

QUEEN'S MATHEMATICAL COMMUNICATOR



SUMMER 1998



Leonhard Euler to Christian Goldbach in a letter of June 19, 1742: That every number which is resolvable into two prime numbers can be resolved into as many prime numbers as you like, can be illustrated and confirmed by an observation which you have formerly communicated to me, namely that every even number is a sum of two prime numbers.

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Queen's Mathematical Communicator
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GOLDBACH'S CONJECTURE AND VARIATIONS

M. Ram Murty*

In a letter dated 7th June, 1742, Christian Goldbach asked the famous mathematician Leonhard Euler if every even number greater than two is the sum of two prime numbers. This question, referred to as Goldbach's conjecture, is still unresolved as of today.

In 1937, I. M. Vinogradov showed that every "sufficiently large" odd number can be written as a sum of three prime numbers. Let us note that Goldbach's conjecture implies such a result since any odd number minus three is an even number and so by Goldbach should be a sum of two primes. Thus, Vinogradov's theorem is a major advance. It also implies that every sufficiently large even number can be written as the sum of four primes since any even number minus three is an odd number. Vinogradov's work led to the development of an important method in analytic number theory called the method of trigonometric sums. It has been successfully used to attack other additive questions such as Waring's problem, about which we shall say more later.

Vinogradov's result still leaves some open questions. Namely, how big is "sufficiently large"? In 1956, Borodzkin calculated "sufficiently large" to mean greater than $10^{7,000,000}$. In 1989, Chen and Wang reduced this lower bound to $10^{43,000}$. Even this reduction is, at present, beyond the range of machine calculations! That is, we have not yet verified that every odd number $< 10^{43,000}$ can be written as a sum of three primes.

There is a celebrated hypothesis, known as the Riemann hypothesis, that has a ubiquitous presence in many questions in number theory. The hypothesis asserts the following:

*This is a summary of the Coleman-Ellis lecture given at Queen's University in November 1997.

if the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} = 0$$

for some $s \in \mathbf{C}$, with $0 < \operatorname{Re} s < 1$, then $\operatorname{Re} s = \frac{1}{2}$.

In 1997, Deshouillers, Effinger, te Riele and Zinoviev proved that if we assume a (generalized)[†] Riemann hypothesis, then every odd number greater than 5 is a sum of three primes.

As a complement to the Vinogradov theorem, Chen proved in 1966 that every sufficiently large even number can be written as the sum of two primes or as a sum of a prime and a number with at most two prime factors. Chen's theorem uses the very sophisticated lower bound sieve method.

In mathematics, when a problem cannot be attacked directly, we consider two fruitful methods of investigation. One is the method of variations of the problem. Namely, are there consequences of the conjecture that can be proved? In a sense, the Vinogradov and Chen results are of this nature. Another method is that of analogy, where we ask the same question but in a different context. Both methods have their virtues and are useful in isolating the difficulties in the original problem. Below, we shall look at both methods of investigation as applied to the Goldbach conjecture.

Let $G(x)$ be the number of even numbers which can be written as the sum of two

[†]Notice that it is not hard to show that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}$ converges for $\operatorname{Re} s > 0$. The generalized Riemann hypothesis is the assertion that if $\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = 0$ for some $s \in \mathbf{C}$, with $\operatorname{Re} s > 0$ and $\chi : (\mathbf{Z}/m\mathbf{Z})^* \rightarrow \mathbf{C}$ is a non-trivial homomorphism of the group of coprime residue classes mod m , then $\operatorname{Re} s = \frac{1}{2}$.

primes. Then, the Goldbach conjecture implies that

$$\lim_{x \rightarrow \infty} \frac{G(x)}{x} = \frac{1}{2}.$$

We will refer to this as the weak Goldbach conjecture. It was proved in 1937 by van der Corput, Estermann and Cudakov. The proof, which we shall not discuss here, is not difficult.

