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AN EXCURSION INTO THE THEORY OF QUANTUM BITS

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1. INTRODUCTION

Quantum Theory associates with each system a possibly infinite dimensional Hilbert space. In these notes I confine myself to systems with *finite dimensional Hilbert space*. The mathematics of quantum systems with finite dimensional Hilbert space is quite simple: It is essentially linear algebra. In order to refresh your memory, I have collected the most important ideas and results of linear algebra which you'll need to understand this paper, in Appendix A. Thus if the ideas presented in Appendix A ring a bell, and you are willing to accept the postulates of finite dimensional Quantum Theory that are listed in Appendix B, you should have no difficulty to understand the following notes in which I attempt to explain such mind-boggling phenomena as the famous (or infamous) Einstein-Podolsky-Rosen (*EPR*-) *Paradox* and quantum mechanical *teleportation*.

2. THE MATHEMATICS (AND PHYSICS) OF ONE QUANTUM BIT

The most simple quantum system is the *quantum bit* or *qubit*. It is the quantum analogue of the "classical bit", a device that can assume only two distinct states, usually termed 0 and 1. A *quantum bit* or *qubit* is a quantum system whose associated *Hilbert space* is two-dimensional. Moreover the Hilbert space is endowed with a distinguished *orthonormal basis* ($|0\rangle, |1\rangle$) (called the *computational basis*). The situation can be mimicked by choosing for the Hilbert

space simply \mathbb{C}^2 and for the members of the distinguished basis the *standard basis* ($|0\rangle := (1, 0), |1\rangle := (0, 1)$). Typically a qubit is a spin-1/2 particle, a two-level atom or a polarized photon. *In these notes we emphasize the interpretation of a qubit as a spin-1/2 particle.*

From Postulate 1 (cf. Appendix B!) it follows that each observable A can be represented by an hermitian operator A in \mathbb{C}^2 . *We identify each hermitian operator with its matrix relative to the computational basis.* A basis for the real vector space \mathcal{H}_2 of all hermitian operators is constituted by the identity matrix

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

together with the three *Pauli-matrices*:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Since, moreover, the Pauli-matrices have zero trace, each hermitian operator A in \mathbb{C}^2 admits the representation

$$(2.1) \quad A = \frac{1}{2}(\tau 1 + \mathbf{x} \cdot \boldsymbol{\sigma})$$

where τ is the *trace* of A , $\mathbf{x} \in \mathbb{R}^3$ and

$$\mathbf{x} \cdot \boldsymbol{\sigma} := a_1 \sigma_1 + a_2 \sigma_2 + a_3 \sigma_3$$

It is easily verified that the Pauli-matrices enjoy the following properties:

$$(2.2) \quad \sigma_k^2 = 1, \quad k = 1, 2, 3$$

$$(2.3) \quad \sigma_1 \sigma_2 = i \sigma_3 = -\sigma_2 \sigma_1$$

and cyclic permutations

As a consequence of these relations we obtain the important formula:

$$(2.4) \quad (\mathbf{x} \cdot \boldsymbol{\sigma})(\mathbf{y} \cdot \boldsymbol{\sigma}) + (\mathbf{y} \cdot \boldsymbol{\sigma})(\mathbf{x} \cdot \boldsymbol{\sigma}) = 2(\mathbf{x} \cdot \mathbf{y})1, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^3$$

Taking the trace on both sides of equation (2.4) gives:

$$(2.5) \quad \text{trace}((\mathbf{x} \cdot \boldsymbol{\sigma})(\mathbf{y} \cdot \boldsymbol{\sigma})) = 2 \mathbf{x} \cdot \mathbf{y}$$

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Putting $\mathbf{y} = \mathbf{x}$ in equation (2.4) gives:

$$(2.6) \quad (\mathbf{x} \cdot \sigma)^2 = \|\mathbf{x}\|^2 \mathbf{1}$$

This formula together with the fact that the Pauli matrices are traceless implies that the operator $\mathbf{x} \cdot \sigma$ has the eigenvalues $\pm \|\mathbf{x}\|$.

By Postulate 2 a *state* of a quantum system is described by a density operator ρ , i.e. by a hermitian, positive semidefinite operator of trace 1. Due to (2.1) every hermitian operator of trace 1 can be written in the form:

$$\rho = \frac{1}{2}(\mathbf{1} + \mathbf{x} \cdot \sigma)$$

The eigenvalues of ρ are $\frac{1}{2}(1 \pm \|\mathbf{x}\|)$ which implies that ρ is positive semi-definite if and only if $\|\mathbf{x}\| \leq 1$, i.e. the states of a qubit are in bijection with the points of the unit ball $B^3 \subset \mathbb{R}^3$. For $\|\mathbf{x}\| = 1$ ρ is a projection operator (=projector), since

$$\rho^2 = \frac{1}{4}(\mathbf{1} + 2(\mathbf{x} \cdot \sigma) + \mathbf{1}) = \rho$$

If \mathbf{a} is a *unit vector* we write for the corresponding projector

$$(2.7) \quad P_{\mathbf{a}} = \frac{1}{2}(\mathbf{1} + \mathbf{a} \cdot \sigma)$$

The density operator $P_{\mathbf{a}}$ corresponds to a state of *maximal information* of the qubit. Such a state is called a *pure state*.

Observe that

$$P_{\mathbf{a}} \cdot P_{-\mathbf{a}} = \frac{1}{4}(\mathbf{1} - (\mathbf{a} \cdot \sigma)^2) = 0$$

and

$$(2.8) \quad \mathbf{a} \cdot \sigma = P_{\mathbf{a}} - P_{-\mathbf{a}}$$

The latter equation is the *spectral representation* for the operator $\mathbf{a} \cdot \sigma$ (cf. formula (A.4)).

Suppose the qubit is realized as a charged spin-1/2 particle, then the observable corresponding to the operator $\mathbf{a} \cdot \sigma$ is the component $\mathbf{a} \cdot \mathbf{S}$ of the spin vector \mathbf{S} in the direction \mathbf{a} . To measure $\mathbf{a} \cdot \mathbf{S}$ means to perform a *Stern-Gerlach experiment*: Particles emanating from a source are sent through a magnetic field of direction \mathbf{a} whose gradient has also direction \mathbf{a} . The magnet serves as a sorter: Due to the internal magnetic

moment associated with the spin, the path of the particles as they enter the magnetic field begins to deviate from their original path: For some particles the deviation is in the direction of \mathbf{a} and for the remaining particles the deviation is in the opposite direction $-\mathbf{a}$. This can be verified by letting the particles impact a photographic plate behind the magnet where the particles form two distinct spots. This result is in agreement with the predictions of Quantum Theory which asserts that the possible outcomes of a measurement of the observable $\mathbf{a} \cdot \mathbf{S}$ are the two eigenvalues ± 1 of the operator $\mathbf{a} \cdot \sigma$ (cf. Postulate 3a) which means physically that given any direction \mathbf{a} the spin-vector \mathbf{S} aligns itself either with \mathbf{a} (spin up!) or with the opposite direction $-\mathbf{a}$ (spin down).

Question 1: What is the probability that the spin-vector of a particle aligns itself with \mathbf{a} ($-\mathbf{a}$), causing the particle ending up on the upper (lower) spot on the plate?

Answer: Assuming that the particles that emanate from the source are in the “state of maximal disorder”, i.e. the state that corresponds to the density operator $\rho = \frac{1}{2}\mathbf{1}$, the probability for each alternative is $\frac{1}{2}$. Indeed, using formula (B.1) we find:

$$\begin{aligned} p_{\pm\mathbf{a}} &= \text{trace}(\rho P_{\pm\mathbf{a}}) \\ &= \frac{1}{2} \text{trace}(P_{\pm\mathbf{a}}) = \frac{1}{2} \end{aligned}$$

Thus, if we replace the photographic plate behind the magnet by two particle detectors situated where the two spots were, in average half of the time the “upper” detector fires and the other half of the time the “lower” detector responds.

Question 2: Suppose we block the lower channel, (i.e the path of those particles which deviate in the direction $-\mathbf{a}$ from the original path) and let only those particles pass which enter the upper channel, thereby creating a new source of particles; what state should we ascribe to a particle emanating from this new source?

