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It is impossible to write a power of degree larger than 2 as a sum of two powers of the same degree. I have discovered a truly wonderful proof of this theorem, which however, does not fit into this margin. Pierre Fermat.

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Fermat's Last Theorem

Ernst Kani

Coleman-Ellis Lecture, November 1996

Introduction

On the 23rd of June 1993, Andrew Wiles concluded a three-day lecture series in Cambridge, England, with the assertion:

Theorem. *Every semi-stable elliptic curve is modular.*

This not only electrified number theorists and mathematicians around the world, but even made the headlines of many major newspapers such as the *New York Times, Le Monde, Frankfurter Allgemeine,...*, a rare event for a mathematical theorem.

The main reason for this excitement and publicity is due to the fact that it had just been shown a few years earlier that the above theorem implies the truth of *Fermat's Last Theorem*,

$$(\mathrm{FLT}_n) \qquad x^n+y^n \neq z^n, \quad xyz \neq 0,$$

for any non-zero integers $x, y, z \in \mathbb{Z}$ and $n \geq 3$; this had been asserted by Fermat 350 years ago!

The purpose of this lecture is to relate some of the history behind FLT (= Fermat's Last Theorem[1]), to explain in simple terms how Wiles's theorem is related to FLT and, above all, to give you a glimpse of the significance of Wiles's result which, in fact, goes far beyond FLT.

1. Early History

Although FLT is an assertion about sums of n-th powers for $n \geq 3$, it was inspired by looking at the case $n = 2$, the so-called Pythagorean equation:

$$x^2 + y^2 = z^2.$$

[1] So called because it was the last of Fermat's many assertions which still had to be resolved.

In high school, every student learns that $(3, 4, 5)$ and $(5, 12, 13)$ are solutions (called *Pythagorean triplets*) of this equation, but few learn that

$$12,709^2 + 13,500^2 = 18,541^2.$$

Indeed, this solution, and many others like it, had been known for almost 4000 years, and were recorded on clay tablets around the era of Hammurabi (ca. 1700 B.C.), more than 1000 years before Pythagoras (ca. 550 B.C.). In fact, from the way the following tablet (Plimpton 322, discovered by O. Neugebauer and Sachs; cf. Figure 1) is arranged, historians are convinced that the Babylonians already knew the following formula (or something close to it) for generating all Pythagorean triplets:

$$x = u^2 - v^2, \quad y = 2uv, \quad z = u^2 + v^2, \quad (1)$$

where $u, v \in \mathbb{Z}$; this formula is usually attributed to Pythagoras or Plato (ca. 400 B.C.).

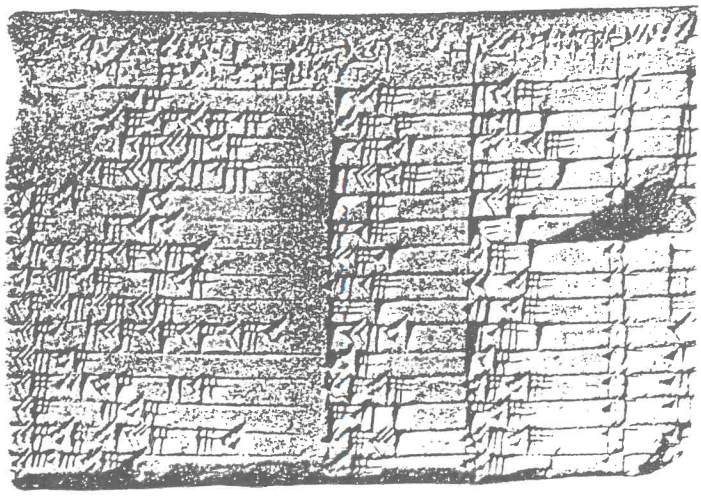
Certainly *Diophantus of Alexandria* (ca. 250 A.D.) was not only aware of this formula, but even based a large number of problems on it in his *Arithmetica*, a very remarkable collection of 13 books of which 9 have survived. (Of these, only 6 were known in the Renaissance; the other 3 were discovered only 20 years ago in a library in Iran.) Thus we find in Book II:

Problem 8: *Decompose a given square into a sum of two squares.*

Diophantus presents the numerical example $4^2 = \left(\frac{16}{5}\right)^2 + \left(\frac{12}{5}\right)^2$, but his method is perfectly general and actually leads to the formula

$$a^2 = \left(\frac{2ma}{m^2+1}\right)^2 + \left(\frac{a(m^2-1)}{m^2+1}\right)^2,$$

Plimpton 322
ca. 1800 - 1650 B.C.



$\frac{d^2}{h^2}$	w	d	n	h
$\frac{28561}{14400}$	119	169	1	120
$\frac{23280625}{11943936}$	3367	4825	2	3456
$\frac{44209201}{23040000}$	4601	6649	3	4800
$\frac{343768681}{182250000}$	12709	18541	4	13500
$\frac{9409}{5184}$	65	97	5	72
$\frac{231361}{129600}$	319	481	6	360
$\frac{12538681}{7290000}$	2291	3541	7	2700
$\frac{8667}{5120}$	799	1249	8	960
$\frac{591361}{360000}$	481	769	9	600
$\frac{66601921}{41990400}$	4961	8161	10	6480
$\frac{25}{16}$	45	75	11	60
$\frac{8579041}{5760000}$	1679	2929	12	2400
$\frac{83521}{57600}$	161	289	13	240
$\frac{10426441}{7290000}$	1771	3229	14	2700
$\frac{2809}{2025}$	56	106	15	90

Figure 1: A clay tablet and its translation:[2] Pythagorean triplets $h^2 + w^2 = d^2$

where a^2 is the square to be decomposed and m is any integer. This, of course, is just a variant of the formula (1).

