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GUERNICA (1937), BY PABLO PICASSO

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The Axiom of Choice: What is it, where did it come from, and why is it important?[1]

Ole A. Nielsen

The axiom of choice

The axiom of choice (AC) is an assumption or a theorem or a piece of theology (your choice) that is occasionally invoked (or, more frequently, swept under the carpet) but rarely discussed in undergraduate mathematics courses. The rationale for giving the Coleman-Ellis lecture on which this article is based was (in one hour) to describe AC and how it enters into undergraduate mathematics, discuss some of its history, explain its current logical status, and suggest why it has gained general acceptance amongst mathematicians.

Let me begin with an analogy due to Bertrand Russell. If I have an infinite number of pairs of shoes there is an algorithm for selecting one shoe from each pair: just select the left shoe. But if I have an infinite number of pairs of socks there is no algorithm for selecting one sock from each pair. The difference is, of course, that the two shoes in any pair can be distinguished from one another whereas the two socks in any pair are indistinguishable. If each pair of shoes and each pair of socks is regarded as a set containing two elements then this problem of selecting one shoe or one sock from each pair becomes that of finding an algorithm that will select, out of each member of a certain family of non-empty sets, one element. The point of this analogy of Russell is that there is no reason to expect to be able find an algorithm which will select, out of each member of a given family of non-empty sets, one element. Of course, if there are only a finite number of sets (or pairs of socks) then there is no difficulty in making the selections; no matter how long it takes us to select one element from each set (or one sock from each pair) we will eventually have made the required finite number of selections.

To put the problem just considered into a more mathematical form it is only necessary to replace 'algorithm' by 'function'. So AC takes the following form:

Given a set $\mathcal{A}$ whose elements are non-empty sets, is there a function f whose domain is $\mathcal{A}$ and with the property that, for each set $A \in \mathcal{A}$, $f(A) \in A$? (AC)

Such a function f, if it exists, selects one element, viz., $f(A)$, from each set $A \in \mathcal{A}$ and is called a *choice function* for $\mathcal{A}$. The point of Russell's story is that there is no reason to expect a choice function will exist for an arbitrary set of non-empty sets.

History of the axiom of choice

In the early 1880's Georg Cantor began studying the convergence of Fourier and trigonometric series and this led him to think about subsets of **R**. Soon after he began this study he formulated the notion of a well-ordering and of a well-ordered set. He believed that any set could be well-ordered but was unable to prove it and was even unable to prove that the real line itself could be well-ordered. Ernst Zermelo introduced AC in a paper published in 1904 in order to better understand Cantor's notion of well-ordering: Zermelo proved that AC was valid if and only if every set could be well-ordered. AC generated a considerable amount of controversy almost as soon as Zermelo's paper was published. A number of the prominent mathematicians of the day (such as Henri Lebesgue, Emile Borel, René Baire, and Giuseppe Peano) objected strenuously, saying that such arbitrary choices could not be made and hence that they saw no need for choice functions to exist. These mathematicians would only allow an infinite number of choices if the choices could be made according to some algorithm or rule. That is to say, these mathematicians would admit a function if its values are defined by some rule or algorithm but not if its values are arbitrary or capricious. To them selecting the left shoe from each pair was legitimate but selecting one sock from each pair required arbitrary choices and so there need not exist such a function (or algorithm).

In 1908 Zermelo wrote another paper in which he vigorously defended his earlier paper. One way in which he did this was to point out that numerous mathematicians, including most of his critics, have used AC implicitly in their own work. For example, both Borel and Lebesgue, in their germinal work on real variables and integration theory, had used AC implicitly. In 1872 Eduard Heine had implicitly used AC to prove that the ε-δ formulation of continuity was the same as the sequential

[1]Based on a Coleman-Ellis lecture given on March 14, 2001

formulation. And at about the same time Cantor implicitly used AC to prove that the union of a countable number of countable set is countable. Of course, the fact that AC had been widely used, albeit implicitly, in the nineteenth century and the first few years of the twentieth was only realized later, following Zermelo's paper of 1904.

In the two decades following the publication of Zermelo's second paper AC gradually received grudging acceptance amongst more and more mathematicians as it became clear that mathematics would be very different without AC. Not only did it seem as though AC was necessary to prove many results, but some of these results were even shown to imply AC. This meant there were a number of mathematical theorems that were logically equivalent to AC and therefore equally reliable or equally suspect. Much of this work was done in the 1920's and 1930's by a group of Polish mathematicians (amongst whom were Waclaw Sierpinski, Alfred Tarski, and Kazimierz Kuratowski) .