Answer: By the von Neumann projection postulate (cf. Postulate 3b) (formula (B.2) of appendix B) the state of a particle emanating from the new source is the (pure) state corresponding to the density operator $P_{\mathbf{a}}$.

Question 3: Suppose, using a second magnet, we measure on a particle emanating from the new source the component $\mathbf{b} \cdot \mathbf{S}$ of the spin vector in the direction \mathbf{b} , what is the probability p that the outcome is $+1$?

Answer: Since a particle emanating from the new source is in the state corresponding to the density operator $P_{\mathbf{a}}$, formula (B.1) of Appendix B in conjunction with equation (2.5) yields:

$$\begin{aligned} p &= \text{trace}(P_{\mathbf{a}}P_{\mathbf{b}}) \\ &= \frac{1}{4} \text{trace}[(1 + \mathbf{a} \cdot \boldsymbol{\sigma})(1 + \mathbf{b} \cdot \boldsymbol{\sigma})] \\ &= \frac{1}{4} \text{trace}[1 + (\mathbf{a} \cdot \boldsymbol{\sigma})(\mathbf{b} \cdot \boldsymbol{\sigma})] \\ &= \frac{1}{4}(2 + 2\mathbf{a} \cdot \mathbf{b}) = \cos^2(\varphi/2) \end{aligned}$$

where φ denotes the angle between the two directions \mathbf{a}, \mathbf{b} .

Question 4: What is the expectation value $\langle \mathbf{b} \cdot \mathbf{S} \rangle$ of the component of the spin vector \mathbf{S} in the direction \mathbf{b} for particles that emanate from the new source?

Answer: Making use of formula (B.3) of Appendix B and again formula (2.5) we obtain:

$$\begin{aligned} (2.9) \quad \langle \mathbf{b} \cdot \mathbf{S} \rangle_{P_{\mathbf{a}}} &= \text{trace}(P_{\mathbf{a}}\mathbf{b} \cdot \boldsymbol{\sigma}) = \\ &= \frac{1}{2} \text{trace}((\mathbf{a} \cdot \boldsymbol{\sigma})(\mathbf{b} \cdot \boldsymbol{\sigma})) = \mathbf{a} \cdot \mathbf{b} = \cos \varphi \end{aligned}$$

3. THE BEHAVIOR OF A PAIR OF QUBITS

Next let us study the behavior of a *pair* of qubits. The Hilbert space of a pair of qubits is $H_4 = \mathbb{C}^2 \otimes \mathbb{C}^2$. A distinguished orthonormal basis in H_4 is the Bell-basis $(\phi_0, \phi_1, \phi_2, \phi_3)$ whose members are given by

$$\phi_0 = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

$$(3.1) \quad \phi_1 = \frac{i}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$\phi_2 = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$\phi_3 = \frac{-i}{\sqrt{2}}(|01\rangle + |10\rangle)$$

where we use the customary abbreviation

$$|jk\rangle := |j\rangle \otimes |k\rangle, \quad j, k = 0, 1$$

for the members of the *computational basis* of a qubit pair.

(For an interesting relationship between the Bell-basis of H_4 and the basis $(1, \sigma_1, \sigma_2, \sigma_3)$ of \mathcal{H}_2 see H.J.Kummer [4]). If a qubit pair is in a pure state whose associated state vector χ (cf. Postulate 2) belongs to the *real* subspace generated by the Bell-basis, then the constituents of the pair are *maximally entangled*. More precisely, let $\alpha_k = \langle \chi, \phi_k \rangle$, $k = 0, 1, 2, 3$ be the four components of χ with respect to the Bell-basis. Then a measure for the entanglement of the two constituents, if the pair has been prepared into the pure state corresponding to χ , is the so-called *concurrence* $\xi(\chi)$ of χ . It is defined by:

$$\xi(\chi) := \left| \sum_{k=0}^3 \alpha_k^2 \right|$$

Using the triangular inequality we find (keeping in mind that χ is a unit vector):

$$\xi(\chi) = \left| \sum_{k=0}^3 \alpha_k^2 \right| \leq \sum_{k=0}^3 |\alpha_k|^2 = 1$$

If the α_k 's are real then $\alpha_k^2 = |\alpha_k|^2$ and the concurrence takes its maximal value: $\xi(\chi) = 1$.

If concurrence is a valid measure of entanglement then it should be 0 when the pair is in a pure state whose state vector is of the product form

$$(3.2) \quad \chi = \chi_1 \otimes \chi_2$$

Indeed, supposing that

$$\chi_j = a_j |0\rangle + b_j |1\rangle, \quad j = 1, 2$$

where $|a_j|^2 + |b_j|^2 = 1$, we have:

$$\chi = a_1 a_2 |00\rangle + a_1 b_2 |01\rangle + a_2 b_1 |10\rangle + b_1 b_2 |11\rangle$$

Therefore

$$\alpha_0 = \langle \chi, \phi_0 \rangle = \frac{1}{\sqrt{2}}(a_1 b_2 - a_2 b_1)$$

$$\alpha_1 = \langle \chi, \phi_1 \rangle = \frac{i}{\sqrt{2}}(a_1 a_2 - b_1 b_2)$$

$$\alpha_2 = \langle \chi, \phi_2 \rangle = \frac{1}{\sqrt{2}}(a_1 a_2 + b_1 b_2)$$

$$\alpha_3 = \langle \chi, \phi_3 \rangle = \frac{-i}{\sqrt{2}}(a_1 b_2 + a_2 b_1)$$

and finally:

$$\begin{aligned} \xi(\chi) &= \left| \sum_{k=0}^3 \alpha_k^2 \right| = |(\alpha_0^2 + \alpha_3^2) + (\alpha_1^2 + \alpha_2^2)| \\ &= |-2a_1 a_2 b_1 b_2 + 2a_1 a_2 b_1 b_2| = 0 \end{aligned}$$

If the qubit pair is in a maximally entangled state (a state of concurrence 1) then the two constituent particles are strongly correlated. Before we can see what this means, we need some further mathematical considerations. Since we want to consider a pair of qubits that has been prepared into the so-called “singlet state”, i.e. the pure state that corresponds to the first ϕ_0 of the four Bell-basis vectors, we would like to determine the (idempotent) density operator $P_0 = |\phi_0\rangle\langle\phi_0|$ that corresponds to this state. We claim that P_0 can be expressed in terms of the Pauli matrices as follows:

(3.3)

$$P_0 = \frac{1}{4}(\mathbf{1} \otimes \mathbf{1} - \sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2 - \sigma_3 \otimes \sigma_3)$$

Using (2.2) and (2.3) it is not hard to show that $P_0^2 = P_0$ and since obviously

$$\text{trace} P_0 = 1,$$

P_0 is a one-dimensional projection operator. To see that ϕ_0 belongs to its range it suffices to observe that ϕ_0 is an eigenvector belonging to the eigenvalue -1 of all three operators $\sigma_k \otimes \sigma_k$, $k = 1, 2, 3$ (E.g. σ_1 simply interchanges the two vectors $|0\rangle$ and $|1\rangle$ of the computational basis and therefore

$$(\sigma_1 \otimes \sigma_1)\phi_0 = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle) = -\phi_0$$

Exercise: Carry out the analogous computations for $(\sigma_2 \otimes \sigma_2)$ and $(\sigma_3 \otimes \sigma_3)$!

Now let us assume that we have prepared a pair of *spin-1/2 particles* into the “singlet state”, which is mathematically represented either by the density operator P_0 or by the state vector ϕ_0 .

Question 1: In what state are the individual constituents of the pair?

Answer: By Postulate 5 the states of the individual particles correspond to the two *reduced density operators*:

$$\rho_1 = \text{trace}_2(P_0) = \frac{1}{2}\mathbf{1}$$

$$\rho_2 = \text{trace}_1(P_0) = \frac{1}{2}\mathbf{1}$$

Thus although the pair is in a state of maximal information (a pure state!) the individual constituents are in a state of minimal information or maximal disorder! Thus Quantum Theory supports the age-old wisdom: *The whole is larger than the sum of its parts!*

Question 2: Suppose we measure the spin-component $\mathbf{a} \cdot \mathbf{S}$ on the first particle and find outcome $+1$; what density operator should we now assign to the pair of particles?