What we will discuss is a remarkable and elementary idea of Shnirelman, which he discovered in 1930 and used to attack Goldbach's conjecture. So, let A be a sequence of integers

$$0 < a_1 < a_2 < \dots$$

Define the S -density of A to be

$$d(A) = \inf_{n \geq 1} \frac{A(n)}{n},$$

where $A(n)$ is the number of elements in A which are $\leq n$. For example, $d(A) = 1$ if and only if A is the set of natural numbers. Also, $d(A) = 0$ if $1 \notin A$. Notice that the sequence of primes has S -density zero. The sequence of squares, or cubes, or more generally, k -th powers also has S -density zero.

Here is a basic observation of Shnirelman:

Theorem 1 Suppose $d(A) > \frac{1}{2}$. Then every natural number can be written as the sum of two elements of A .

Proof Since $d(A) > \frac{1}{2}$, we have $A(n) > \frac{n}{2}$ for every n . The elements

$$a \in A, a \leq n$$

and

$$n - a, a \in A, a \leq n$$

cannot all be distinct for otherwise $2A(n) \leq n$, a contradiction. Thus, for some $a' \in A$, we must have $n - a' \in A$. Thus $n = a + a'$, with $a, a' \in A$.

Shnirelman then asked the question: given two sets A and B , how many elements are there in

$$C = A + B = \{a + b : a \in A, b \in B\}?$$

For a natural number n , let us compare $C(n)$ with $A(n)$ and $B(n)$. Let $A(n) = k$ and write

$$0 < a_1 < a_2 < \dots < a_k < n.$$

Then $C = A + B$ certainly contains the elements $a_i + r$ where $r \leq a_{i+1} - a_i - 1$ and $r \in B$. These are clearly distinct. Taking the elements of A also into account, we get

$$C(n) \geq A(n) + B(a_1 - 1) + B(a_2 - a_1 - 1) + \dots + B(a_k - a_{k-1} - 1) + B(n - a_k).$$

Since

$$B(a_1) \geq d(B)(a_1 - 1)$$

and for $i \geq 1$,

$$B(a_2 - a_1 - 1) \geq d(B)(a_2 - a_1 - 1)$$

...

$$B(n - a_k) \geq d(B)(n - a_k),$$

we obtain

$$\begin{aligned} C(n) &\geq A(n) + d(B)\{(a_1 - 1) + (a_2 - a_1 - 1) \\ &\quad + \dots + (a_k - a_{k-1} - 1) + (n - a_k)\} \\ &= A(n) + d(B)(n - k) \\ &= A(n)(1 - d(B)) + d(B)n \end{aligned}$$

because $A(n) = k$. Simplifying, we get:

Theorem 2 $d(A + B) \geq d(A) + d(B) - d(A)d(B)$. Another way to state this is:

$$d(A + B) \geq 1 - (1 - d(A))(1 - d(B)).$$

Let us agree to write $A^{(2)}$ for $A + A$, $A^{(3)}$ for $A + A + A$, and so on. Then we have:

Theorem 3 If $0 < d(A) < 1$, then for some k , $d(A^{(k)}) > \frac{1}{2}$.

Proof From

$$d(A + B) \geq 1 - (1 - d(A))(1 - d(B))$$

we have by induction,

$$d(A^{(k)}) \geq 1 - (1 - d(A))^k$$

from which the result follows.

If we modify the definition of S -density as

$$d(A) = \inf_{n \geq n_0} \frac{A(n)}{n},$$

then all of the above results hold qualified by “for all integers $n \geq n_0$.”

Combining Theorem 3 with Theorem 1 yields the important:

Theorem 4 If $0 < d(A) < 1$, then there is a constant k_0 such that every natural number can be written as a sum of at most k_0 numbers from A .

Naturally, we would like to apply this to Goldbach’s conjecture. Let P be the set of primes with the number 1 also included. What is the S -density of $P + P$? If the weak Goldbach conjecture is true, then the natural density is $\frac{1}{2}$. Recall that $G(x)$ is the number of even numbers which can be written as a sum of two primes. Then

$$G(x) > (.49)x$$

for all x sufficiently large. It is then “somewhat” clear that $R = P + P$ has S -density $> .49$. Therefore $R + R$ has S -density $> .49 + .49 - (.49)^2 > \frac{1}{2}$. Hence, we can expect that every sufficiently large natural number can be written as a sum of at most 8 primes.