While studying this problem, Pierre De Fermat (1601 -1665) wrote the following text in the margin of his copy of the *Arithmetica* (which had recently been translated from Greek to Latin by Bachet):

Cubum autem in duos cubos, aut quadrato-quadratum in duos quadrato-quadratos, et generaliter nullam in infinitum ulta quadratum postestatem in duos ejusdem nominis fas est dividere; cujus rei demonstrationem mirabilem sane dexteri. Hanc marginis exiguitas non caperet.

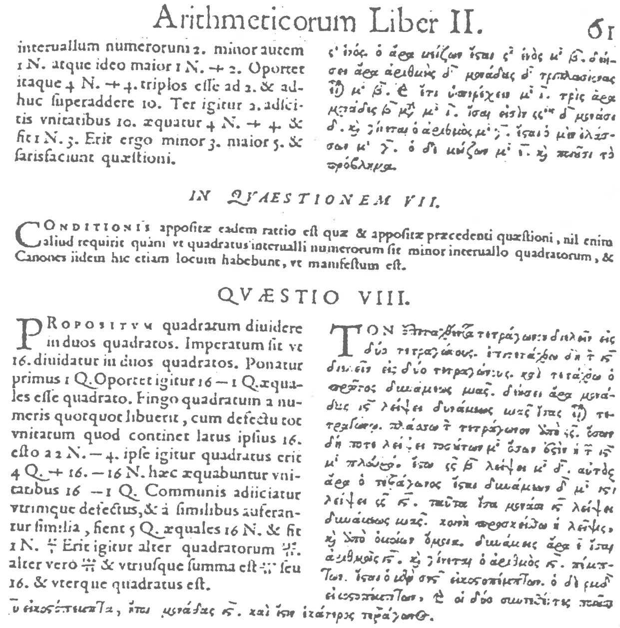
Translation[He]: On the other hand it is impossible to separate a cube into two cubes, or a biquadrate into two biquadrates, or generally any power except a square into two powers with the same exponent. I have discovered a truly marvellous proof of this, which however the margin is not large enough to contain.

[2]This translation includes some corrections. In addition, the column h, which does not appear in the original, was added for convenience.

We do not know the exact date of this entry, but in 1638 he challenged Jumeau de Saint-Croix to find two cubes whose sum is a cube (and similarly for biquadrates), so it seems likely that he became convinced of the truth of FLT around that time.

Fermat himself gave a proof of FLT for $n = 4$ which he wrote in the margin at the end of the last book of Diophantus. However, since he does not seem to refer to this conjecture in his correspondence (except in the case $n = 4$), it might well have been lost to posterity had not his son Samuel published in 1670 another edition of Diophantus, interspersed with his father's comments (cf. Figure 2).

Now that I have dwelt in such detail on the birth of the conjecture, I will be much briefer with subsequent early developments. The case $n = 3$ was done by L. Euler in 1753 (with some additional details furnished later by C.F. Gauss). In 1825/28 Dirichlet and Legrendre (independently) settled the case $n = 5$ and in 1832 Dirichlet also did the case $n = 14$. The latter result became superfluous when G. Lamé proved FLT for $n = 7$ in 1839. Later, in 1847, Lamé also presented



Arithmeticorum Liber II. 61

interuallum numerorum 2. minor autem 1 N. atque ideo maior 1 N. + 2. Oportet itaque 4 N. + 4. triplos esse ad 2. & adhuc superaddere 10. Ter igitur 2. adscitis vnitatibus 10. æquatur 4 N. + 4. & fit 1 N. 3. Erit ergo minor 3. maior 5. & satisfaciunt quæstioni.

[illegible]

IN QVAESTIONEM VII.

CONDITIONIS appositæ eadem ratio est quæ & appositæ præcedenti quæstioni, nil enim aliud requirit quam vt quadratus interualli numerorum sit minor interuallo quadratorum, & Canones iidem hic etiam locum habebunt, vt manifestum est.

QVÆSTIO VIII.

PROPOSITVM quadratum diuidere in duos quadratos. Imperatum sit vt 16. diuidatur in duos quadratos. Ponatur primus 1 Q. Oportet igitur 16 − 1 Q. æquales esse quadrato. Fingo quadratum a numeris quotquot libuerit, cum defectu tot vnitatum quod continet latus ipsius 16. esto a 2 N. − 4. ipse igitur quadratus erit 4 Q. + 16. − 16 N. hæc æquabuntur vnitatibus 16 − 1 Q. Communis adiiciatur vtrimque defectus, & a similibus auferantur similia, fient 5 Q. æquales 16 N. & fit 1 N. $\frac{16}{5}$ Erit igitur alter quadratorum $\frac{256}{25}$. alter vero $\frac{144}{25}$ & vtriusque summa est $\frac{400}{25}$ seu 16. & vterque quadratus est.

[illegible]

OBSERVATIO DOMINI PETRI DE FERMAT.

Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos & generaliter nullam in infinitum vltra quadratum potestatem in duos eiusdem nominis fas est diuidere cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.

QVÆSTIO IX.

RVRSVS oporteat quadratum 16 diuidere in duos quadratos. Ponatur rursus primi latus 1 N. alterius vero quotcunque numerorum cum defectu tot vnitatum, quot constat latus diuidendi. Esto itaque 2 N. − 4. erunt quadrati, hic quidem 1 Q. ille vero 4 Q. + 16. − 16 N. Cæterum volo vtrumque simul æquari vnitatibus 16. Igitur 5 Q. + 16. − 16 N. æquatur vnitatibus 16. & fit 1 N. $\frac{16}{5}$ erit

[illegible]

Figure 2: Fermat's comment in the margin of the *Arithmetica* by Diophantus

a proof for general primes $p \geq 3$ based on the arithmetic of the number ring (called the *ring of cyclotomic integers*)

$$\mathbb{Z}[\zeta_p] = \{n_0 + n_1\zeta_p + \ldots + n_{p-2}\zeta_p^{p-2} : n_i \in \mathbb{Z}\},$$

where $\zeta_p = e^{\frac{2\pi i}{p}} = \cos(\frac{2\pi}{p}) + i\sin(\frac{2\pi}{p})$, but this proof was wrong since he assumed that the property of unique factorization holds in $\mathbb{Z}[\zeta_p]$ for all p. (In fact, it is now known that this holds if and only if $p \leq 19$.)