Answer: Measuring $\mathbf{a} \cdot \mathbf{S}$ on the first particle means measuring the observable of the particle pair that corresponds to the hermitian operator

$$\mathbf{a} \cdot \sigma \otimes \mathbf{1} = P_{\mathbf{a}} \otimes \mathbf{1} - P_{-\mathbf{a}} \otimes \mathbf{1}$$

Since we assumed that we found outcome $+1$, according to the von Neumann projection postulate (cf. Postulate 3b), the density operator associated with the state after the measurement must be:

$$\begin{aligned} P'_0 &= 2(P_{\mathbf{a}} \otimes \mathbf{1})P_0(P_{\mathbf{a}} \otimes \mathbf{1}) \\ &= \frac{1}{2}(P_{\mathbf{a}} \otimes \mathbf{1} - \sum_{k=1}^3 P_{\mathbf{a}} \sigma_k P_{\mathbf{a}} \otimes \sigma_k) \end{aligned}$$

But

$$P_{\mathbf{a}} \sigma_k P_{\mathbf{a}} = a_k P_{\mathbf{a}}$$

(Since the range of the operator on the left hand side of the equation must be contained in the range of $P_{\mathbf{a}}$ we have at first $P_{\mathbf{a}}\sigma_k P_{\mathbf{a}} = \gamma_k P_{\mathbf{a}}$ for some real number γ_k . Taking the trace on both sides and using the defining equation (2.7) of $P_{\mathbf{a}}$ and formula (2.5) we find that $\gamma_k = a_k$). Inserting this expression into the formula for P'_0 we finally obtain:

$$\begin{aligned} P'_0 &= \frac{1}{2}(P_{\mathbf{a}} \otimes \mathbf{1} - \sum_{k=1}^3 a_k P_{\mathbf{a}} \otimes \sigma_k) \\ &= \frac{1}{2}(P_{\mathbf{a}} \otimes \mathbf{1} - P_{\mathbf{a}} \otimes (\mathbf{a} \cdot \sigma)) = P_{\mathbf{a}} \otimes P_{-\mathbf{a}} \end{aligned}$$

Physically this means that the two particles are now *disentangled* (cf. Postulate 5). Moreover the state of the first particle, as to be expected, is now described by the density operator $P_{\mathbf{a}}$. The second particle is in the state corresponding to $P_{-\mathbf{a}}$, i.e. the spin vector of the second particle is aligned with the opposite direction $-\mathbf{a}$ *independently of the choice of the direction \mathbf{a}* . This is quite surprising for the two particles may have moved far apart from each other. Imagine that the two particles emanate from a central source in opposite directions, the first particle moving towards Alice and the second towards Bob. Imagine that both Bob and Alice are equipped with a Stern-Gerlach magnet whose magnetic field has direction \mathbf{a} and a pair of detectors. If at Alice's position the upper detector fires then at Bob's position the lower detector responds and vice versa. Moreover this behavior is independent of the choice of the direction \mathbf{a} and also independent of *how far apart Alice and Bob may be located!* Thus if a pair of qubits is in a maximally entangled state, such as the singlet state, the behavior of the individual qubits is highly correlated. Einstein could not accept the explanation which Quantum Theory offers for this non-local behavior of entangled particles. He saw in it a paradox, which in the physical literature is referred to as the *EPR-Paradox* (cf. reference [2]). In his correspondence with Max Born, Einstein expressed his misgivings about the above implications of Quantum Theory (cf.

reference [3]). In a letter dated the 3rd of March 1947 he writes:

"I cannot seriously believe in it [Quantum Theory] because the theory cannot be reconciled with the idea that physics should represent a reality in time and space, that is free from spooky actions at a distance..."

and in March 1948 he returns a manuscript entitled "Metaphysical Conclusions" which Max Born had sent to him previously. Einstein writes several comments on the margin, the last of which contains the following remarks:

"But whatever we regard as existing (real) should somehow be localized in time and space. That is, the real in part of space A should be (in theory) somehow 'exist' independently of what is thought of as real in space B. When a system in physics extends over the parts of space A *and* B, then that which exists in B should somehow exist independently of what exists in A. That which really exists in B should therefore not depend on what kind of measurement is carried out in part of space A; it should also be independent of whether or not any measurement is carried out at all in space A..."

4. THREE QUBITS; TELEPORTATION

Imagine there are three qubits 1, 2, 3 whereby the pair (1, 2) is at Alice's position and qubit #3 is at Bob's position. Assume that the particle pair (2, 3) is in the singlet state described by the state vector $\phi_0 \in H_4 = \mathbb{C}^2 \otimes \mathbb{C}^2$ and therefore maximally entangled. Furthermore assume that particle #1 at Alice's position is in a unknown pure state corresponding to the state vector

$$\varphi = a |0\rangle + b |1\rangle \in \mathbb{C}^2$$

so that the original state of the qubit triple is a pure state that corresponds to a unit vector in $H_8 = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ of the form $\varphi \otimes \phi_0$. Imagine that Alice has a pair-observable \underline{A} at her disposal whose corresponding hermitian operator A has the spectral representation

$$A = \sum_{k=0}^3 \alpha_k P_k$$

where $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ is a quadruple of four *distinct* real numbers and $P_k = |\phi_k\rangle\langle\phi_k|$ denotes the projector onto the ray generated by the k -th Bell-basis vector. We pose the following

Problem: Suppose Alice measures on her pair $(1, 2)$ of particles the observable \underline{A} .

Question 1: With what probability will she find the outcome α_k ?

Question 2: Suppose she actually finds the outcome α_k ; in what state will Bob's particle be after the measurement?

Solution: In order to answer both questions we first observe that to measure the observable \underline{A} on the pair $(1, 2)$ of qubits means to measure the observable of the *qubit triple* whose corresponding hermitian operator is $A \otimes \mathbf{1} = \sum_{k=0}^3 \alpha_k (P_k \otimes \mathbf{1})$. Thus the answer to the first question according to Postulate 3a is:

$$(4.1) \quad p_k = \|(P_k \otimes \mathbf{1})(\varphi \otimes \phi_0)\|^2$$

The answer to the second question will be obvious once we have found the state vector of the *qubit triple* after the measurement. But by the von Neumann Postulate (cf. Postulate 3b) the state vector of the qubit triple after the measurement is a unit vector belonging to the ray generated by:

$$(4.2) \quad (P_k \otimes \mathbf{1})(\varphi \otimes \phi_0)$$

In order to carry out the explicit computations we first express the original state vector $\varphi \otimes \phi_0$ in terms of the *computational basis* of the qubit triple:

$$(4.3) \quad \begin{aligned} \varphi \otimes \phi_0 = & \frac{1}{\sqrt{2}}(a(|00\rangle \otimes |1\rangle - |01\rangle \otimes |0\rangle) \\ & + b(|10\rangle \otimes |1\rangle - |11\rangle \otimes |0\rangle)) \end{aligned}$$

Inverting the equations (3.1) we obtain:

$$|00\rangle = \frac{1}{\sqrt{2}}(\phi_2 - i\phi_1)$$

$$|01\rangle = \frac{1}{\sqrt{2}}(i\phi_3 + \phi_0)$$

$$|10\rangle = \frac{1}{\sqrt{2}}(i\phi_3 - \phi_0)$$

$$|11\rangle = \frac{1}{\sqrt{2}}(\phi_2 + i\phi_1)$$

Substituting these expressions for the first factors of (4.3) and ordering with respect to the members of the Bell-basis we find:

$$\begin{aligned} & \varphi \otimes \phi_0 \\ = & \frac{1}{2}[(\phi_2 - i\phi_1) \otimes a | 1\rangle - (i\phi_3 + \phi_0) \otimes a | 0\rangle) \\ & + (i\phi_3 - \phi_0) \otimes b | 1\rangle - (\phi_2 + i\phi_1) \otimes b | 0\rangle)] = \\ & \frac{1}{2}[-\phi_0 \otimes (a | 0\rangle + b | 1\rangle) - \phi_1 \otimes i(b | 0\rangle + a | 1\rangle) \\ & - \phi_2 \otimes (b | 0\rangle - a | 1\rangle) - \phi_3 \otimes i(a | 0\rangle - b | 1\rangle)] \\ = & \frac{1}{2}(-\phi_0 \otimes \varphi - \phi_1 \otimes u_1 \varphi - \phi_2 \otimes u_2 \varphi - \phi_3 \otimes u_3 \varphi) \end{aligned}$$

where the u_k 's, $k = 1, 2, 3$ denote the following unitary matrices

$$u_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \quad u_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad u_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

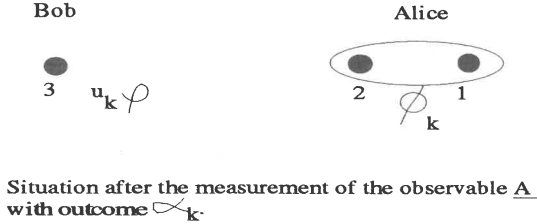
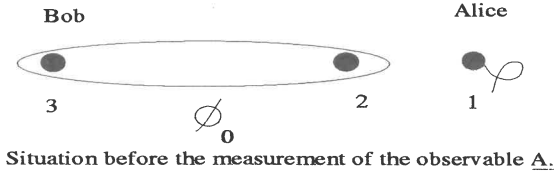
of determinant 1. Observe that $u_k = i\sigma_k$! (If we consider the u_k 's as elements of the group $G = SU(2)/\{-1, 1\}$ they generate a subgroup of G that is isomorphic to the so-called *four-group*.)