In 1942, Mann proved an important refinement of Theorem 2.

Theorem 5 (Mann, 1942) $d(A + B) \geq \min(1, d(A) + d(B))$.

With this theorem, it becomes plausible by the above reasoning that every number can be written as a sum of at most six primes. This result was proved rigorously by O. Romaré in 1995.

It is remarkable that Shnirelman’s method, which is very elementary, can yield such powerful results. It is therefore tempting to apply the method to other questions.

Recall a classical theorem of Lagrange which states that every natural number can be written as a sum of four squares. A lesser known theorem is that of Wieferich that says every natural number can be written as a sum of nine cubes. This suggests the following question: does there exist a number $n(k)$ such that every natural number can be written as a sum of $n(k)$ k -th powers? This question is known as Waring’s problem. For exam-

ple, by Lagrange and Wieferich, $n(2) = 4$ and $n(3) = 9$.

Shnirelman’s method can be applied to solve Waring’s problem and this was done by L. K. Hua. The problem was originally solved by Hilbert using a more complicated method.

As mentioned at the outset, the method of analogy is a useful path of investigation of difficult questions. If we view Goldbach’s conjecture as an assertion about the ring of integers, then we can ask if the analogue of Goldbach’s conjecture is true for other rings.

In particular, one can consider $\mathbf{F}_p[x]$, the polynomial ring over the finite field of p elements. Effinger and Hayes have shown that Vinogradov’s method can be extended to show that every polynomial can be written as a sum of three irreducible polynomials. At present, it is unknown if every polynomial in $\mathbf{F}_p[x]$ can be written as the sum of two irreducible polynomials. It seems, therefore, relevant to tackle this “easier” question before tackling the Goldbach conjecture.

If instead of $\mathbf{F}_p[x]$, we consider $\mathbf{Z}[x]$, then it is not difficult to show that Goldbach’s conjecture holds in this context. This is a nice application of Eisenstein’s criterion. The solution has been written up by three undergraduate students of mine, and their work appears in *Comptes Rendus/Mathematical Reports*, Vol. 20, No. 3 (1998).

Our Alumni

David Hamilton graduated from Queen’s in Statistics some 30 years ago. At present he is director of Statistics at Dalhousie University. David took up the invitation to our alumni to write to us about how their education at Queen’s influenced their subsequent careers by sending us the following contribution.

My education at Queen’s started in 1968, just before the department moved into Jeffery Hall. My early academic career in Mathematics was undistinguished. I couldn’t see

the relevance of much of the material I was exposed to. Those who were not in the honours stream seemed to learn how to do much more useful things than we did.

Fortunately, the department included a vigorous group of statisticians at that time, and my first course in that subject was enough to lead me to concentrate in that area. Don Watts, among others, made it apparent that Statistics was a viable and interesting discipline on which to base a career. I subsequently went on to do a Master's and Ph.D. under his supervision, and apart from a short stint at Statistics Canada between those degrees, have been a professor of Statistics at Dalhousie University ever since.

As a faculty member, I am involved in teaching Statistics at all levels, in advising undergraduate and graduate students, in research in Statistics, and in collaboration with scientists in Biology, Chemistry and Medicine. Most of the Statistics I do, including the research, involved Mathematics which I learned in my first two years at Queen's. One can never know enough matrix theory for Statistics, it seems, and Norm Pullman's fourth year course has stood me in good stead. Mathematics certainly plays a role in Statistics, but many of the most important ideas, like the importance of randomizing experiments, are non mathematical. The evolution of computers has led to a decreased reliance on many mathematical techniques, and the introduction of important new statistical ideas like the bootstrap.

My case is not unusual. Others who were graduate students with me have gone on to be chairs of Departments of Statistics (Doug Bates, University of Wisconsin), prominent researchers in civil service (Bill Ross, Health Protection Branch; Rick Burnett, Health Canada) and in health research (Andrew Willan). More recent graduates working in eastern Canada include Debbie Dupuis (Daltech) and Tess Astatke (NS Agriculture College).