The most notable advance was made by Ernst Edward Kummer. Already a year before Lamé's hasty announcement, he had invented his "ideal numbers" (precursors of Dedekind's theory of ideals, which we use today) in order to rescue the unique factorization property for $\mathbb{Z}[\zeta_p]$. In 1847 (published 1850) he applied his methods to prove that (FLT_p) is true for all *regular* primes p. Furthermore, he devised a method (based on Bernoulli numbers) in order to test whether a given prime number is regular or not. Using this, he found in 1874 that of the 37 primes $p < 163$, only 8 are irregular: $p = 37, 59, 67, 101, 103, 131, 149$, and 157.[3]

After Kummer, there were many partial results on FLT for which I refer you to P. Ribenboim's excellent book [Ri1]. By combining these with Kummer's general result, S. Wagstaff was able to verify in 1976 with the help of a computer that Fermat's Last Theorem is true for all integers $n \leq 125,000$.

2. Recent Results

In the 1970's and '80's, the deep and powerful methods of Algebraic Geometry (as developed by A. Grothendieck and his school) lead to many significant advances in the theory of *Diophantine Equations* (named after Diophantus). Most of these were not (or did not seem to be) directly connected with FLT. A notable exception was the *Mordell Conjecture*, which had been formulated by L.J. Mordell in 1922, and which was then established through the work of G. Faltings in 1983, for which he received the Fields Medal[4] in 1986. Specialized to Fermat equations, Faltings' theorem yields the following finiteness result.

[3] This evidence suggests that there are more regular primes than irregular ones, and this is borne out by further calculations (e.g. 8399 of 13848 primes $p < 150,000$ are regular; cf. [Ri2]). In addition, C.L. Siegel proved in 1964 that if certain (unproven) random distribution property of Bernoulli numbers holds, then $\frac{1}{\sqrt{e}} \approx 0.61$ of all primes are regular. However, unconditionally it is still unknown whether there exist infinitely many regular primes (whereas it was shown in 1915 by Jensen that there are infinitely many irregular primes).

[4] The Fields Medal, named after the Canadian mathematician John Charles Fields (1863 - 1932), is the most prestigious prize for mathematical research. It is awarded every 4 years at the International Congress of Mathematicians to the top 2-4 researchers under the age of 40.

Theorem (Faltings, 1983) *For each $n \geq 4$, the set*

$$\{(x,y,z) \in \mathbb{Z}^3 : x^n + y^n = z^n \text{ and } (x,y,z) = 1\}$$

is finite.

While this was clearly a very significant result (particularly in its more general form), it did not convince skeptics about FLT. Indeed, there did not seem to be *any (conceptual) reason whatsoever* that the equation $x^p + y^p = z^p$ should not have the same five solutions (say) for all primes $p > 2$!

This changed drastically in the mid 1980's when not only one but two separate reasons were advanced. On the one hand D. Masser(1985) and J. Oesterlé(1988) proposed a very remarkable general conjecture (called the *ABC-Conjecture*) from which it would follow that not only the Fermat equation but also the twisted Fermat equation $ax^n + yb^n = zc^n$ (where $a, b, c \in \mathbb{Z}$ are fixed relatively prime integers) has only "trivial" solutions (in particular, only finitely many solutions) for all *sufficiently large* exponents n. This conjecture, as well as the statement about twisted Fermat equations (called the *asymptotic Fermat Conjecture*) is still open at present.[5]

On the other hand, in a Paris seminar in 1985, G. Frey suggested a method (based on some (vague) conjectures of Serre) that a certain well-known conjecture, called the *Taniyama Conjecture* (or (TWS)- Conjecture), should imply FLT. I cannot resist the temptation of relating a personal anecdote about this discovery. Indeed, I can still remember the day (but not the date - probably in the spring of 1982) when Gerd Frey, who is a good friend of mine, phoned me up and said: "I've just proved FLT, can you find the mistake?" Of course I couldn't, but after giving me an hour lecture he himself saw that there were a number of gaps to be filled. These gaps were formulated in terms of a precise conjecture by J.P. Serre in a letter to Frey in 1985 and became known as the "ε-Conjecture"; this was published as part of a far more general conjecture by Serre[Se] in 1987. In the meanwhile, Ken Ribet succeeded in 1986/87 to prove the ε-conjecture in an ingenious way; cf. Ribet[R].

By this time number theorists were (for the most part) convinced of the truth of FLT, for the contrary meant to deny the Taniyama Conjecture which, in turn, would involve a major rethinking of what we know (or conjecture to be true) today. Nevertheless, it was not expected to be proved soon, and so Wiles' announcement in 1993 came as a big surprise!

3. A Basic Principle

Before explaining the method of Frey/Ribet /Wiles, let me first formulate some basic principles that have evolved over the years concerning the nature of solutions of Diophantine equations and which are a partial motivation for the method. First, let me formulate the basic problem of Diophantine equations:

Problem: Find all the integer solutions $(x,y,z) \in \mathbb{Z}^3$ of a given Diophantine equation

$$F(x,y,z) = 0, \tag{2}$$

where $F \in \mathbb{Z}[x,y,z]$ is an integral polynomial.

Examples: 1) Fermat polynomials:

$$F(x,y,z) = F_n(x,y,z) = x^n + y^n - z^n.$$

2) Elliptic curves:

$$F_{a,b}(x,y,z) = y^2z - x^3 + axz^2 + bz^3,$$

where $a, b \in \mathbb{Z}$ and the discriminant $\Delta(F_{a,b}) = 16(4a^3 + 27b^3) \neq 0$.