Now we are in the situation to solve our problem. Indeed, letting $u_0 := \mathbf{1}$ we can say that all four outcomes of Alice's experiment have the same probability, since by formula (4.1)

$$\begin{aligned} p_k &= \|(P_k \otimes \mathbf{1})(\varphi \otimes \phi_0)\|^2 \\ &= \left\| -\frac{1}{2}(\phi_k \otimes u_k \varphi) \right\|^2 = \frac{1}{4} \end{aligned}$$

is independent of k . However, the state vector of the triple of qubits after the measurement will *depend on the outcome*; in fact it is simply $\phi_k \otimes u_k \varphi$. This means that now the pair $(1, 2)$ is in a maximally entangled state, whereas Bob's particle ($\#3$) is in a pure state that is *related to the original state of particle 1 in a simple way*: Its state vector is obtained from the original state vector φ simply by application of an element

u_k of the four-group. To each u_k there corresponds a certain physical operation performed on the particle (cf. reference [5]). Thus if Alice communicates to Bob which of the four outcomes actually occurred (and for this she needs only to communicate one of the numbers 0, 1, 2, 3, which can be done using two ordinary bits!) Bob can revert this operation and in this way create a qubit at his position that is in the same state as Alice's particle #1 was originally: *The quantum state with state vector φ has been "teleported" over the distance between Alice and Bob with the help of an entangled pair of qubits* (cf. references ([1] and [6])).



APPENDIX A. FINITE DIMENSIONAL HILBERT SPACES AND ALL THAT

A (finite dimensional) *Hilbert space* is a complex linear space endowed with an inner product $\langle \psi, \phi \rangle$ that is linear in one factor and antilinear in the other. We adopt the physicist's convention and assume that $\langle \psi, \phi \rangle$ is *linear in the second factor*.

1. Given any pair of vectors $\phi, \psi \in H$ we define the *transition operator* $|\phi\rangle\langle\psi|$ from ψ to ϕ as the following linear operator on H :

$$|\phi\rangle\langle\psi| \chi = \langle \psi, \chi \rangle \phi, \quad \chi \in H$$

The following useful facts concerning transition operators are immediate:

- (a) If A is any linear operator in H then

$$(A.1) \quad A |\phi\rangle\langle\psi| = |A\phi\rangle\langle\psi|$$

and

$$(A.2) \quad |\phi\rangle\langle\psi| A = |\phi\rangle\langle A^*\psi|$$

where A^* denotes the *adjoint operator* which is defined by the condition

$$\forall \phi, \psi \in H : \langle A^*\psi, \phi \rangle = \langle \psi, A\phi \rangle$$

- (b) If ϕ is a unit vector then $|\phi\rangle\langle\phi|$ is the projector onto the ray (one-dimensional subspace) generated by ϕ .
- (c) The trace of the transition operator $|\phi\rangle\langle\psi|$ computes as:

$$(A.3) \quad \text{trace}(|\phi\rangle\langle\psi|) = \langle \psi, \phi \rangle$$

To prove the last formula let $(\phi_1, \phi_2, \dots, \phi_n)$ be an orthonormal basis of H . Then

$$\text{trace}(|\phi\rangle\langle\psi|) =$$

$$= \sum_{k=1}^n \langle \psi, \phi_k \rangle \langle \phi_k, \phi \rangle = \langle \psi, \phi \rangle$$

2. A linear operator A in H such that $A^* = A$ is called *self-adjoint* or *hermitian*. **Spectral Theorem:** Every hermitian operator A admits a spectral representation:

$$(A.4) \quad A = \sum_{k=1}^m \alpha_k P_{\alpha_k}$$

where $(\alpha_1, \alpha_2, \dots, \alpha_m)$ are the (real!) eigenvalues of A and P_{α_k} denotes the projector onto the eigenspace belonging to α_k . The P_{α_k} 's form an orthogonal resolution of the identity, i.e

$$P_{\alpha_j} P_{\alpha_k} = \delta_{jk} P_{\alpha_k}$$

and

$$\sum_{k=1}^m P_{\alpha_k} = 1$$

3. Given two Hilbert spaces H_1, H_2 of dimensions n_1, n_2 we can form their *tensor product* $H := H_1 \otimes H_2$ whose dimension is $n_1 n_2$. H is spanned by the set of all vectors of *product form*

$$(A.5) \quad \phi = \phi_1 \otimes \phi_2, \quad \phi_1 \in H_1, \phi_2 \in H_2$$

The inner product on H is defined in such a way that it satisfies the rule:

$$(A.6) \quad \langle \phi_1 \otimes \phi_2, \chi_1 \otimes \chi_2 \rangle = \langle \phi_1, \chi_1 \rangle \langle \phi_2, \chi_2 \rangle$$

Given two linear operators A_1 in H_1 and A_2 in H_2 there is a unique linear operator $A_1 \otimes A_2$ on H called the *tensor product* of A_1 and A_2 , characterized by the property:

$$(A_1 \otimes A_2)(\phi_1 \otimes \phi_2) = A_1 \phi_1 \otimes A_2 \phi_2,$$

$$\phi_1 \in H_1, \phi_2 \in H_2$$

For example, using formula (A.6) we can easily compute the tensor product of the transition operators $|\phi_1\rangle\langle\psi_1|$ and $|\phi_2\rangle\langle\psi_2|$

$$|\phi_1\rangle\langle\psi_1| \otimes |\phi_2\rangle\langle\psi_2|$$

$$= |\phi_1 \otimes \phi_2\rangle\langle\psi_1 \otimes \psi_2|$$

The complex vector space $\mathcal{L}(H)$ of all of all linear operators in H is spanned by the set of all operators of the tensor product form $A = A_1 \otimes A_2$. The *partial traces*

$$\text{trace}_j : \mathcal{L}(H) \rightarrow \mathcal{L}(H_j), \quad j = 1, 2$$

are defined on operators of the tensor product form by the rules:

$$(A.7) \quad \text{trace}_1(A) = \text{trace}(A_1)A_2$$

$$(A.8) \quad \text{trace}_2(A) = \text{trace}(A_2)A_1$$

and linear extension to all of $\mathcal{L}(H)$.

APPENDIX B. POSTULATES OF QUANTUM MECHANICS OF A SYSTEM WITH FINITE-DIMENSIONAL HILBERT SPACE.