There is an urgent need for more statisticians in industry, government and universi-

tites. When our department advertised two positions, one in Statistics, the other in Mathematics, there were less than 20 applicants in Statistics and well over 100 in Mathematics. Students graduating from Dalhousie in Statistics have no trouble finding employment.

When I look at the current state of Statistics at Queen's I am saddened and angered. What was once a vibrant group of 8 or 10 statisticians has been reduced to 3 or 4. Many of those who have retired have not been replaced, and those who have been hired have found the environment hostile or unsatisfying, and have moved on. Incoming students to Queen's may no longer be exposed to this interesting and rewarding discipline. At a time when Statistics programs are flourishing across the country, Queen's is going in the other direction.

[editorial note: For some reasons why Statistics at Queen's is temporarily in difficulties read the "Head's Report" on page 6]

Louis H. Broekhoven 1932-1998

Terry Smith

A personal reminiscence given at the Memorial Service on 7 February 1998

Louis was a statistician, as I am. I knew Louis almost as long as I have been in Kingston, and we were colleagues during the years he spent in the Department of Mathematics and Statistics at Queen's.

First, I want to touch on Louis' professional life. I first knew Louis as the person in charge of statistical computing at the Computing Centre. In that position he was heavily involved with statistical consulting for members of the Queen's community. He joined the Department of Mathematics and Statistics during the period when Statistics was developing as a separate discipline at Queen's with the creation of honours, Master's and PhD programs, the establishment of STAT-LAB and the beginnings of applied statistical

research. Louis was a key player in this development. With his extensive experience as a statistician in industry, he brought a unique perspective and considerable wisdom to our deliberations.

Louis could always be counted on to bring common sense to any discussion, and to state his views directly and frankly. He would not hesitate to tell you if he thought an idea you were spouting was "a load of rubbish". He was passionate about things he held to be important. One of his ideas that made a lasting impression on me was the importance of the collaborative role of the statistician as a partner with other scientists in the pursuit of knowledge. He worked hard and expected the same standard of others. During his time at Queen's he helped countless students, faculty and administrators with statistical problems, first at the Computing Centre and later as Director of STATLAB. He supervised a number of graduate theses and played an important role in numerous research projects in a wide spectrum of departments. Louis believed in total involvement in the project; I am reminded of his adventures, airborne and otherwise, placing fox baits containing rabies vaccine in the Eastern Ontario bush for the rabies study led by Rollie Tinline of the Geography Department.

When I learned that Louis had worked for Guinness Brewery in Dublin, I was inclined to regard him with a degree of reverence deserving of one who has trod on hallowed ground. W. S. Gossett, one of the fathers of modern statistics best known as the inventor of the t-test, spent his career at Guinness Brewery, and I had no doubt that working in that exalted place gave one special statistical insight. In fact, I never did ascertain that Louis had not spent his time there in a warehouse stacking barrels of the famous stout.

Louis' presence in the department meant a lot to me personally. While I was on sabbatical in Glasgow in 1993-94, the department held a retirement party for several colleagues, Louis included. Here is part of a letter I wrote to Louis on that occasion:

"In thinking about the significance of this occasion, I am reminded that you were my colleague for most of my working career to date. I am also aware that you had a substantial influence on my thinking and professional development throughout that time. When I came to Queen's, I was the product of a mathematical statistics education. From you I learned about the real statistical work of answering scientific questions and helping researchers to understand their data. At times when I had to make major career decisions, I was fortunate to be able always to count on your advice and support."

It was Louis' suggestion and encouragement that prompted me to establish a course in statistical consulting; I think he knew rightly that the course was one I would find particularly rewarding. Louis got me involved in STATLAB and subsequently turned over the reins. I looked to him for advice and guidance in my career choices; I valued his wisdom and honesty; I knew I could rely on him for a verbal "kick in the pants" if that was necessary. I missed him after he retired. He laughed when I once suggested that he come back and help us out. It was disappointing but not surprising that he showed little interest in that idea. I didn't blame him in the slightest and I envied him his woodlot, his winemaking, his woodworking, and his time with his wife, children and grandchildren.

Good friends and trusted colleagues are not plentiful. To lose such a one as Louis is cause for great regret.