To give you an impression of the difficulty of this problem, let me remark that at present **no general algorithm is known** which decides in a finite amount of time whether a given polynomial $F(x,y,z)$ has at least one non-trivial integer solution $(x,y,z) \neq (0,0,0)$ or not,[6] let alone an algorithm that finds all

[5] For comprehensive discussion of how these and other conjectures fit together, cf. Frey[Fr3].

[6] In fact, it is known that for integer polynomi-

the solutions! Let us, therefore, consider the following

Easier Problem: For each prime number p, solve the congruence

$$F(x,y,z) \equiv 0 \pmod{p}. \tag{3}$$

Clearly, this is a *finite problem* (for each p), for we need to check only p^3 values. In particular, the number of solutions modulo p,

$$\begin{aligned} N_p^*(F) &= \#\{(x,y,z) \in (\mathbb{Z}/p\mathbb{Z})^3 : \\ &\qquad F(x,y,z) \equiv 0 \pmod{p}\} \\ &= \#\{(x,y,z) \in \mathbb{Z}^3 : 0 \le x,y,z < p \\ &\qquad \text{and } p | F(x,y,z)\}, \end{aligned}$$

is finite: $N_p^*(F) \le p^3$. Put:

$$\begin{aligned} N_p(F) &= (N_p^*(F) - 1)/(p-1) \\ &= \#\text{of essentially distinct solutions} \\ &\quad \text{of (3) (excluding (0,0,0))}. \end{aligned}$$

Question: Do these numbers shed any light on the solutions of equation (2)?

The naive interpretation of this question is blatantly false: there exist polynomials $F(x,y,z)$ with only trivial integral solutions, yet $N_F(p) \neq 0$ for all primes p. In addition, it follows from a theorem due to H. Hasse and A. Weil that $N_F(p) \approx p$, for p large, so the mere existence of solutions modulo p cannot yield any information about the existence of integral solutions. Nevertheless, we have the following

Basic (Conjectural) Principle: the sequence of numbers

$$a_p(F) \overset{def}{=} (p+1) - N_p(F), \text{ as } p \to \infty, \tag{4}$$

should determine the nature of the solutions of (2).

For elliptic curves, this principle assumes the form of two very precise conjectures which have been partly verified:

(TWS)–Conjecture: - due to Y. Taniyama (1955), A. Weil (1967), G. Shimura (1971)

(B/SwD)–Conjecture: - B. Birch, H.P.F. Swinnerton–Dyer (1960's)

The (TWS)-Conjecture will be explained in the next section. I will not discuss the (B/SwD)- conjecture in detail here, but only mention the following recent result (which at the same time shows the importance of the (TWS)-conjecture):

Theorem 1 (V. A. Kolyvagin (1988), K. Murty, R. Murty (1991)).[7] *Let $E : F_{a,b}(x,y,z) = 0$ be an elliptic curve satisfying (TWS). Then the sequence of numbers*

$$a_p(E) = p + 1 - N_p(F_{a,b}), \quad p \to \infty,$$

determines a ("computable") real constant $L_E(1) \in \mathbb{R}$. If

$$L_E(1) \neq 0,$$

then the equation $F_{a,b}(x,y,z) = 0$ has only finitely many integral solutions $(x,y,z) \in \mathbb{Z}^3$ with $gcd(x,y,z) = 1$, and these can be explicitly calculated.

Note. The above theorem constitutes an explicit algorithm which has been implemented on a MAPLE package called APECS.

Example (Frey). The above leads to a *computer proof* (a true proof!) of FLT_3 and FLT_4, using only *four* short computer commands.

4. The TWS–Conjecture

Roughly speaking, the TWS-Conjecture may be viewed as stating that the numbers $a_p(E)$ possess many "hidden symmetries"; in particular, the knowledge of the a_p's for the first few p's determines *all the others.*

Before explaining this more precisely, let us look at the elliptic curve E defined by the equation

$$y^2 + y = x^3 - x^2.$$

als $F(x_1,\ldots,x_r)$ in $r \ge 13$ variables, no such algorithm can exist, as was shown by Matijasevič in 1970, thereby supplying a negative answer to Hilbert's 10th problem; cf. [DMR].

[7] This theorem was first proven by Kolyvagin under an additional hypothesis, which was then later removed by Murty-Murty and, independently, by D. Bump, S. Friedberg and J. Hoffstein.

The Elliptic Curve $E : y^2 + y = x^3 - x^2$

The number $N_p(E)$ of solutions of E over $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ and the number $a_p(E) = p+1-N_p(E)$ are given by:

p	2	3	5	7	11	13	17	19	23	29	31	37	41
$N_p(E)$	5	5	5	10	11	10	20	20	25	30	25	35	50
$a_p(E)$	-2	-1	1	-2	1	4	-2	0	-1	0	7	3	-8

On the other hand, the unique newform $f(z) \in S_2(\Gamma_0(11))$ of level 11 is:

$$f(z) = q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11n})^2 \;=\; \sum_{n=1}^{\infty} a_n(f)q^n$$
$$= q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 - 2q^{10} + q^{11} - 2q^{12} + 4q^{13} + 4q^{14}$$
$$-q^{15} - 4q^{16} - 2q^{17} + 4q^{18} + 2q^{20} + 2q^{21} - 2q^{22} - q^{23} - 4q^{25} - 8q^{26} + 5q^{27}$$
$$-4q^{28} + 2q^{30} + 7q^{31} + 8q^{32} - q^{33} + 4q^{34} - 2q^{35} - 4q^{36} + 3q^{37} - 4q^{39} - 8q^{41} + \ldots$$

Its first few Fourier coefficients at prime indices are:

p	2	3	5	7	11	13	17	19	23	29	31	37	41
$a_p(f)$	-2	-1	1	-2	1	4	-2	0	-1	0	7	3	-8

In this case, the numbers $a_p(E)$ have a very remarkable interpretation: each turns out to be equal to the p-th Fourier coefficient of the function f defined by product expansion

$$f(z) = q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11n})^2,$$

where $q = e^{2\pi iz}$ (see the insert on the top of this page). Now it can be shown that this function has many "hidden symmetries", i.e. it satisfies the transformation law (5) below (with $N = 11$), and that this characterizes the function f uniquely.

