quite a bit of anxiety about the midyear and final exams. The large lecture setting also requires a great deal of adjustment from students who have just come from high school. We will continue to work on minimizing the stress level in the course while requiring students to work to a high standard.

Faculty teaching the course are pleased with the students' success rate in a course that is not a snap for very many of them. Answers on final exams reveal that many students are able to say something sensible even about questions that they can not completely solve. Students express glee over being able to solve problems that their colleagues in our other (formerly harder) calculus course find difficult.

News of Graduates

Vijay K. Bhargava, a 1970 graduate of our Mathematics and Engineering program is the winner of the 1995 McNaughton Gold Medal. The McNaughton Medal, named after general Andrew McNaughton, is a prestigious award, which the Canadian chapter of the Institute for Electric and Electronic Engineers (IEEE) bestows annually onto those of its members who made outstanding contributions to the field of electric or electronic engineering. At present Dr. Bhargava, who obtained his Ph.D. from the Electrical Engineering department at Queen's, is running for the post of the president-elect of the Canadian chapter of the IEEE.

Robert McCann who in 1989 completed his undergraduate studies in mathematics at Queen's with a B.Sc. and in 1994 received his Ph.D. from Princeton University under the direction of Elliot Lieb, was awarded an AMS Centennial Fellowship. In his doctoral thesis McCann developed a convexity theory which led to the solution of two problems from mathematical physics: the first modelled an ineracting gas, while the second involved the shape of crystals in an externel field. During his Centennial Fellowship he plans to visit the Courant Institute of Mathematical Sciences in New York city and the University of California, Berkeley.

Jmin Chen who graduated in 1991 from Queen's University with a degree in pure mathematics received his D. Phil. from Oxford University in July 1996. Shortly afterwards he has been awarded a three year NSERC postdoctoral fellowship, the first year of which he will spend at the University of California in Berkeley, beginning this September. The remaining two years he will subsequently spend at McGill University in Montreal.

Ontario Student Opportunity Trust Fund

The Ontario Government has announced a plan to match funds donated between May 7, 1996 and March 31, 1997 to Queen's University directed towards student financial aid.

The Department of Mathematics and Statistics will establish a bursary for third and fourth year students in Mathematics and Statistics.

All donations to the bursary fund will be used to establish an endowment and the interest from this fund will be used to provide bursaries to students needing financial assistance.

In order to get started, the fund must reach \$2,500. Please send donations to the Mathematics and Statistics Bursary Fund, Department of Mathematics and Statistics, Queen's University, Kingston, Ontario, K7L 3N6.

Problems

Peter Taylor

Sum of cubes is square of sum

Ed Barbeau of the University of Toronto is an undisputed master of good recreational mathematics problems. I was at a conference last month and he ran a problem solving session with a number of lovely examples—the neatest of these, in my view, I will share with you.

You are perhaps all familiar with the formula for the sum of the first n natural numbers:

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}.$$

And perhaps also the sum of the first n squares:

$$1+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}.$$

But have you also met the remarkable formula for the sum of the first n cubes?

$$1+2^3+3^3+\cdots+n^3=\left(\frac{n(n+1)}{2}\right)^2.$$

Of course, the reason this is so remarkable has to do with its relationship to the first formula. I have puzzled over why this happens to be true, trying to find some intuitive justification, but haven't come up with anything as yet.

Anyway, let's look at it another way—we have obtained a collection of numbers for which the sum of the cubes equals the square of the sum. Well now that's interesting enough, and it leads us to ask the question: can you find any other "natural" collection of numbers for which the sum of the cubes equals the square of the sum?

Well here's such a collection. Take any positive integer n. Now make two columns. In column one make a list of all the divisors of n, and beside each of these, in column two, put the number of divisors of the column one number. Then it turns out that column two is always such a collection of numbers!

I'll illustrate with $n = 12$.

Divisors of 12	number of divisors
1	1
2	2
3	2
4	3
6	4
12	6

Take the numbers in the second column.
Square of sum: $(1+2+2+3+4+6)^2 = 324$
Sum of cubes: $(1+8+8+27+64+216) = 324$
The problem of course is to show that it always works.

Perfect shuffles.

At the same conference I encountered another result which blew me right out of the water. You can try to see why this one works too.

You know what a perfect shuffle is?—you divide the deck exactly in half, and then riff-shuffle the deck so that the cards fall alternately. Go try it out! Right. It sounds hard to do and it is, but competent card artists can do it reliably.