Mathematics and Engineering Reunion '97

Robert Burke Sc '99

(Robert Burke was employed by the Department of Mathematics and Statistics in the

summer of 1997 to help organize the reunion, and to help with other tasks including preparing some materials for classes in the Fall of '97. We're happy to have him working for us again this summer)

Last summer's Math and Engineering 30th Anniversary Celebration brought more than 60 alumni back to Queen's for a weekend of fun with old friends. Graduates came from as far away as British Columbia (Vijay Bhargava '74) and as far back as the second graduating class (Gregory Gauld '68). Math and Engineering's rapid growth in recent years, as well as overwhelming interest from alumni, made last summer the perfect time to invite our alumni back for a romp around Kingston.

Activities kicked off on Friday night with a reception in Victoria Hall. Saturday afternoon began with a barbeque, followed by a series of demonstrations: Drs Ron Hirschorn and Jon Davis took the helm with the high-tech robotics of the Control Lab; Dr Alajaji and grad student Ali Nazer explained the inner workings of the Communications Lab; and undergrad Robert Burke ('99) presented The Legend of the Greasepole, a computer game being developed by a team of students. The afternoon's events culminated in a talk by Dean of Engineering Tom Harris, who discussed his vision of the role of Math and Engineering in Queen's Applied Science program.

Saturday evening's banquet took place in the newly revamped Ban Righ dining room. With Dr Dan Norman acting as master of ceremonies, we were entertained by after-dinner speakers representing the first three decades of "Apple Math." Professor Jacke Hogarth, founder of the Math & Eng program, talked about the program's beginnings; and graduates David Simmons ('73), Marilyn Lightstone ('85) and Andrew McKellips ('92) reminisced about their years as students, and reflected on their experiences since graduation.

Leonarg Segall ('79) sent us a letter because he was unable to attend. You may enjoy reading his reminiscences by visiting our Alumni Home Page, <http://mast.queensu.ca/alumni>, and click on Alumni News.

The reunion wound down on Sunday afternoon with a brunch on the patios of Princess Street. Responses from alumni suggest the reunion was an overwhelming success. Many thanks to Marge Lambert, who helped ensure that things ran smoothly, and to all the alumni who took the time out to visit.

As for another reunion? Dr Eddy Campbell, Head of Math and Stats, has hinted it's a definite possibility. Keep your ear to the ground.

Math & Engineering Grads Are Great Speakers!

Dan Norman

Since 1990, we have had a fourth-year seminar course MATH 494. The main component of the course consists of 50 minute talks by people about aspects of their work as engineers. We have been lucky to have mostly Math and Engineering graduates as our speakers – and students really enjoy hearing about the work that these graduates are doing. Some of the talks are very hi-tech, some are quite mathematical, some focus on entrepreneurial aspects, and some emphasize engineering applications of fairly basic engineering science.

Volunteers are most welcome! Get in touch with Dan Norman (Phone 613-545-2431, Fax 613-545-2964, E-mail normand@mast.queensu.ca, Mail Dan Norman, Department of Mathematics and Statistics, Queen's University, Kingston, Ontario, K7L 3N6).

Head's Report

Eddy Campbell

We have made two outstanding appointments this year: Tamás Linder and Andrew Lewis. Tamás studies communications theory, joining Fady Alajaji, Jon Davis and Glen

Takahara. Tamás' degrees are in Electrical Engineering, an MSc at the Technical University of Budapest and a PhD at the Hungarian Academy of Science. Tamás arrived in Kingston earlier this month from the US where Tamás was visiting scholar at the University of California, San Diego. His family will be joining him later. Andrew works in nonlinear control theory, joining Jon Davis and Ron Hirschorn. Andrew did his undergraduate work in Mechanical Engineering at the University of New Brunswick, and obtained his MSc and PhD at the California Institute of Technology. He is currently on a PostDoctoral Fellowship at the University of Warwick and will join us in November.