Statements equivalent, weaker, or stronger than AC

There is a wonderful book by Gregory Moore of McMaster University that has a list of 73 statements each of which is equivalent to AC. Some of these are

(a) **Axiom of Choice** If $\mathcal{A}$ is a non-empty set whose elements are non-empty sets there is a function f with domain $\mathcal{A}$ and with the property that $f(A) \in A$ for all $A \in \mathcal{A}$.

(b) **Russell's multiplicative axiom** If $\mathcal{A}$ is a non-empty set whose elements are pairwise disjoint non-empty sets then there is a set T such that $T \cap A$ consists of one element for each $A \in \mathcal{A}$.

(c) **Cantor's well-ordering principle** Every non-empty set can be well-ordered.

(d) **Zorn's lemma**

(e) **Burali-Forti property** If A and B are any two sets there is a function from A into B which is either one-to-one or onto.

(f) Every vector space has a basis.

(g) Any two basis for a vector space have the same number of vectors.

(h) If S is a subspace of a real vector space V then there is a subspace S' of V such that $S \cap S' = \{0\}$ and $V = S + S'$.

(i) If A is a subset of a vector space V and if A spans V then A contains a basis for V.

It is easy to see that (a) and (b) are equivalent. Nowadays in algebra and analysis the most common way of invoking AC is by means of Zorn's lemma. The last four conditions are well-known from linear algebra and, from the point of view of AC, are only interesting for infinite-dimensional vector spaces.

Moore's book also contains two other much shorter lists, one of statements that are stronger than AC and the other of statements that are weaker than AC. The two statements

(j) **Gödel's axiom of constructibility**

(k) **Generalized continuum hypothesis**

are strictly stronger than AC and the six statements

(l) In a ring with identity every ideal is included in a maximal ideal.

(m) Every abelian subgroup of a group is contained in a maximal abelian subgroup.

(n) Every field is a subfield of an algebraically closed field.

(o) The equivalence of continuity and sequential continuity.

(p) **Countable union theorem** The union of a countable number of countable sets is countable.

(q) Every infinite set contains a countable set.

are strictly weaker than AC. Statements (o)-(q) (and especially (q)) appear to be innocuous enough and it is not at all evident that they depend on AC. To show how easy it was for nineteenth century mathematicians to implicitly use AC it will be worthwhile to consider (o) and (q) in some detail.

Let us first try to prove (q). Suppose that A is an infinite set and let us try to construct a countable subset of A. Since A is infinite it must contain an element, say, a_1. Now $A \setminus \{a_1\}$ is non-empty since A is infinite; let a_2 be an element of this set. Then $a_1 \neq a_2$. Next, $A \setminus \{a_1, a_2\}$ is non-empty since A is infinite and so must contain an element, say, a_3. Continuing in this manner, it appears that we will construct a sequence $a_1, a_2, a_3, \ldots$ of pairwise distinct elements of A and hence a countable subset $\{a_1, a_2, a_3, \ldots\}$ of A.

Now let us try to prove (o). Consider a function $f : \mathbf{R} \to \mathbf{R}$ and a point $x_0 \in \mathbf{R}$. The function f is said to be *continuous* at x_0 if

$$\left.\begin{array}{l}\text{for each } \varepsilon > 0 \text{ there is a } \delta > 0 \text{ such that} \\ \text{if } x \in \mathbf{R} \text{ and } |x - x_0| < \delta \text{ then} \\ |f(x) - f(x_0)| < \varepsilon\end{array}\right\} \quad \text{(C)}$$

and to be *sequentially continuous* at x_0 if

$$\left.\begin{array}{l}\text{if } (x_k) \text{ is a sequence in } \mathbf{R} \text{ such that} \\ \lim_{k\to\infty} x_k = x_0 \text{ then } \lim_{k\to\infty} f(x_k) = f(x_0).\end{array}\right\} \quad \text{(SC)}$$

The proof that (C) implies (SC) is easy and uncontroversial. To prove that, conversely, (SC) implies (C) let us argue by contradiction and begin by assuming that (C) is false. That means that there is an $\varepsilon > 0$ with the property that, for each $\delta > 0$, there is an $x \in \mathbf{R}$ satisfying $|x - x_0| < \delta$ and $|f(x) - f(x_0)| \geq \varepsilon$. So taking $\delta = 1/k$, where k is a natural number, there must be an $x_k \in \mathbf{R}$ satisfying $|x_k - x_0| < 1/k$ and $|f(x_k) - f(x_0)| \geq \varepsilon$. But these numbers x_k constitute a sequence (x_k) for which $\lim_{k\to\infty} x_k = x_0$ and $\lim_{k\to\infty} f(x_k) \neq f(x_0)$, contradicting (SC). This shows that if (SC) holds then (C) too must hold.