This phenomenon can be generalized to arbitrary elliptic curves, but for this we need the following two concepts:

1) The *conductor* $N = N_E$ of an elliptic curve $E = E_{a,b}$: this is a positive integer

$$N \mid \Delta_{a,b}$$

which is closely related to the discriminant $\Delta_{a,b}$ (and which is explicitly computable).

2) The space $S(N) = S_2(\Gamma_0(N))$ of *modular forms of level* N: this consists of (complex-valued) functions of the form

$$f(z) \;=\; \sum_{n=1}^{\infty} a_n(f)q^n, \quad \text{with } q = e^{2\pi iz},$$

where the $a_n(f) \in \mathbb{C}$ and the sum converges for $Im(z) > 0$; these are to satisfy certain additional properties such as the rule

$$f\left(\frac{az+b}{cz+d}\right) \;=\; (cz+d)^2 f(z), \qquad (5)$$

where $a, b, c, d \in \mathbb{Z}$ are any integers with $ad - bc = 1$ and $N|c$.

Properties: 1) $S(N)$ is a finite-dimensional $\mathbb{C}$-vector space. There is an explicit formula for its dimension $g_N := \dim_{\mathbb{C}} S(N)$, which is approximately $g_N \approx \frac{N}{12}$.

2) Each $f \in S(N)$ is uniquely described by its first $2g_N \approx \frac{N}{6}$ Fourier coefficients $a_1(f), \ldots, a_{2g_N}(f)$.

3) The space $S(N)$ has a *distinguished* $\mathbb{C}$-basis $\mathfrak{B}(N) = \mathfrak{B}^+(N) \cup \mathfrak{B}^-(N)$. The functions in $\mathfrak{B}^+(N)$ are called *newforms*, those in $\mathfrak{B}^-(N)$ *oldforms*. For each N, these forms are explicitly computable (and have been computed for $N \leq 10^6$).

The above properties show that for each N, the set of functions $\mathfrak{B}(N)$ is determined by a finite amount of data, and hence may be viewed as being explicitly known. The (TWS)-Conjecture relates the Diophantine numbers $a_p(E)$ to these functions as follows.

Conjecture (TWS): *For every elliptic curve E of conductor N, there is a (unique) newform $f(z) = \sum a_n(f)q^n \in \mathfrak{B}^+(N)$ of level N such that*

$$a_p(E) = a_p(f), \quad \text{for all primes } p \nmid N. \quad (6)$$

At first sight, this seems to be a rather daring and mysterious conjecture: why should the numbers $a_p(E)$ have anything to do with modular forms?

The first major piece of evidence for this conjecture was provided by A. Weil who showed in 1967 that its falsity would contradict a main principle of Number Theory (the principle that certain arithmetically defined functions (called L-functions) should have a functional equations). Shortly thereafter, G. Shimura[Sh] showed that the converse to the conjecture is in fact true:

Theorem 2 (Shimura, 1971). *For each $f \in \mathfrak{B}^+(N)$ with integral Fourier coefficients there is an elliptic curve E (of conductor N) such that (6) holds.*

Although this result provides us with many explicit (numerical) examples for which the (TWS)-Conjecture is true, it is too weak to prove that there are infinitely many elliptic curves which satisfy (TWS), for there is no way to guarantee that there are any modular forms *with integral coefficients* for large N. This, however, and much more, follows from the important theorem proven by Wiles[W] (with the help of R. Taylor[8]):

Theorem 3 (Wiles, 1995). *The conjecture (TWS) is true if N_E is squarefree.*[9]

As should be evident from the above discussion, Wiles's result goes much further than merely proving (FLT): it should be viewed as an important step towards realizing the goal of finding a general algorithm for solving Diophantine problems involving elliptic curves.

5. $\mathrm{TWS}_{ss} \Rightarrow \mathrm{FLT}$

Although the work of Wiles[10] clearly advances our understanding of the arithmetic of elliptic curves, it is less evident how it relates to FLT, and indeed, the deduction of FLT from Theorem 3 constitutes another major step in the proof of FLT. Here is a brief sketch of the ideas involved:

Proof of $\mathrm{TWS}_{ss} \Rightarrow$ FLT: Since FLT_3 and FLT_4 are known to be true, it is elementary to see that we can restrict attention to primes $p \geq 5$.

Suppose, therefore, that FLT_p is false, i.e. that there exist $a, b, c \in \mathbb{Z}$ with $abc \neq 0$ such that

$$a^p + b^p = c^p.$$

[8] The original proof of Wiles and Taylor is 130 pages long, and fills an entire issue of the Annals. Since its publication, a number of simplifications have been suggested by a number of people such as G. Faltings, H. Lenstra and F. Diamond; cf. [Di]. For an overview of the original proof, together with a lot of background information, the reader is encouraged to consult [DDT].

[9] Recently (February, 1997), Conrad, Diamond and Taylor have announced that they can prove that (TWS) is true as long as 27 does not divide N_E.

[10] Due to the age restriction, Wiles just missed getting the prestigious Fields Medal for his work. However, he has received many other awards, including an *Honourary Doctorate* from Queen's University in May 1997.