Among their duties, both Tamás and Andrew are expected to support the Mathematics and Engineering program. This program is accredited by the Canadian Engineering Accreditation Board (CEAB) and faces its next review in the year 2000. We have had trouble with accreditation in the past because the small number of faculty with status as Professional Engineers. The concern is that courses with significant Engineering Science and Design content be taught by Professional Engineers. The appointments of Tamás and Andrew bring us to five faculty members registered or eligible to register as Professional Engineers and this will allow us to satisfy this requirement of accreditation. We have also cross-appointed Jim McLellan into the Department. Jim is a Mathematics and Engineering graduate ('81), now on faculty with the Department of Chemical Engineering.

The Mathematics and Engineering program continues to be attractive to students in the Faculty of Applied Science. Our incoming second-year class of 50 students, third year has 25 students, while fourth-year has 45 students, not including seven students in the experience option. The total makes us the third largest department in the Faculty behind Electrical and Computing Engineering and Mechanical Engineering.

It is certainly worth mentioning that the experience option is available to all students

at Queen's. In the experience option students finishing their third year can take degree-related work in industry for up to sixteen months before returning to finish their degrees. This is the Queen's version of co-operative programs in place at other institutions. From an industry point of view, it offers a close-up look at some outstanding students at a relatively low cost. The program is administered by Sandra McCance here at Queen's: her email address is mc-cances@post.queensu.ca

There is a new memorandum of understanding governing relations between departments in Arts and Science with programs in the Faculty of Applied Science and that Faculty. It was viewed as necessary given the turmoil which resulted from the CEAB's decision to terminate the engineering degree offered by the Geology Department. This decision has since been rescinded because of massive changes to the program.

I'm delighted to tell you that Ram Murty has been recognized with a Killam Research Fellowship. The award allows him two years off of his regular duties to concentrate on his research into sieve methods. Sieve methods have been used for centuries to distinguish between prime numbers (numbers divisible by only one and themselves) and other numbers. Recently Ram and others have discovered exciting new applications of sieve methods to problems in cryptography. Modern cryptography uses fast computers and algorithms to encode and protect confidential data from prying eyes. Ram plans to write a book to help make complex sieve techniques more accessible and indicate how such methods can be applied to solve problems in number theory, arithmetic geometry, cryptography and computer data security. Ram was also awarded a distinguished lectureship at Brown University.

I'm also delighted to tell you that Morris Orzech was recognized with an Ontario Council of University Faculty Association teaching award. I helped put the nomination together. It was great fun, and an inspiration, to talk

with so many former and current students who were so enthusiastic and happy to recall their time with Morris in the classroom. And David Cardon, a Postdoctoral Fellow with us, and Leo Jonker won awards for their teaching to first-year students in Applied Science.

This summer we have three students working on various projects. Robert Burke, MTHE '99, is back working on the Department's database, and in setting up a jobs network for our students. And so is Erik Jensen, ArtSci '99, working as research assistant for Ram Murty, and the invariant theory group consisting of Ian Hughes, Jim Shank, David Wehlau, and myself. Sarah Sumner is also working as a research assistant for Ram.

The Mathematics and Statistics Library has become a Reading Room with some 12,000 volumes, thanks to the extraordinary generosity of Graham and Stevie Keyser, who donated a quarter of a million dollars, and many others who donated smaller amounts. We hope to create a research facility in the Reading Room for our graduate students and faculty. This will involve creating good quality study space and providing appropriate wiring for workstations.

This has been a particularly tough few years for universities in Ontario, and it has forced us to make difficult choices. While we have been able to retain and enhance our Mathematics and Engineering program, our Statistics program has suffered. In my first year as Head, we lost two faculty in Statistics: Tom Stroud took early retirement and Ed Chow accepted a position in industry in California. That brought our faculty complement of statisticians to five. This year, Duncan Murdoch accepted a position in the Department of Statistics at the University of Western Ontario. We certainly wish Duncan all the best while at the same time regretting his loss.

We do have plans to return to good health in Statistics. The Queen's National Scholars program provides a competition among departments to appoint extraordinary members of faculty. There are usually two to

four such appointments each year. The program provides "bridging" funds to expected retirements. We are advertising in the hope of attracting an outstanding statistician by means of this program. In addition, we will be seeking permission this fall to advertise for a senior statistician. We may be able to use some portion of the Mathematics and Statistics Trust Fund to build our bridge to an expected retirement in support of this initiative.