For quantum systems with a finite dimensional Hilbert space H the postulates of Quantum Theory take the following simple form:

1. To an observable \underline{A} there corresponds an *hermitian (self-adjoint) operator* A in H .
2. To each state $\underline{\rho}$ of the system there corresponds a *positive semidefinite hermitian operator* ρ of trace 1 a so-called *density operator*. The pure states (=states of maximal information) of the system correspond to the *idempotent* density operators (=one-dimensional projectors). A pure state may also be labeled by a *unit vector* ϕ that belongs to the range of ρ , in which case ϕ is called the *state vector*. (It is important to note that *the state vector is only determined up to a phase factor!*). If the state vector ϕ is given then the corresponding idempotent density operator $\rho = |\phi\rangle\langle\phi|$ is the projector onto the ray generated by ϕ .
3. In a *measurement experiment* one prepares the system into a state $\underline{\rho}$ and subsequently measures an observable \underline{A} :
 - (a) the possible outcomes of the experiment are the eigenvalues $(\alpha_1, \dots, \alpha_m)$ of the hermitian operator A corresponding to the observable \underline{A} , whereby the eigenvalue α_i occurs with probability

$$(B.1) \quad p_k = \text{trace}(\rho P_{\alpha_k})$$

(cf. formula (A.4)). In case $\underline{\rho}$ is a pure state with state vector ϕ , we obtain by substituting the density operator $\rho = |\phi\rangle\langle\phi|$ into the formula (B.1) and using (A.2), (A.3) and (2.5):

$$p_k = \text{trace}(|\phi\rangle\langle\phi| P_{\alpha_k})$$

$$= \text{trace}(P_{\alpha_k} | \phi \rangle \langle \phi | P_{\alpha_k}) = \|P_{\alpha_k} \phi\|^2$$

- (b) von Neumann's Projection Postulate: At the conclusion of an experiment that resulted in the outcome α_k , the the system is in a new (well-defined) state, whose corresponding density operator is ρ' is given by:

$$(B.2) \quad \rho' = P_{\alpha_k} \rho P_{\alpha_k} / \text{trace}(P_{\alpha_k} \rho)$$

In case ρ is a pure state with state vector ϕ , we obtain by substituting the density operator $\rho = |\phi\rangle\langle\phi|$ into the formula (B.2):

$$\rho' = |P_{\alpha_k} \phi\rangle \langle P_{\alpha_k} \phi| / \|P_{\alpha_k} \phi\|^2,$$

i.e. ρ' is the projector onto the ray generated by $P_{\alpha_k} \phi$. Hence the new state vector is given by:

$$\phi' = P_{\alpha_k} \phi / \|P_{\alpha_k} \phi\|$$

This can be summarized as: *If the measurement of the observable \underline{A} of a system that has been prepared into the pure state corresponding to the vector ϕ yields the outcome α_k , the new state vector ϕ' is a unit vector belonging to the ray generated by the vector $P_{\alpha_k} \phi$.*

4. Using (B.1) and (A.4) we find for the expected value $\langle A \rangle_\rho$:

$$(B.3) \quad \langle A \rangle_\rho = \sum_{k=1}^m p_k \alpha_k = \text{trace}(\rho A)$$

which in case that $\rho = |\phi\rangle\langle\phi|$ simplifies to:

$$\langle A \rangle_\rho = \langle \phi, A \phi \rangle$$

5. If \mathcal{S}_1 and \mathcal{S}_2 are two quantum systems whose respective Hilbert spaces are H_1 and H_2 then the Hilbert space of the combined system is given by the tensor product:

$$H = H_1 \otimes H_2$$

If the combined system has been prepared into the state corresponding to the density operator ρ in the Hilbert space H then the states of the individual constituents are described by the

reduced density operators $\rho_1 = \text{trace}_2 \rho$ and $\rho_2 = \text{trace}_1 \rho$ respectively (cf. formulas (A.7) and (A.8)). A (pure) state of the combined system whose state vector has the form (A.5) is said to be *separable*. In this case the corresponding density operator has the form

$$\rho = |\phi_1 \otimes \phi_2\rangle \langle \phi_1 \otimes \phi_2|$$

$$= |\phi_1\rangle \langle \phi_1| \otimes |\phi_2\rangle \langle \phi_2| = P_1 \otimes P_2$$

i.e. ρ is the tensor product of the two one-dimensional projectors corresponding to the rays generated by ϕ_1, ϕ_2 respectively. The most general separable state is characterized by a density operator that is a weighted mean of such tensor products of one-dimensional projectors. If the combined system is in a separable state, the two subsystems \mathcal{S}_1 and \mathcal{S}_2 are *disentangled*, i.e. the measurement of an observable of \mathcal{S}_1 will not influence the state of \mathcal{S}_2 and vice versa.

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Head's Report

Bob Erdahl

This was the year for teaching awards – more than ever before. For example, Leo Jonker and Jim Whitley were the two winners of the Applied Science First Year Teaching Awards for the winter term – this Award has been in place for only four years, and members of this Department have accounted for *eight of the first thirteen winners*. We take pride in our teaching, and are always tinkering with new ideas and strategies to test in our classrooms – it's part of our culture. We are enthusiastic about students and about teaching, so are delighted when faculty members win teaching awards. Our faculty has won so many awards in recent years that we think we are one of the best teaching departments at Queen's – possibly the very best! At least that's what we say when we greet each other in the Department lounge. Judge for yourselves! Here are some of the highlights of what the Department has been achieving in teaching and learning.

Many alumni will remember **Leo Jonker**, and the courses they took from him. He makes complicated things simple, and does this at all levels – for seventh and eighth graders at a local elementary school, all the way to research seminars for grads in our doctoral program. This spring he received the *Alumni Teaching Award* for his spectacular performances in the classroom. He made the following comment during the presentation ceremony: "Teaching is about making connections – connections between teacher and student, connections between student and subject, connections between theory and applications, connections provided in explanatory metaphors, and of course, ultimately, synaptic connections between students' brain cells." About connections, one of his students had this to say about Leo's lectures: "It is like painting an entire picture for us rather than just drawing one object in the middle of the canvas. It helps us understand the concepts behind the method we are using and the very nature of the problem itself. I walk out of his lectures thinking to myself – Wow! I understand this!"

Besides winning the Alumni Teaching Award, Leo was awarded an Ontario Council of University Faculty Associations Teaching Prize, or *OCUFA Prize*, in May. Each year there are only ten such awards, across all disciplines, in Ontario. This and the Alumni Teaching Award were given, in part, for Leo's innovative approach in introducing high school students to the world of university mathematics. Many high school students come to us believing that success in a mathematics course involves memorizing a large

number of "recipes" which are applied by rote on exams and tests. Some of these students are initially hostile to the idea that the skills needed and learned in this way are but a small part, perhaps a quarter, of what we mathematicians view as a university-level mathematics course. Leo quickly turns things around, and has these students looking behind the recipes and techniques they learn, building confidence and sophistication. We are continually astonished by Leo's ability to connect effectively with students in classes as large as 450 students.

Leo is equally successful in small classes with upper year students, grad students, or even elementary school students. One of our recent grads spoke of an upper year analysis course where "details are important, yet can easily become overwhelming." She commented – "Fortunately, Dr. Jonker is an expert at breaking up a complicated mathematical proof into smaller, manageable steps, and is able to do so without oversimplifying the matter, so the full power of the theory still shines through." Leo has also brought his passion for mathematics to grade seven and eight students by running weekly enrichment sessions at a nearby elementary school. The problems he poses do not introduce high school mathematics, but rather open the eyes of all students to the beauty of mathematics at a level they can appreciate and understand.

Morris Orzech, another of our super-star teachers, sees teaching as somewhat "akin to medical practice – both deal with situations made complex by a multitude of factors: prior conditions, often unknown ones; individual variation in response to the same input; unintended side-effects. Good practice in both areas requires a desire to be helpful, a willingness to pay attention to people, an openness to ideas, care not to do harm through thoughtless experimentation, and a sense of one's limitations." Morris is known around the department for his flexible approach to teaching, and his imaginative use of computer technology. One of his students remarked that – "One of the reasons why Dr. Orzech holds my attention is his varied use of teaching techniques." He went on to say that – "Course work devised by Dr. Orzech comes in a variety of forms, which helps to ensure that people with different learning styles are not disadvantaged." Morris was named a *3M-Teaching Fellow* this spring, a prestigious award given to 10 outstanding university teachers each year, across Canada. The Society for Teaching and Learning in Higher Education and 3M Canada established this Award in 1986. It is for exceptional contributions to teaching and learning at Canadian Universities.

Morris won an OCUFA award in the spring of 1998, so the addition of the 3M-Fellowship places him in an

elite group at Queen's – there is only one other faculty member that has won both the top provincial and national teaching prizes. Morris has been a leader at Queen's for his work in improving the quality of education for students, and for promoting effective teaching among his colleagues. He introduced "Interactive Notes" for first year linear algebra, which helps students keep involved during class while freeing them from mindless note taking. The notes allow students to focus completely on the whole picture rather than on insignificant detail, and push students to become independent learners. He also introduced "MathChat", a computer-based bulletin board that promotes discussions between students, teaching assistants and faculty on issues that arise in the classroom, or in problem sets. This pioneering use of the Internet is now used in more than 80 courses on Campus.