Perhaps you've noticed that there will be two kinds of perfect shuffles depending on which half of the deck you let fall first, in one case card #1 will stay on top, and in the other card #1 will go to position #2 and card #27 will wind up on top. The first kind is called an OUT-shuffle (card #1 stays "out") and the second kind is an IN-shuffle.

Okay, suppose you've got the deck with a certain card, say the spade ace, on the top and you want to move it to a designated position in the deck with a sequence of shuffles. Can you do it, and if so how? This is the sort of question that might arise in the execution of a card trick.

The answer is that you can do it with a few perfect shuffles, using a suitable sequence of IN's and OUT's. Maybe that's not too surprising, but wait till you hear how to find that sequence.

To work with an example, suppose you want to move the spade ace to position #14. That means you want to move it down 13 places. Write 13 in binary:

$$13 = 1101.$$

Okay (fasten your seat belt) replace each 1 by an I and each 0 by an O (well what else?):

IIOI.

The recipe calls for two IN's followed by an OUT followed by another IN. The ace will move down 13 places to position #14—evidence for sure that god did a joint major in math and english.

Problems From Last Issue

Peter Taylor

Here are two nice little probability problems.

1. I have an urn which contains 100 balls, each one either black or white. If you draw two balls at random from the urn, then the probability that they are the same colour is exactly 1/2. How many balls of each colour were there in the urn?

2. A flips a fair coin 100 times, and B flips a fair coin 101 times. What is the probability that B gets more heads than A?

Solutions

Solutions by Alan Donald, M.Sc. '79 now Associate Professor of Health Management at the Atlantic Veterinary College, University of Prince Edward Island.

1. Suppose we have n balls of which x are white and $n-x$ are black. Then the probability that we choose a white followed by a black (or a black followed by a white) is

$$p = \frac{x}{n} \cdot \frac{n-x}{n-1}$$

The probability that we get balls of different colours is $2p$. If we set this equal to 1/2, we get a quadratic in x whose solution is

$$x = \frac{n \pm \sqrt{n}}{2}$$

If $n = 100$, for example, $x = 45$ or 55.

2. This a very nice problem. I suppose there is a complicated solution that uses messy binomial coefficients. But I gave up on those after about 15 minutes. (*Good decision!*) Here are two dodgy solutions. (*They're not really dodgy but they are quite clever. Solution (i) is nice and is the standard "nice" solution. Solution (ii) is spectacular.*)

Solution (i)

Suppose A tosses n times and B $n+1$ times. Hold off, for the time being, on B's last toss and consider only her first n tosses. There are three exhaustive, mutually exclusive outcomes.

Event X: A has more heads than B
Event Y: B has more heads than A
Event Z: A and B are tied.

Let p_X, p_Y and p_Z be the probabilities of these events. Then the exhaustiveness and exclusiveness gives $p_X + p_Y + p_Z = 1$. And, by symmetry, $p_X = p_Y$. Thus

$$2p_Y + p_Z = 1$$

Now, B gets more heads than A if Y is true after the first n tosses or if Z is true and B gets a head on the last toss (a probability of 1/2). Thus the probability of B getting more heads is

$$p_Y + p_Z \cdot \frac{1}{2}$$

From the first equation, this is exactly 1/2.

Solution (ii)

Let R be the event that B gets more heads than A; let S be the event that B gets more tails than A. We note that these two events are mutually exclusive and exhaustive.

They are mutually exclusive since they could only both hold if B had at least 2 more tosses than A. They are exhaustive since B tosses more than A.

Thus if p_R and p_S are the probabilities of the two events, $p_R + p_S = 1$. By symmetry, the two probabilities are equal and therefore both equal 1/2.

Problem 1 was also solved by Greg Baker, Math '98.

Head's Report

Eddy Campbell

1997 will mark the 30th anniversary of the first Mathematics and Engineering class to graduate from Queen's. We are inviting all graduates of that program and their families, to visit us for a reunion August 1-3, 1997. We will have a Friday reception, a public lecture Saturday afternoon, a banquet Saturday night and a Sunday Brunch. We will arrange activities for children of all ages, and babysitting as appropriate. All our alumni should have received a letter from me: if you have not, please get in touch with us at the addresses provided. We'd love to see you.

Ram Murty of McGill University accepted a position as full professor in our department as of July 1, 1996. Ram was honoured as a Queen's National Scholar. He has been a Steacie Fellow, one of the Natural Sciences and Engineering Research Council (NSERC) of Canada's highest awards, and a Fellow of the Royal Society of Canada. Ram is one of the best number theorists in Canada, and his addition makes our department one of the best in the country. We now have three Queen's National Scholars among our faculty.