By interchanging a and b we may suppose without loss of generality that $2|a$, and so we have in particular that $16|a^p$. Consider the elliptic curve

$$E: \qquad y^2z = x(x - a^pz)(x + b^pz),$$

called a *Frey curve.*[11] Then:

1) $\Delta_E = (abc)^{2p}$

2) N_E is squarefree (this uses the fact that $16|a^p$).

Thus, by Wiles's theorem, there is an $f = f_E \in \mathfrak{B}^+(N_E)$ such that (6) holds.

Claim: Such an f_E does not exist!

The verification of this claim is really the heart of the proof. For this, Ribet[R] proves the following "Lowering the Level Principle" (also known as Serre's ε-Conjecture) which is a special case of Serre's general conjecture (cf. [Se]):

Theorem 4 ("Lowering the Level" - Ribet, 1991). *Suppose $f = f_E \in \mathfrak{B}^+(N)$ is a newform of level N. For a fixed prime number $p > 3$ let M_p denote the product of the prime numbers $q > 2$ such that $p|expt_q(\Delta_E)$. Then there exists $g \in \mathfrak{B}^+(N/M_p)$ such that*

$$a_n(g) \equiv a_n(f) \pmod{p},$$

for all $n \geq 1$ with $\gcd(n, N) = 1$.[12]

Conclusion. Apply this to f_E as above. Then by 1) we obtain that $M_p = \frac{N}{2}$, so by Ribet's theorem there is a newform $g \in \mathfrak{B}^+(2)$. But this is impossible since $\dim S(2) = 0$. Thus, no such modular form f_E can exist, so neither can E and hence no such Fermat triplet (a, b, c) exists!

[11] In his fundamental paper, Frey[Fr1](see also [Fr2]) showed how many Diophantine statements can be reduced to the study of elliptic curves by means of certain elliptic curves now called Frey curves.

[12] This theorem should be read with a grain of salt, for one cannot assume that g has coefficients in $\mathbb{Z}$. Thus, while the precise statement of the theorem is somewhat more technical, the basic flavour is the same.

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Queen's Putnam Team Ranks 10th in Competition

The 57th William Lowell Putnam Mathematical Competition was held December 7, 1996. In the 1996 Putnam Competition, a mathematical problem solving contest open to universities and colleges in Canada and the U.S.A., the Queen's team stood tenth. The Queen's members were Joanna L. Karczmarek, Michael A. Levi, Allan J. Roberts.

A total of 2,407 students from 408 colleges and universities in Canada and the United States participated in the Competition. There were teams from 294 institutions.

CEC '97

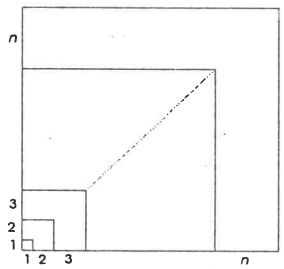
Over the weekend of March 6-9, Sunjay Nath (Mathematics & Engineering, Applied Mechanics, 97) and Dilip Andrade (Mathematics & Engineering, Control and Communications, 98) went to Moncton New Brunswick to compete in the Canadian Engineering Competition (CEC). After having previously placed second in the Debate category of the Ontario Engineering Competition held at McMaster University, Feb.14-16, the team qualified to enter CEC '97, hosted by the Université de Moncton. Sunjay and Dilip were pleased to have finished second in the nation in the category of Extemporaneous Debate, after competing against the best teams that each region of the country provided.

Sum of Cubes

Peter Taylor

The problem I posed last issue turned out to be so interesting that I'm including a small article about it. This is actually excerpted from my recent draft high school text book called IN PROCESS, which I distributed at the OAME meetings in Toronto in May. These are interesting times in high school curriculum reform, and I find myself spending a lot of time on it, time which I am hoping will contribute to some real changes.

We start with the familiar formulae for the sums of consecutive powers. First power: $1+2+3+\ldots+n=\frac{n(n+1)}{2}$ Second power: $1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$ Third power: $1^3+2^3+3^3+\ldots+n^3=\left(\frac{n(n+1)^2}{2}\right)$ Maybe the third is not quite so familiar, but we are certainly struck by the fact that:



$$1^3+2^3+3^3+\ldots+n^3=(1+2+3+\ldots+n)^2$$

What we have is a collection of numbers for which the sum of the cubes equals the square of the sum. By the way, there's a very nice geometric proof that this must hold, contained in the picture that sits above the formulae. [I am grateful to Halip Saifi for showing me this construction.]

Well now that leads us to ask the question: can you find any other "natural" collection of numbers for which the sum of the cubes equals the square of the sum?

Well here's such a collection. Take any positive integer N. Now make two columns. In column one make a list of all the divisors of N, and beside each of these, in column two, put the number of divisors of the column one number. Then it turns out that column two is always such a collection of numbers! Below I have presented the table for N=72.

Table for $N = 72$

Divisors of 72	# of divisors	cubes of col 2
1	1	1
2	2	8
3	2	8
4	3	27
6	4	64
8	4	64
9	3	27
12	6	216
18	6	216
24	8	512
36	9	729
72	12	1728
SUM:	60	3600

The question to ask is: what does this new example have to do with the old one? Is it really new, or is this just the old consecutive integer property dressed up to go out on the town? Let's examine a bunch of small cases

Table for $N = 5$

Divisors of 5	# of divisors	cubes of col 2
1	1	1
5	2	8
SUM:	3	9

Table for $N = 9$

Divisors of 9	# of divisors	cubes of col 2
1	1	1
3	2	8
9	3	27
SUM:	6	36

Table for $N = 8$

Divisors of 8	# of divisors	cubes of col 2
1	1	1
2	2	8
4	3	27
8	4	64
SUM:	10	100

Table for $N = 10$

Divisors of 10	# of divisors	cubes of col 2
1	1	1
2	2	8
5	2	8
10	4	64
SUM:	9	81

Table for $N = 12$

Divisors of 12	# of divisors	cubes of col 2
1	1	1
2	2	8
3	2	8
4	3	27
6	4	64
12	6	216
SUM:	18	324

Table for $N = 16$

Divisors of 16	# of divisors	cubes of col 2
1	1	1
2	2	8
4	3	27
8	4	64
16	5	125
SUM:	15	225

Well, there are certainly some patterns to be accounted for. For example, the N=5 table is small, having just a 1 and a 2. When will that happen?–precisely when N is prime. There's one result.