Problem

Peter Taylor

This nice problem was shown to me by Peter Harrison at a high school curriculum meeting we had in May at the Fields Institute. It emerges directly from the writings of Galileo in trying to understand the effects of gravity on a falling object. That was interesting to me, as the work I've been doing this year with high school students has involved doing experiments which re-create the investigations of centuries ago into the workings of water flow, gravity, air pressure and heat transfer. Anyway, there's some nice geometry. Getting the answer is only the beginning—the challenge is to see and explain "why".

A whole family of planks radiate from a point at different angles down to the ground below. On each plank there is a ball sitting at that top point (so the different balls are all coincident at the beginning). At $t=0$, each ball begins to roll down its plank in a frictionless manner. [There is a vertical plank which doesn't even have to be there, and its ball simply falls to the ground.] The problem is to find the locus of the family of balls at any time. That is, at any moment the balls will all lie along a natural curve. What is that curve?



The two ants problem

(from Summer '97 issue)

Peter Taylor

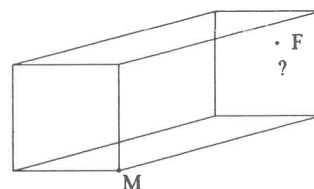
A rectangular room has dimensions $12 \times 12 \times 24$. That is, the floor and ceiling and both the side walls are 12×24 and the two end walls are 12×12 . In the room there are two ants, a male and a female. The male ant is on the floor at one of the corners.

Now the female has positioned herself to be as far as possible from the male. That is, she has located herself at a point so that the male will take the longest possible time to get to her, given that he has to crawl along the walls, floor or ceiling of the room and will (of course) choose his path so that he gets to the female in the shortest possible time. The question is: where is the female?

Well there's an obvious answer—the diametrically opposite corner. That's certainly the point which is farthest from the ant “as the crow flies.” But an ant is not a crow.

Solution

Solutions were received from Robert Thomas (Applied Mathematics at Manitoba), Ross Ethier (Mechanical and Industrial Engineering at Toronto) and Uri Fixman (Mathematics Emeritus at Queen's). There are two main parts to the solution. The first is to argue, using some form of symmetry, that the female has to be on the main diagonal of the far wall, and the second is to locate the exact point on that diagonal. The first part can be tricky as it's not so hard to convince yourself it must be true, but it's hard to see how to make the argument neatly, and I didn't expect many students to have great success with it. The second part is more computational, though you still have to be aware of what you're doing.



First part. I partition the end wall into three separate parts, the diagonal D , the part below the diagonal A , and the part above the diagonal B . The diagram at the right gives 4 ways to open up the box. On each version of the end wall, the diagonal is drawn and the A region is shaded. We first argue that for any point in A the shortest path must go “up,” that is, it must go to one of the top two versions of the end wall. Here’s the argument. Below right I mark four versions of a random point x in A , and my assertion is that the shortest path to x will be a straight line drawn to x_1 or to x_2 . This will follow if I can show that the line joining M to x_3 is longer than the line to x_2 and the line joining M to x_4 is longer than the line to x_1 . And the argument for these uses symmetry. For example consider x_3 . Let y_3 be its mirror image in the diagonal. Then y_3 is clearly closer to M than x_3 . But y_3 is the same distance from M as x_2 (by symmetry). So x_2 is closer than x_3 .