These two proofs, one of (o) and the other (q), are pretty simple. But notice that each of them required an infinite number of arbitrary choices. In the case of (o) the choices were in selecting the $x_1, x_2, \ldots$ and, in the case of (q), in selecting the $a_1, a_2, \ldots$. This means, of course, that both of the proofs depend on AC. However, it seems as though these proofs just barely used AC and (o) and (q) themselves appear so compelling that one is left wondering if there might not be clever proofs of (o) and (q) that avoid AC.

A similar careful analysis of the standard proof of (p) will show that it too uses AC.

Banach-Tarski paradox

The so-called Banach-Tarski paradox first appeared in a paper by Stefan Banach and Alfred Tarski (two Polish mathematicians) in 1924. They were independently examining some work done by Felix Hausdorff a few years earlier and, having obtained similar results, published their work jointly. One popular version of their paradox goes as follows:

> A solid ball may be decomposed into a finite number of pieces in such a way that the pieces, by means of suitable rotations and translations, can be reassembled so as to form two solid balls each with the same diameter as the original ball.

The usual reaction to this is (i) it is manifestly wrong and totally nonsensical and (ii) let's try it with a ball made of gold. In fact, Banach and Tarski showed much more: given any two objects in $\mathbf{R}^3$ with interior points (such as small ball and a large tetrahedron or a mouse and the CN tower), it is possible to decompose one of them into a finite number of pieces and, by means of rotations and translations, reassemble the pieces so as to form the other one. In doing so it is necessary to use AC; the pieces have such bizarre shapes that they must be defined by means of AC and the decomposition cannot actually be carried out with real objects. It should be pointed out that the arguments employed by Banach and Tarski work in $\mathbf{R}^k$ for $k \geq 3$ but not in $\mathbf{R}$ or $\mathbf{R}^2$.

The paper by Banach and Tarski contained the kind of results that the opponents of AC were looking for and they, not unexpectedly, latched on to it as soon as it was published. They argued that the consequences of AC given in that paper were so bizarre that surely AC had to be abandoned as being illegitimate.

One reason for calling the Banach-Tarski paradox paradoxical is that it appears as though volume (or mass) is not preserved. But do we have any right to expect that decomposing an object into a finite number of pieces and reassembling them should preserve volume? Only if we are able to measure the volume of an arbitrary subset of $\mathbf{R}^3$. But are we?

Let us think about the volume of subsets of $\mathbf{R}^3$ for a minute or, more to the point, let us think about the properties that a volume function on $\mathbf{R}^3$ should have. First of all, a volume function on $\mathbf{R}^3$ should be a function, say ν, whose domain is all subsets of $\mathbf{R}^3$ and whose value $\nu(A)$ at a subset A of $\mathbf{R}^3$ can be interpreted as the volume of A. Such a function ν must surely satisfy the following conditions: (i) its values should be non-negative numbers or infinity, (ii) its values should be rotation and translation invariant, meaning that if A, $B \subseteq \mathbf{R}^3$ and if B is obtained by either rotating or translating A then $\nu(A) = \nu(B)$, (iii) if A and B are disjoint sets then $\nu(A \cup B) = \nu(A) + \nu(B)$, and (iv) $\nu(C) = 1$ for any cube C with side-length 1. These properties imply that if a subset A of $\mathbf{R}^3$ is decomposed into pieces $A_1, A_2, \ldots, A_m$ and, by rotations and translations, these pieces are reassembled into a set B then

$$\nu(A) = \sum_{i=1}^{m} \nu(A_i) = \nu(B).$$

In view of this equation the Banach-Tarski paradox quite clearly implies that there is no volume function on $\mathbf{R}^3$.

The conclusion just drawn can be paraphrased in a more positive way: The Banach-Tarski paradox suggests that we may have AC or we may have a volume function on $\mathbf{R}^3$ but we may not have both. It was conclusions of this sort that Banach and Tarski touted in their original paper rather than the paradoxical nature of their discoveries. In fact, they regarded their results as theorems and not as paradoxes and stressed this point of view in their paper.