But the striking observation belongs to the tables for N = 8, 9 and 16. For these, column two is a set of the original type (consecutive integers) and so the "sum of cubes" property we have here is just the original one. Well now that's very encouraging. But there's more-Look! *Look at tables 8 and 9. What are the sums?–10 and 6. And their product is 60, which is the sum for 72. And 8 and 9 of course have product 72. Holy cow.*

Okay, let's slow down here. There's something very nice happening and we don't want to blow it. First of all, when will column two have this simple form?–being exactly a set of consecutive integers? And the answer is that this will happen N is a prime power. If N = pk, for some prime p, then column one will consist of the powers of p from 0 to k (these are the divisors of pk), and column two will therefore consist of the integers from 1 to k+1.

So at any rate, this is an important "start" on my question-is this "divisor" phenomenon new or not? What we've learned so far is that if N is a prime power, then it isn't new at all because the sum we get in column two is of the familiar form

$$\boxed{1+2+3+\ldots+n}$$

Now for the next observation. How are we to organize and understand the relationship between the 72-table and the 8- and 9-tables?

Let's start by asking how the divisors of 72 are related to the divisors of 8 and 9. Of course 72 is the product of 8 and 9, but more than this is true-this decomposition of 72 follows the prime factorization. That is, there are two primes in the factorization of 72, 2 and 3, and the 8 collects the 2's and the 9 collects the 3's:

$$\boxed{72 = 8.9 = 2^3.3^2}$$

What this means if you think about it is that the divisors of 72 are exactly the products of the divisors of 8 and the divisors of 9. It's important to be careful here. Certainly if I take a divisor of 8 and a divisor of 9 and multiply them together, I'll get a divisor of 72, but I'm saying more than that. I'm saying that if we make a list of the divisors of 8 and of the divisors of 9, and then take all

possible products of one list with the other, we'll get exactly the divisors of 72, with no repeats.

So now, let's ask about column two. To take an example, consider the divisor 12 of 72. I ask how many divisors 12 has, but I am going to try to find the answer, not by looking in column two in table 72 (where there's a 6 sitting right beside it) but by trekking over to tables 8 and 9. To do that, I write 12 as the product 4×3 so we look at the 4-row in the 8-table and the 3-row in the 9-table. Now column two in those two tables tells us how many divisors there are of each factor-4 has 3 divisors and 3 has 2 divisors. How many divisors does that give for 12?–well, $3 \times 2 = 6$ because the divisors of 4×3 are all the products of the divisors of 4 and the divisors of 3.

What has this told us?–*that each column-two entry in the 72-table is the product of the corresponding column-two entries in the 8- and 9-tables.* **YES!**

To emphasize this, I rewrite the 72-table replacing the entries by the appropriate products. This also prompts me to reorder the rows to make the structure easier to see.

The way we have written the 72-table shows in a very precise sense what it means to assert that this table is the "product" of the 8-table and the 9-table.

Table for $N = 8$

Divisors of 8	# of divisors	cubes of col 2
1	1	1
2	2	8
4	3	27
8	4	64
SUM:	10	100

Table for $N = 9$

Divisors of 9	# of divisors	cubes of col 2
1	1	1
3	2	8
9	3	27
SUM:	6	36

Table for $n = 72$

Divisors of 72	# of divisors	cubes of col 2
1=1·1	1=1·1	1=1·1
2=2·1	2=2·1	8=8·1
4=4·1	3=3·1	27=27·1
8=8·1	4=4·1	64=64·1
3=1·3	2=1·2	8=1·8
6=2·3	4=2·2	64=8·8
12=4·3	6=3·2	216=27·8
24=8·3	8=4·2	512=64·8
9=1·9	3=1·3	27=1·3
18=2·9	6=2·3	216=8·27
36=4·9	9=3·3	729=27·27
72=8·9	12=4·3	1728=64·27
SUM:	60=10·6	3600=100·36

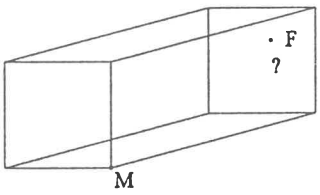
This article was cobbled together from comments received from many readers: Brian Boe and David Penney, both at the University of Georgia, Dan Haggarty in Etobicoke, and most particularly, Halip Saifi, now in Boston, but who was here last term and offered some great ideas about possible generalizations. In this vein, Dan Haggarty produced a remarkable general result. He observed that the sum of the kth powers of the integers from 1 to N is a polynomial in N of degree k+1 (check it out from the above formulae) and he even produced a general and quite lovely formula for the coefficients of this polynomial. It was a nice exercise in using a computer to generate data and then searching for "the pattern."

Problem: Two ants.

A rectangular room has dimensions $12 \times 12 \times 24$. That is, the floor and ceiling and both the side walls are 12×24 and the two end walls are 12×12. In the room there are two ants, a male and a female. The male ant is on the floor at one of the corners.

Now the female has positioned herself to be as far as possible from the male. That is, she has located herself at a point so that the male will take the longest possible time to get to her, given that he has to crawl along the walls, floor or ceiling of the room and will (of course) choose his path so that he gets to the female in the shortest possible time.

The question is: where is the female?



Well there's an obvious answer-the diametrically opposite corner. That's certainly the point which is farthest from the ant "as the crow flies." But an ant is not a crow.