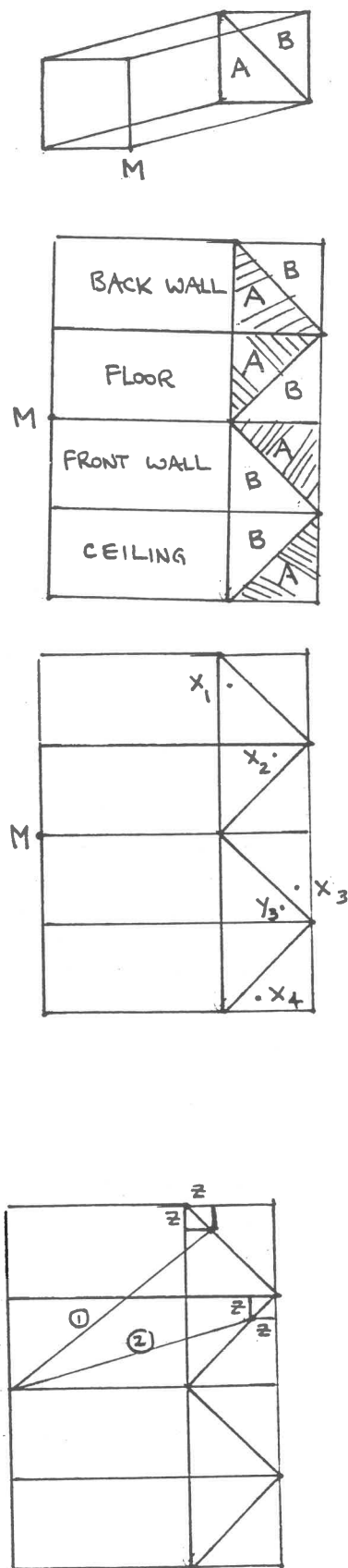
Now I argue that the female could not be anywhere in A . Indeed, no matter where x_1 or x_2 are there will always be points in A that are farther from M . Indeed, since x is not on D , the perpendicular from x to D will contain points that are farther from M by both the x_1 and the x_2 routes.

A similar argument shows that the female could not be anywhere in B . We conclude that the female must be on the diagonal D .

Second part. So let her be at distance z (horizontally and vertically) from the opposite corner. I now argue that the point at $z = 3$ is the farthest from M —that’s the point that’s a quarter the way down the diagonal. First I calculate the distance² from M to the female by each of the two paths:

- (1) $(24 + z)^2 + (24 - z)^2 = 1152 + 2z^2$.
- (2) $(36 - z)^2 + (12 - z)^2 = 1440 - 96z + 2z^2$.

Now which path gives the shortest distance to any particular point? Well, (1) is shorter than (2) when:



$$\begin{aligned}
1152 + 2z^2 &< 1440 - 96z + 2z^2 \\
96z &< 1440 - 1152 = 288 \\
z &< 3.
\end{aligned}$$

So for points near the top end of the diagonal ($z < 3$) we use path (1) and for points near the bottom end of the diagonal ($z > 3$) we use path (2). Now which of the “path-1” points is farthest from M ? Well, observe that path (1) *increases* in length as we move down the diagonal because expression (1) clearly increases as z increases. Hence the point at $z = 3$ is the farthest of all path-1 points. Secondly observe that path (2) *decreases* in length as we move down the diagonal. Indeed that’s clear from the picture but it can also be seen by rewriting expression (2) (“completing the square”) as

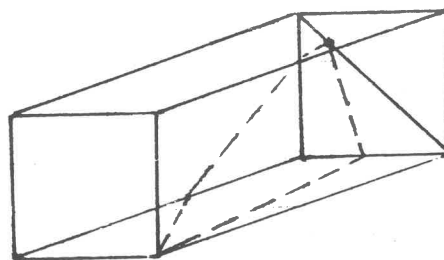
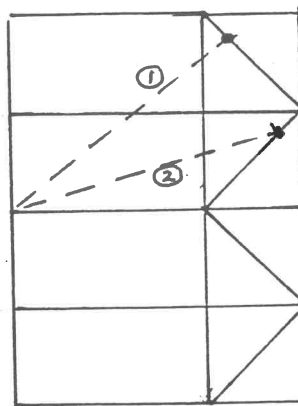
$$1440 - 96z + 2z^2 = 2(24 - z)^2 + 288.$$

The second expression clearly decreases as z increases. We conclude that the point at $z = 3$ is also the farthest of all path-2 points.

So the $z = 3$ point must be the farthest from M of all points and (1) and (2) both give shortest paths to it. We get the distance² from M to this point by plugging $z = 3$ into either (1) or (2):

$$\begin{aligned}
d^2 &= 1152 + 2(3^2) = 1170 \\
\text{So } d &\sim 34.2.
\end{aligned}$$

The male will travel a distance of 34.2 to get to the female (if she doesn’t move in the meantime.)



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