Tony Geramita was appointed to the Chair of Geometry at the University of Genoa under the provision of Italian Law known as "chiara fama" or "clear fame". This is a highly unusual honour, to say the least. Tony is internationally known for his work in commutative algebra and algebraic geometry.

Oleg Bogoyavlenskij, appointed to our department as a Queen's National Scholar in 1992, received a major increase in his operating grant from NSERC, in recognition of his outstanding work on dynamical systems.

Agnes Herzberg became Vice-President of the International Statistics Institute.

Ed Chow, one of our young statisticians, has decided to leave us as of August 1, 1996. Ed has taken a job designing neural net software in San Diego. Ed is an excellent teacher, and we regret very much losing him. We wish him the best of luck in his new career.

Two years ago the department decided to devote a position to postdoctoral fellows. We have been able to attract four brilliant young scholars to Queen's to work with us: Eduardo Aranda-Bricaire, working with Jon Davis and Ron Hirschorn; Keith Pardue, working with Tony Geramita and Leslie Roberts; Jim Shank, working with Ian Hughes, David Wehlau and myself, and Helena Verril working with Noriko Yui and Ram Marty. Ram will bring with him another young post doctoral fellow, David Cardon, who just graduated from Stanford. The presence of all these young researchers has made and will make a tremendous difference to the life of the department.

Due to government cutbacks the department faced a budget cut of 10% this year. Queen's made selective cuts — our department's share of the pain was slightly below average relative to other departments in the faculty of Arts and Science. Four positions which were vacant for one reason or another were closed, and the position of one of our statisticians, Tom Stroud, who took early retirement as of June 30, 1996 (cf. Notes on this Year's Retirees) was collapsed. The department, which had some 48 faculty members only fifteen years ago, now has just 36. Under Leo Jonker's leadership, we have cut the number of courses offered from 220 half courses five years ago to just 127 in the year past. Math 121, our first year calculus course for roughly 700 Arts and Science students, represents an example of the effects of the reduction in the number of sections in our service courses. A few years ago the course was taught in ten sections, each containing roughly seventy students. This year the same course was offered in only *three* sections, one containing four hundred and the other two containing two hundred and one hundred students, respectively. Such large classes are now common at many universities. Our instructors, lead by Grace Orzech, have worked long and hard to make improvements. As a result, I believe that this is a better course today, and you can read more about it elsewhere in this issue of the communicator.

We are continuing to rethink and redesign our curriculum across the board. Morris Orzech has led some really interesting experiments in the application of technology in the class room. He has worked long and hard on a shoestring budget to make our semi-public computing site, Jeffery201, a success. Morris has received funding through our Dean's curriculum development funds — which the faculty has been able to protect in spite of the severe cutbacks.

A really tragic consequence of these government cutbacks was a decision by Senate (the vote was 20-18) to force the relocation of the Mathematics and Statistics library from Jeffery Hall to a newly renovated Douglas Library as of April 1997. The Mathematics and Statistics library is one of nine branch libraries to be closed in order to reduce service costs. There is some hope that we will be able to salvage a small research collection from the wreckage. Real damage will be caused to the sense of community which the library has served to create among students and faculty over the past thirty years — a cost that is very difficult to quantify. Don Akenson of the History Department said it best: "The real danger is that we will come to value most that which is most easily counted". A number of our alumni wrote to express their concern and some even responded with donations. One of our graduates from 1964, now a retired teacher, offered to work one day a week in the library, a gift of time of great value.We are very grateful to every one who responded.

Our strategic plan calls for the creation of Industrial Research Chairs through a program partially funded by the Natural Sciences and Engineering Research Council of Canada. We seek two such chairs, one in Statistics and the other in Mathematics and Engineering. The program requires the participation of an industrial partner or consortium and each such chair requires total funding over a five to ten year period of some $750K-$2M. We are still in the early stages of our search — identifying high-quality candidates with whom to attract industrial partners. Please get in touch with me if you think you or your company can help in this effort.