In 1995 Morris created the Mathematics and Statistics Learning Seminar, which brings together highly motivated teachers from our Department, the Faculty of Education, the Instructional Development Center on Campus, and occasional visitors from other universities and high schools. The topics range from "The Mathematics Classroom of the 21st Century" through to the range of grades awarded to our students. There are few departments at Queen's that have a regular seminar series on teaching issues, and it is a tribute to Morris's dedication and enthusiasm that his seminar is a regular fixture – its influence has spread far beyond Jeffery Hall.

Peter Taylor is another outstanding teacher. He believes that "now is the time to innovate with the high school mathematics curriculum", and is in a good position to influence the direction in which our high schools are moving. Peter is a big part of MSTE, the *Mathematics, Science and Technology Education Group*, based in the Faculty of Education, which just received a one million-dollar grant from Imperial Oil to study current trends in high school mathematics. This Group is particularly interested in reintroducing geometry, the oldest of topics, into the curriculum because it allows kids to think like mathematicians. In addition to his role as teacher and researcher at Queens, Peter has played an active role in developing high school mathematics curriculum, and has frequently taught a calculus course in one of the local high schools.

Peter organized a two-day workshop in August, at the Fields Institute, on *Re-inventing the Math Teacher*. The mix of participants included the future leaders in high school mathematics in Ontario – there were students, young teachers, experienced teachers, coordinators and textbook authors. The agenda was to

discuss and reflect upon three aspects of the profession of mathematics teacher: teacher as scholar, teacher as student, and teacher as teacher. This was the kick-off event for the activities of the MSTE group as they start their deliberations on the high school curriculum, discharging their responsibilities to Imperial Oil.

Many alumni have fond memories of Peter's interdisciplinary course *Mathematics and Poetry*, which he initiated in 1982 with the late Bill Barnes from the English Department. They were fascinated by both poetry and mathematics, so decided to combine these passions in a single course. They argued that a poem should be read, listened to, played with, studied, and talked about in order to appreciate how the poet achieves his remarkable effect – and mathematical argument, or proof of a theorem, should be approached in the same way. Peter was awarded a 3M-Fellowship in 1994 for his many innovations in the classroom, and of course the *Mathematics and Poetry* course. Only four faculty members at Queen's have been honored in this way. Peter has also won a *Distinguished Teaching Award* from the Mathematical Association of America (1992), and the *W. J. Barnes Award for Teaching* in the Arts and Science Faculty at Queen's (1986).

There are many other teachers who have played an important role, and I am sure you will remember them: Jon Davis, Ron Hirschorn, Dan Norman and Bruce Kirby on Math and Engineering side, and David Gregory, Hans Kummer, Ole Neilsen, Dave Pollack and Norman Rice on the Arts side. There are many others. There are also many teaching awards, distinguished awards given by various groups at Queens to outstanding teachers. The length of the following list of winners from the Department gives a measure of our enthusiasm and dedication to teaching, and our success as a teaching department.

- **Jim Whitley** is as famous a teacher on Campus now, five years after his retirement, as he was 30 years ago. He was awarded a *First Year Teaching Prize* this past year for the outstanding effort he has put into teaching calculus and linear algebra to first year engineers. Anyone who has gone through Queen's in engineering will know of Jim's popularity among students, his marathon tutorials on Saturdays and Sundays before exams, and his policy of always having his door open. I have rarely gone past his office without seeing it crowded with students seeking tips on homework problems or old exam questions. The first year teaching prize for engineers was instituted four years ago and Jim's name has now appeared twice in the list of awardees – he was a runner-up in winter term of 1999.

In the spring of 1991 Jim received the Alma Maters Society's *Frank Knox Award*, one of two awards given each year for outstanding commitment to students and excellence in teaching. This is the most prestigious award given by students to teachers at Queen's. Jim is also distinguished as being a multiple winner of the *Golden Apple Award* given each year to one or two teachers of engineering classes for excellence in teaching. Also given by students, this is the top award from the Engineering Society.

- **Eight of thirteen** – that's our share of the First Year Teaching and Learning Awards that have gone to faculty over the last four years. This puts us way out in front as a teaching department in the Applied Science Faculty! This award is for outstanding contributions to the teaching and learning environment in the first year engineering program. Several of our faculty have been multiple winners: Leo Jonker won the prize four times, and was also a runner-up; Jim Whitley both won the prize and was a runner-up; Alan Ableson, a doctoral student, was a runner-up; David Cardon, a post-doc, won the prize three times.
- The Arts and Science Undergraduate Society's *W.J. Barnes Teaching Prize* went to **Joan Geramita, Grace Orzech and Leo Jonker** in 1997 for their innovative approach to teaching calculus to Arts and Science Students. They completely retooled first year calculus, devoting additional time to modeling natural phenomena so that the power of the theory is vividly brought to the students. Earlier winners of the prize were Ed Chow in 1996 and Peter Taylor in 1985.
- **Leo Jonker** is our latest winner of the *Golden Apple Award*, given each year to a few Queen's faculty for outstanding teaching in the Applied Science Faculty. Over the years we have had many winners of this Award, stretching back to the early 70s when the award was instituted. Ron Hirschorn, Jon Davis, Hans Kummer, Bill Woodside, Jim Verner, Peter Taylor, have all played a prominent role in our Mathematics and Engineering Program, and their names appear on the distinguished list of winners of this award.
- **Lucian Haddad** won the *Teaching Excellence Award* at the Royal Military College this past year. This award is the highest honor for teaching at RMC, and an analogue of the Alumni Teaching Award at Queen's. Lucian is a discrete mathematician, is an important part of our Discrete Mathematics Group, and is cross-appointed to our graduate faculty from RMC.

- **Jim McLellan** is a graduate of Mathematics and Engineering, was cross-appointed Chemical Engineering to our graduate faculty five years ago, and has played an active role in our graduate program in Mathematics and Engineering ever since. He is another winner of the prestigious *OCUFA Award*. His 1999 Award makes it three in a row for the Department – Morris 1998, Jim 1999, Leo 2000. Jim also won the Frank Knox Award in 1997.
- **Bruce Kirby**, known to many of you as fabulous teacher in the Mathematics and Engineering Program, was one of the first winners of the OCUFA award. He won the award in 1973, the very first year this prize was given. To further emphasize the mark this Department is making, 10 OCUFA awards have gone to Queen's in total, and five of these have gone to either regular faculty of this Department, or cross-appointed faculty. Although retired, Bruce is still a regular fixture in the classrooms of Jeffery Hall – he's teaching differential equations to second year engineers this fall.

The undergraduate programs: With the continual tinkering that goes on behind the scenes by the faculty listed above, our undergraduate programs are continually being improved. The most striking piece of evidence that shows we are moving in the right direction is that numbers in Mathematics and Engineering Program have increased dramatically. There are 65 incoming students in the second year class, the biggest class we have ever had, so big that it is third largest in the Faculty of Applied Science.

The Provincial Government's Access to Opportunities Program (ATOP) is beginning to play an important role in supporting our Mathematics and Engineering Program. The ATOP funds will be used for new faculty positions, and for improving facilities in critical areas. The ATOP program is intended to double the number of graduates in high technology programs, and in response to this initiative we have developed a new sub-program called "Computers and Communications." Another initiative is the Communications Lab, which is taking its place along side the Control Lab as being another important facility for our Math-Eng students.

Two other important initiatives are WAMS (Workplace Applications of the Mathematical Sciences) and the **Jobs Network**. See Joan Geramita's article in this issue for more information.

Research and graduate studies is another important focus area for the Department. The most important change here is the steady increase in size of doctoral

and the post-doctoral programs. Our graduate program has run at about 45 for about five years, but the proportion of doctoral students has increased steadily over this period to about 70% of the whole. When I introduced the post-doctoral students to the Department at the beginning of term I was surprised to learn that the number had grown to 10, one of the biggest figures for any department on campus. About a third of our graduate students have either National or Provincial Fellowships, a proportion that places us near the top at Queen's.