Set theory and AC

The first axiomatization of set theory is known as **Z** and was contained in Zermelo's 1908 paper as part of his defense of his 1904 paper. By the early 1920's it was realized that the axioms of **Z** were inadequate for the development of set theory and needed to be supplemented by one additional axiom. This axiom was proposed independently by Thoralf Skolem and Abraham Fraenkel and the enlarged set of axioms was known as **ZF**; these axioms and this theory are used today by virtually all mathematicians and logicians. AC is not one of these axioms and both the logical status of these axioms (are they consistent?) and their relation to AC (do they imply AC or its negation?) was an open question for many years. In 1931 Kurt Gödel proved his so-called second incompleteness theorem and one of the consequences of this theorem is that if **ZF** is consistent then there is no proof of the consistency of **ZF**. (On the other hand, any inconsistent theory can prove any statement and so if **ZF** were inconsistent it would be able to prove its own consistency.) In 1937 Gödel was able to prove that if **ZF** is consistent then so is $\mathbf{ZF} + \mathbf{AC}$, thereby showing that AC cannot be refuted by **ZF**. Gödel did this by using the assumption that **ZF** is consistent to build a model of $\mathbf{ZF} + \mathbf{AC}$, the so-called constructible model. (It is in this way that (j) is strictly stronger than AC.) And in 1963 Paul Cohen proved that if **ZF** is consistent then there are models of **ZF** with any of the following properties: (i) **R** does not have a well-ordering and so AC does not hold for the set of non-empty subsets of **R** and (ii) there is a countable family of two-element sets with no choice function (cf. Russell's socks). This means that AC cannot be proven from **ZF**.

In summary, then, the work of Gödel and Cohen shows that **ZF** can neither prove nor refute AC and hence that either AC or $\neg$AC may be adjoined to **ZF** without fear of introducing an inconsistency. Or, putting it a little more succinctly, **ZF** and AC are logically independent of one another.

The technique Cohen discovered and which he used to prove his theorem about **ZF** and AC is known as *forcing* and is still being used today to prove theorems about set theory. In particular, this technique was used in the mid-1960's by Robert Solovay to construct a model of **ZF** in which (i) there is a function on **R** which is sequentially continuous but not continuous and (ii) **R** is the union of a countable number of countable sets. At about the same time forcing was used to construct a model of set theory in which there is an infinite set with no countable subset. (In connection with (ii) recall that, by Cantor's theorem, **R** is uncountable.) So each of (o)-(q) really do depend on AC and there is no way to prove any of them without invoking AC. (And it had appeared as though AC was being used in the proofs of (o) and (q) in such an innocuous way!)

Current status of AC

Today virtually all working mathematicians accept AC and, as has been suggested, there is one overriding reason for this: there are numerous theorems that 'ought' to be true and virtually characterize mathematics as we know it today and which cannot be proven without AC. Examples of such theorems are conditions (d)-(i) and (n)-(q). Conditions (f)-(i) and (n) are regarded as 'facts of life' by algebraists, (o) and (p) are regarded similarly by analysts, and (q) is just plain 'obviously true'. Without these and similar theorems mathematics would be a very different subject; its usefulness to engineers and scientists and its intrinsic aesthetic appeal to mathematicians would be greatly diminished.

Finally, it should be said that the full force of AC is not needed to obtain the theorems alluded to and that some mathematical logicians are troubled by the seemingly immense deductive strength of AC. Some of these logicians are wondering if it might not be possible to replace AC by a weaker statement, one that is more palatable but still sufficiently strong to imply the theorems that are at the heart of mathematics (remember that (l)-(q) are strictly weaker than AC). Research in this direction has been ongoing for several decades now, with most attention being focused on assumptions related to winning strategies for various infinitary games.

Head's Report

Bob Erdahl

In her 2003 Killam Lecture, delivered at the University of British Columbia, Shirley Tilghman spoke on *The Challenges of Education for the Next Generation of the Professoriate.* Central to her message was the importance of attracting the brightest and ablest of our undergraduates into careers in scientific research and into our university faculties. *"The reasons are straightforward enough. First, research universities have assumed the role of research engines for our countries; they are the sources of innovation and future prosperity. If the universities falter, so do the future health and wellbeing of our countries. Second, as I reminded members of Princeton's board of trustees recently when they were questioning why we spend so much time and resources vying with other universities for the very best faculty, a university in which the students are smarter than the faculty is not an attractive model for excellence in education."*

Because there is a longstanding faith in the value of an education, research universities such as Queen's hold a privileged position in our society. This faith is based on the conviction that a good portion of Canada's vitality, including its robust economy, owes much to our universities. This conviction is expressed in many ways, including the investment of our Federal and Provincial Governments through a variety of programs promoting excellence such as the Canada Research Chairs Program. This conviction is also expressed at Queen's by the generous giving of our alumni, which has established the Queen's endowment as one of the top two per student in Canada.