Peter Taylor

An Invitation to our Alumni

The Communicator is an annual tradition. We send these few pages with information about what the department has been doing over the last year. This year, we would like to turn the tables, and ask you what you have been doing since you left Queen's.

We want to create a collection of the experiences of our alumni. This information will be made available to our undergraduate students, to give them an idea of where they can go and what they can do with their mathematics and statistics degrees.

We hope that having this material available will help our graduating students discover some of the opportunities available to them. The information will be posted on the department's web pages, so that our students can get at it easily. This is similar to a project undertaken a few years ago. This time, we hope that we will get a wider range of responses, and, due to the popularity of the web, the information will be more accessible to students.

Also, if you would like, we will make your e-mail address or phone number available to our students, so they can ask any questions that they may have about your profession or experiences. This would be a less involved version of Alumni Affairs' Mentorship program.

If you would like to take part, the following information would be useful to us:

- your name and the year you graduated
- your concentration (eg. major in statistics, medial in math and chemistry)
- your phone number. Also, note whether or not you would like this made available to students.
- your e-mail address (if you have one), again with a note indicating whether or not you would like this made available to students.
- the address of you web home page (if you have one)
- a few paragraphs describing you experiences with your mathematics or statistics degree
- any other information that you think would be relevant to our undergraduates.

You can send us the information via e-mail by sending to baker@mast.queensu.ca. It can be submitted on the web by visiting the departmental web page at http://www.mast.queensu.ca/ and following the "Alumni" link. Or, finally, it can be sent to Eddy Campbell, Head, Department of Mathematics and Statistics, Queen's University, Kingston, Ontario, K7L 3N6.

Head's Report
Eddy Campbell

First, some remarks about the present status of our library. The Mathematics and Statistics Library has become a Reading Room, with some 12,000 volumes and journals, mostly in Pure Mathematics, thanks to the extraordinary generosity of Graham and Stevie Keyser, who donated a quarter of a million dollars, and many others who donated smaller amounts. The interest from these donations will help pay for the Department's share of the service costs associated to the Reading Room. We are very grateful to the Keyers and to everybody who gave so generously. The balance of the former Library, some 25,000 volumes of monographs and journals, moved to the newly renovated Douglas Library.

By the time you read this, the Math and Engineering Reunion will have taken place, with many of our thirty years worth of graduates returning to campus for the long weekend at the beginning of August. We will try to involve our alumni in mentorship programs for our continuing students and seek out those alumni interested in partipating in a jobs network for continuing and graduating students.

This summer we have five students working on various projects in the Department. Kitty Lee, MTHE'99, is working on a curriculum development project with Norman Rice. We're examining our service teaching in the faculty of Applied Science seeking ways to

incorporate technology and encourage independent learning. Students in Applied Science have extraordinary demands made upon their time, so when we think of independent learning for this group of students, it is important to provide structure and feedback to ensure that students don't fall behind. Robert Burke, MTHE'99, is helping Kitty and also helping out with the Math and Engineering Reunion. Scott Siegler, MTHE'98, is helping out in the Control and Communications Lab. Greg Baker, ArtSci'98, is helping prepare updated web pages for students and faculty as well as helping Morris Orzech on a curriculum development project. Finally, Erik Jensen, ArtSci'99 is working as a research assistant with Ram Murty, and the invariant theory group consisting of Ian Hughes, Jim Shank, David Wehlau and myself.

Thanks are due to the new Dean of Applied Science, Tom Harris, for funding Norm's curriculum development project. Morris' team has received funding from the Dean of Arts and Science for some years now - our current Dean is Bob Silverman. I should also mention that we were able to hire Erik as a research assistant with the help of generous donation to the Trust Fund.

In December, the Canadian Engineering Accreditation Board (CEAB) "terminated" the engineering degree offered in the Faculty of Applied Science by the Department of Geological Engineering. Various nasty repercussions are still reverberating throughout campus. Changes are required in the way the four Arts and Science Departments administer their engineering degrees but it is too soon to predict the effect on this Department. We do not wish to make changes to our program without first seeking input from interested and concerned alumni, as well as our students and faculty. Please write us with your current address, by snailmail or email so that we may consult with you.

The Dean of Arts and Science collapsed 8 of 12 open positions in the Faculty this year as a result of further cutbacks in funding. Many departments suffered non-salary budget cutbacks as well. For the first time in many years, Mathematics and Statistics was spared. We do have one open position which we use in support of PostDoctoral Fellows. This position was not collapsed. However, partly in anticipation of further cutbacks, the Department has sought permission from the Dean to advertise this position. We will seek a mathematician or statistician eligible for registration as a Professional Engineer. In addition, we are still currently advertising for a communications or information theorist eligible for registration as a Professional Engineer. This will bring the faculty complement of Professional Engineers to five, which we believe will ensure that our Mathematics and Engineering program is fully accredited by the CEAB at our review in the year 2000.

In addition to the CEAB review, we also face a review by the Ontario Council on Graduate Studies, as well as an internal academic review, all of these to occur in the year 2000. We are preparing for those reviews now, in the hope that we will be able to collect sufficient information sooner rather than later.

We are very much trying to attract more students in our honours programs by recognizing that a degree in Mathematics or Statistics is valuable beyond preparation for graduate training in these disciplines. With this in mind, we'd love to have our home page link to alumni home pages. This seems like a good way to tell potential students of the value of a degree in Mathematics and Statistics.

I take this opportunity to ask our graduates to get in touch with us, perhaps by email. There are many issues we would enjoy discussing with you, and perhaps you would like to stay in touch with each other. We can help you to do that and you can help us. For example, we recently designed a new stream for our control and communications option offered with the Department of Electrical and Computer Engineering as part of the Mathematics and Engineering degree. The stream is designed to produce graduates interested in high technology industries. We'd like feedback from alumni involved in those industries.

IF UNDELIVERED RETURN TO:
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