One of the bright spots in the department these past few years has been our Gill tutors. A modest bequest from Sun-Life honouring the memory of their former president, Mr. Ernest Clark Gill, Queen's graduate of 1923, was used to create these positions. We hire one or two of our best upper year students to help tutor our younger students. We select those who we feel will contribute most to help create a sense of family among all of our students. The program has been a great success. One of the first Gill tutors was Sumit Oberai, son of Kirti Oberai, who is one of the best teachers in our department. Another was Laura Scull who just has finished her first year in the PhD-program at the University of Chicago. She has decided to work in the area of algebraic topology. I am rather proud of having initiated her into the subject during her last year at Queen's. Two years ago Greg Smith, who is just finishing his first year in the PhD-program at Brandeis, was a Gill Tutor. Greg, Ian Hughes, Tony Geramita and I are presently writing a paper, describing the results that arose from Greg's work with us last summer. Serge Mister, BscEng 96 will stay on to do his Master's degree in the Computer and Electrical Engineering department here. Serge will spend the summer studying network security at Nortel, under the supervision of Brad Ross, BScEng 73. This year we had Peter Zion as a Gill Tutor. Peter is off to the University of Toronto to do a Master's degree, and he plans to go from there to Oxford for a PhD. Another Gill Tutor was Steve Nielsen, the son of Ole Nielsen, who wrote the leading article in this communicator. Steve is going to do graduate work in theoretical chemistry at the University of Toronto. Finally, this year we had Paula Dow, one of our best third year students, as a Gill Tutor. She will enter her fourth year this Fall.

I wrote to Mrs. Gill this winter to tell her about the success of the program, and I received a truly lovely reply.

The department once again sponsored a public lecture. In late November, we enjoyed a wonderfully witty and articulate lecture by S. Abhyankar on algebraic curves. Professor Abhyankar is from Purdue University where he is appointed to three different departments: mathematics, computer science, and electrical engineering.

We are genuinely interested in hearing from you, not only if you are able to make a financial contribution! (although it goes without saying that given the present constraints in government funding such a contribution is always deeply appreciated). My E-mail address is eddy@mast.queensu.ca.

Notes on this Year's Retirees

Uri Fixman joined the department in September 1961. Uri is a mathematician with a broad interest and a knowledge which encompasses wide areas of mathematics, including branches of algebra and analysis. While working at Queen's his research interests focused on systems of linear operators in infinite dimensional vector spaces. He supervised two MSc-theses and four PhD-theses on this and related topics. Furthermore, he served on several departmental committees.

Uri, we wish you a long and happy life in retirement. We hope that as an emeritus professor you shall still visit our department occasionally and stir our laughter with your sharp and lively humour!

Tom Stroud It was with both regret and relief that members of the Mathematics and Statistics Department learned of the early retirement of Tom Stroud in March 1996. Regret at losing an able statistician and relief that the Department would meet its quota of required retirements and would not have to carry a crippling debt ($70,000 per year accumulating until the next retirement).

After graduating from the University of Toronto with an MA, Tom taught for four years at Acadia University. He did his PhD at Stanford under the supervision of Ingram Olkin, before coming to Queen's in 1968. In his 28 years at Queen's he supervised three PhD and seven MSc theses as well as a number of MSc projects. He spent years visiting at each of the Educational Testing service, Ecole Polytechnique in Lausanne and at the University college of Wales in Aberystwyth. In 1992 he was promoted to Full Professor.

His 21 published papers appear in a wide variety of journals ranging from *Annals of Mathematical Statistics*, to *Journal of Applied*

Probability and *The Journal of Educational Statistics.*

During his career he has worked as a consultant in Educational Testing and on sample surveys, particularly in the field of Small Area Sampling, for Statistics Canada.

Many students at Queen's will remember his very practical courses in Sample Surveys, and note a few have been grateful to Tom, because this course enabled them to ace the Stats Canada tests and ensure them of employment opportunities with Statistics Canada.

Inhabitants of Jeffery Hall will miss his invitations to Choral Concerts and Barber Shop Quartet Singing.

We all wish Tom a relaxing and satisfying retirement.

Lorne Campbell retires this summer after 33 years in the Department of Mathematics and Statistics. He served as Head from 1980 to 1990 and has been actively involved with the Mathematics and Engineering program (He became a licensed Professional Engineer in 1990). Lorne's mathematical research in information theory has been of great benefit to the engineering community. In 1990 he became a Fellow of the Institute of Electrical and Electronics Engineers for his contributions to the understanding of signals and noise in nonlinear devices. The IEEE is the world's largest professional engineering organization and awards fellowships for outstanding contribution to Electrical Engineering. In 1994 he was presented the Canadian Award in Telecommunications Research for his contributions to information and communications theory and for a lifetime of leadership in these fields in Canada.

Lorne is a dedicated teacher who has taught pure and applied mathematics, statistics, and engineering at the undergraduate and graduate levels. He supervised 14 MSc and 5 PhD students and has published over 54 scientific papers in refereed journals. Lorne plans to continue his research and supervision of graduate students and is currently co-supervising a PhD student. He also plans to do some travelling and hopes to visit Kenya, Tanzania, and South Africa this fall.

IF UNDELIVERED RETURN TO:
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