Another important measure of research and graduate studies is the success of our students. *Kostya Rybnikov* was nominated by the Science Division Graduate School this past year for the Governor General's Gold Medal, the award for the best doctoral thesis at Queen's. Kostya is a geometer, and worked with Bob Erdahl – he is now an NSERC Post-Doctoral Fellow at Cornell. *Leo Butler*, another star grad, has just left for North Western University where he will be an NSERC Post-Doctoral Fellow. He did his thesis with Oleg Bogoyavlenskij in the area of dynamical systems. In fact, our graduate students are all good. This past year we graduated a total of nine MSc students, and 6 doctoral students.

Transitions: *Boris Levit* has just joined faculty as a senior statistician. He did his undergraduate and graduate work at the University of Moscow, and following that has pursued a research career at a variety of leading research universities in Europe and North America. He has been a visiting scholar at the Universities of Bielefeld, Maryland, Paris VII, at the Weierstrass Institute, and Humbolt University. He came to us from the University of Utrecht where he had been on faculty since 1991. He works on non-parametric statistics, and on asymptotic efficiency of estimations.

Roland Speicher is an operator algebraist, and has just joined the Analysis Group. He did his undergraduate and graduate work at the University of Heidelberg, starting out as a physicist as an undergrad, then switching to mathematics as a grad. For the last five years he has been a Heisenberg Fellow, a prestigious position that allows leading young researchers to pursue their passion (mathematics), with no other formal duties. Roland works on free probability theory, a theory which combines operators algebras and random matrices – and gives information on the mysteries of space and time.

Eddy Campbell was appointed as associate Dean of the Faculty of Arts and Science, and has been working in that capacity since April 1. He is overloaded with work, but nevertheless maintains a strong research presence in the Department.

Hans Kummer, Dan Norman and Jim Verner will be retiring at the end of this year. Jim and Dan have been mainstays of the Math and Engineering Program, and Hans has played a double role with both Arts and Engineering student. All three have had very creative careers with us, and we will miss them.

The Campaign for Queen's. Our most ambitious project is the creation of the *Research Fellows Program* – six positions for outstanding young mathematicians and statisticians fresh from their doctoral studies. The crucial period in the life of any aspiring mathematician or statistician is the three years immediately following the completion of a doctorate. This is a time to focus, where research advances made during doctoral studies are exploited, where teaching skills are honed, and where the steady transition from expert to leader begins. New PhDs are bursting with ideas and need a period of consolidation to write and to lecture. The Research Fellows Program will provide such an opportunity.

We hope to initiate this program next year – this will require an endowment of \$500,000 and we are almost there. To have the program fully launched, with a full complement of six fellows, will take four million dollars of endowment. This is our ultimate goal, which will take several years to achieve.

In addition to this, there are a number of more modest objectives. For several years, we have employed Gill Tutors, senior undergraduates chosen for their ability and personality to help foster a sense of community among our junior undergraduates. We would like to have support in order to hire outstanding students for work within the department, either in administrative, research or curriculum development positions. Each \$100,000 in endowment will generate roughly \$5,000 in salary support. I note that the initial bequest from the Gill family was \$25,000.

A very generous donation by Graham and Stevie Keyser has provided us to the opportunity to develop the Keyser Research Center. This is the hub of our research activities. There are several strategically placed discussion rooms with good blackboards. The Keyser's support is intended to help provide the kind of environment in which research flourishes, where chance conversations can lead to profound discoveries.

“But what is it good for?”

Joan Geramita

Believing as we do that mathematics and statistics are inestimably good in themselves, it is sometimes a challenge to find the exact application, which will capture a student's interest or address a parent's concern. Actuarial science and quality control are two old favourites; public key cryptography and creating internet search engines are two new ones. It is better, of course, if we can point to real live people who use the mathematical sciences “on the job”, and it is best if we can offer students the opportunity to use maths and stats “on the job” themselves.

Two initiatives within the Department approach these goals from slightly different directions. **The Jobs Network** was inaugurated approximately 18 months ago by then Head Eddy Campbell using a generous donation from Mark Baker ('91) which has been supplemented by the Math Trust Fund. He saw a twofold goal to the program: developing connections with alumni in industry who could offer the Department (especially the students) advice and networking connections, and developing sources of positions that could be used with the Work Experience Program at Queen's (a kind of co-op for 16 months between May of third year and September of fourth). Some of the goals of the project have been achieved.

On www.mast.queensu.ca/~jobs, the Jobs Network website, several of the Department's graduates offer students their answers to “what it is good for?”. The website also offers practical advice and help for students looking for employment. There are links to job postings at Queen's Career Planning and Placement and to the employment opportunities sections of websites of companies who have hired our graduates.

The Department itself has employed undergraduate students in a variety of tasks. In our experience, the relationships developed and the teaching and learning done in these settings are different and complementary to those generated in a classroom. Some of the employment situations have provided the students with direct experience of at least one facet of the work of an academic mathematician or statistician. Students have been enthusiastic about these positions and faculty have been pleased with the quality and breadth of the students' contributions. To offer our students increased opportunities we hope to enlist our graduates as a source of internships and summer jobs outside academia. This Fall, one of our graduate students, Monica Cojocar, will be calling graduates to see if they can help us in this way.

Our second initiative – the **Workplace Applications of the Mathematical Sciences** project (WAMS for

short) – will allow us to offer students an opportunity to integrate the various parts of their mathematical and statistical knowledge in the context of applications, while still in the university environment with the resources of faculty available to them. Scientists in industries whose research establishments do not include depth in the mathematical sciences often have interesting problems that require more time or specialization than they have at their disposal. Experience at Queen's (in the Department of Chemical Engineering) and elsewhere has shown that well-motivated student teams can often solve these problems or at least make extremely helpful beginnings to solutions.

WAMS will support the mission of the Department by providing a quality, interdisciplinary learning experience for interested undergraduate students; by establishing a framework to facilitate research collaboration between faculty and students and various industrial partners; and by exposing the curriculum to the influences of the ideas and the activities of mathematical scientists outside academia.

Discussions with scientists at Dupont in Kingston have generated a project to be undertaken by a team of students during this academic year. As well, we are also undertaking a series of colloquia where members of the department and scientists from Dupont can meet to talk about the work they are doing. These exchanges may lead to research collaborations. At the very least, we will learn a few more things that “it” is good for.

BE PART OF WAMS OR THE JOBS NETWORK!

Initiatives like these obviously require the support of people outside the university. We are looking to our graduates (and others) to provide us with suitable jobs, internships and projects. However, we can also make good use of contacts – if you don't have a job or project, talk to your relative or colleague who might. (And, of course, you can always send us money to help defray the expenses of contacting graduates.)

Contact us by email at [<jobs@mast.queensu.ca>](mailto:jobs@mast.queensu.ca). Ask Monica to contact you or send us the name of a person in your company who is willing to be called. If you are willing to share information on how your undergraduate education and your subsequent career and educational choices fit together, let us know that, too. Sometimes students want to know that doing a math or stats degree will not hinder their choice of a non-math related job.

If you prefer, write to the Department of Mathematics and Statistics, Queen's University, Kingston, ON K7L 3N6, with your name, address, and daytime phone. Of course, you are totally welcome to include a cheque (made out to Queen's University) also.

Report on the Mathematics and Engineering Program

Ron Hirschorn

The Mathematics and Engineering program is alive and well. Over the last year we have improved our lab facilities, hired new faculty members, added a new option, and experienced an increase in enrollment.

Labs: Our Control-Robotics Lab was upgraded last summer with funds provided by the Ontario Government's ATOP initiative. We now have a networked system of IBM workstations running LINUX and connected to our custom built robotic experiments. We also obtained a small computer controlled milling machine. Many of our 4th year Engineering projects use the lab to design, construct and control various robotic devices. The third year labs were completely revised this summer by Andrew Lewis, our newest faculty member. Our lab will also be used to support the design projects which we will be offering as our contribution to the new first year Applied Science design course APSC100. Projects we are offering include the Design of an Infrared Communications System (Jon Davis); Design and Optimization of an Amusement Ride (Andrew Lewis, Ron Hirschorn); The Role of Feedback in Controlling Mechanical Devices (Ron Hirschorn, Andrew Lewis).

The Communications laboratory is devoted to studies of communication systems. We have obtained (again through ATOP funding) several four-way symmetric multiprocessing machines, on which the comsim environment can be run. This lab is used by undergraduates in connection with their fourth year projects.

Research Interests of Faculty: Our Communications Research group is strong and growing. It now has six members, five from this department (Fady Alajaji, Tamás Linder, Glen Takahara, Jon Davis, Lorne Campbell) and one from ECE. Our Control Theory group has three members (Andrew Lewis, Ron Hirschorn, Jon Davis) and is active in the area of nonlinear control, mechanics and robotics. In addition we are advertizing for a new faculty member with research interests in mechanics.

New Option: Our new Computing and Communications option combines the basic communication systems content of our Control and Communications option, but replaces some hardware and control systems material with a package of courses (from CISC) in the areas of formal analysis, protocols, algorithm analysis and software engineering methods.

New Courses: We now offer three 4th year level courses in Communications: Telecommunication and Data Network Modelling, Information Theory, and Source Coding and Quantization. We have completely revised our third year control course and added a new fourth year course – Lagrangian Mechanics, Dynamics, and Control.

Enrollment: The trend in student enrollment is sharply upward from a total of 56 in 1994-95 to 126 in 1998-99 and 138 in 2000-01.

In summary, the program is in a good state of health. Areas of concern are the lack of ongoing funding for the operation of our labs and the very small number of scholarships in place for Mathematics and Engineering students.

New Problem

Peter Taylor

Problem: Even or Odd?

Eeyore and Owl play the following game: they flip 10 coins, and Eeyore wins if the number of heads is even, and Owl wins if it's odd. The question is, is the game fair, or does it favour one or the other? Well that's not so hard to settle if the coins are unbiased. The game is fair and there are a number of simple arguments for that, some quite clever. But the situation here is that the coins are biased and each come up heads with probability $2/3$ and tails with probability $1/3$. Is the game still fair?

Send your solution, or new problem suggestions, to:

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Tossing the water filled balloon

(from Summer '99 issue)

Peter Taylor

This amusing problem was provided by Norm Rice. It seems he was at a church picnic and one of the games was tossing a water-filled balloon back and forth and of course because of the delicate nature of the missile and the fact that you are after all dressed up in church picnic clothes, you want to minimize the chance that it will break when you catch it. What that means is that you want the balloon to arrive with the minimum speed.

- (a) Let's suppose the balloon is released and caught at the same height. The problem is what angle should the balloon be projected at? If the angle is just above zero, you have to give it a lot of speed as it can't fall too much. If the angle is large, it will go quite high and then gravity will give it lots of speed on the way down. It seems there will be an intermediate angle which minimizes the speed of arrival. What would that be? Take the acceleration due to gravity to be $g = 10 \text{ m/s}^2$.

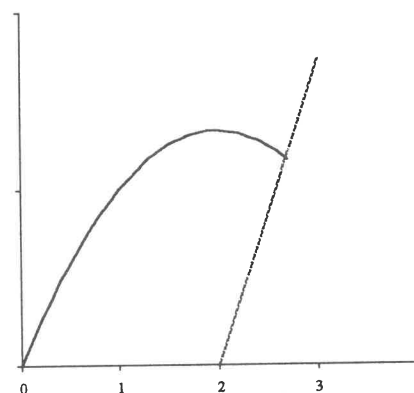
Okay. It turns out that this is really an old problem (but a good one) in disguise. In this case the final speed will be the same as the initial speed (by symmetry) and the problem of choosing the angle to minimize this (with a fixed horizontal distance) has the same answer as the problem of choosing the angle to maximize the horizontal distance traveled with a fixed initial speed. And that's 45 degrees. By all means work it out if you want. It will provide good technical preparation for what follows.

So how are we to make this an interesting problem? Well what if the ground isn't flat and the thrower and receiver are at different heights? Well, that makes things more interesting but it's still standard high school physics. But here's a thought. Suppose the ground isn't flat and I can position the receiver anywhere I want subject to some simple constraints (like he can't be too close to me). Where should he stand so he receives the balloon with the minimum speed. The idea here is that the more I get him above me, the more I am able to get the balloon to him at its maximum height which is when the speed of any trajectory is a minimum. So maybe the problem should be:

- (b) Find an interesting problem along these lines.

Well, (b) is the problem I took for myself and here's what I came up with. So maybe your real job is to solve the following:

- (c) I am standing on flat ground, but 2 meters away a hill starts to rise in a straight line at an angle of 60 degrees to the horizontal. I can put the receiver (who'd better be a competent climber) anywhere on the hill. Where should he be so that (if I project the balloon optimally) he will receive it with minimum speed?



Solution

I am standing on flat ground, but at a distance d a hill starts to rise in a straight line at an angle of 60 degrees to the horizontal. I want to gently toss a water-filled balloon so that it can be caught by a receiver R standing somewhere on the hill. Where should he stand so that (if I project the balloon optimally) he will receive it with minimum speed? (Take $g=10$).

This is an interesting situation because at first you think that R should be as close as possible which would put it at the bottom of the hill. But in going farther away, R also gains height and that can decrease the arrival velocity. For example, if R was directly above me, I could arrange for the balloon to arrive with zero speed.

In fact I got no solutions to this problem. That's a pity because it's a nice and very natural 2-variable optimization problem, and there aren't all that many

nice natural such problems around. What are the two variables? Well there's two degrees of freedom here, where he stands and how I toss the balloon. He has to be on the hill, so there's really only one variable there. Now in tossing the balloon there are two things I can vary, the initial speed and angle, but the balloon has to land at R (in true mathematical fashion we ignore the height of the receiver) and that cuts these two variables down to one. So in all there are two variables. But how should we assign them? Well there are different possibilities here, and it takes a bit of playing to see what choices will keep the algebra simple. Here are the candidates:

- x : the horizontal distance from the bottom of the hill to R
- y : the vertical distance from the bottom of the hill to R
- w : the horizontal (initial and final) velocity of the balloon
- u : the initial vertical velocity of the balloon
- v : the final vertical velocity of the balloon
- t : the time of travel of the balloon
- d : the distance from me to the bottom of the hill (given).
- m : the slope of the hill (given as $\sqrt{3}$)

I choose x to keep track of R and t to keep track of the balloon's trajectory. Now express everything in terms of these two.

$$\begin{aligned} y &= mx \\ w &= (x+d)/t \\ u &= 5t + y/t \quad [\text{This comes from } y = ut - 5t^2.] \\ v &= u - 10t = -5t + y/t = -5t + mx/t \end{aligned}$$

Now we will minimize the square of the final speed which is

$$w^2 + v^2 = [(x+d)/t]^2 + (-5t + mx/t)^2$$

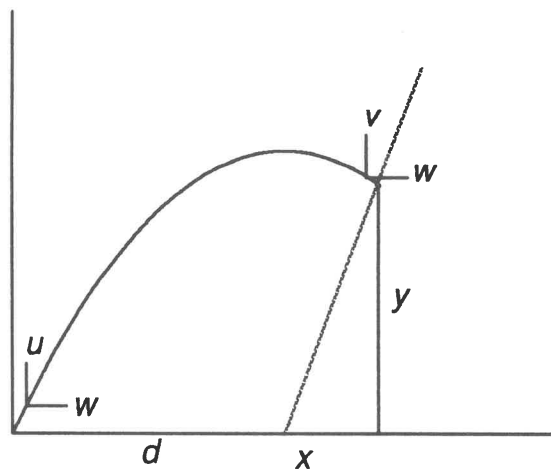
We take the two partial derivatives with respect to x and t and set them to 0. After some reorganization, we get the two equations:

$$\begin{aligned} 5mt^2 &= (x+d) + m^2x \\ 25t^4 &= (x+d)^2 + (mx)^2 \end{aligned}$$

It turns out that these have a solution with positive x precisely when $m > 1$. In this case we get the nice

formula $\frac{x}{d} = \frac{m^2 - 1}{m^2 + 1}$ which for $m = \sqrt{3}$ gives us $x =$

$d/2$. By the way, in case $m < 1$, the optimal turns out to place R at the bottom of the hill. That makes sense; if the hill is not very steep, the "payoff" of increasing height does not compensate enough for increasing distance.



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