In return for this broad support, society rightfully expects from universities, and in particular from Queen's, the generation of new ideas and knowledge that will fuel economic growth and prosperity, and create jobs, but most importantly prepare the next generation of citizens and leaders.

Shirley Tilghman was elected President of Princeton in 2001. She is a Queen's graduate, receiving her Honours B.Sc. in chemistry in 1968. She is an exceptional teacher and world-renowned scholar, and leader in the field of molecular biology.

Challenges for the Department

There have been many changes in the Department over the last ten years – in particular, the way we teach has changed. Class sizes are much larger than they were, and we have had to scramble to maintain the quality of the programs we offer – we have had to scramble to ensure that the courses we teach are as attractive today as they were twenty years ago. We think we have managed to achieve this. In fact, our students are thriving – doing better than ever.

As emphasized by Shirley Tilghman, our challenge is to prepare the next generation of leaders in the sciences – our focus being on mathematics and statistics, the teachers and researchers. I want to describe in this article why we think our students are well-prepared as the next generation of leaders. I will discuss in the context of the three groups that make up the Department's community, our students, our faculty, and our alumni.

Our students are well prepared

The number of academic prizes won by our students in recent years has been remarkable. The example I like is the Prince of Wales Prizes – over the past 12 years there were 24 Prince of Wales Science Prizes awarded, and the Department won 14 of these. The two highest academic awards for science students are the Prince of Wales, and the Runner-Up, given to the two top graduating students at spring convocation. There was only one year in the last twelve when we did not win either the Prince of Wales or Runner-Up Prize, and there were three years when our students earned both. A stunning accomplishment since each year there are over 600 science graduates, but only 30 Honours Math graduates.

Our graduate students have also shown success – over the last five years our doctoral students have twice won the doctoral prize from the Canadian Mathematics Society, won the doctoral prize from the Canadian Applied and Industrial Mathematics Society, and won the Statistical Society of Canada's Pierre Robbillard Prize.

More details on the high achievement of our students is given in *Academic Prizes Won by Mathematics and Statistics Students*, which follows. We have been able to give a rather compete picture back to 1985, but it was difficult to go further back. With help from our alumni I'm sure we can go much further back.

To finish reporting on students I must say a few words about the Department's enthusiasm for the current first year class, the class of 2007. Shortly after the start of term last September the reports started rolling in, *best incoming class ever*. Students linger after each of Leo Jonker's Calculus for Engineers class to talk math; they come in greater numbers to his evening sessions devoted to *challenge problems* – hard engineering problems that require three or four hours just to get started; the grades on the Christmas exams were four percentage points higher than the year before. This is the year that the percentage of incoming students with first class standing shot up to 99.5 percent, four percentage points greater than last year, and twenty percentage points greater than at any other Ontario University.

Many of these students will sign on as Honours Mathematics or Statistics students, or as Mathematics and Engineering students, and we'll see a lot of them throughout the next three years. We will enjoy teaching these kids.

The faculty measures up

Since the last issue of the *Communicator* there have been many awards for teaching and scholarship that have gone to the faculty. To keep you up to date these are listed in *Significant Events*, which follows.

The long list of awards shows that we have an outstanding teaching department, and leading research department. It's hard to keep both balls in the air, teaching and research, but we consider it our responsibility to do so. Our students demand that we push hard in these fundamental directions, teaching and research; Queen's has the top undergraduate body in the country, by a mile, and they demand the same excellence of our faculty.

We have spent time and energy, and resources, competing with universities across North America to assemble the faculty we now have. Shirley Tilghman gave the reason – *"a university in which the students are smarter than the faculty is not an attractive model for excellence in education."* We have met the demands of our students for good courses taught by leading scientists.

Our alumni

In a 1998 survey of our undergraduate alumni we discovered that the third most frequent career track was company president. Well, … we always thought our students would do well out there, but this seemed an exaggeration. Some quipped that coming face-to-face with the murky problems tossed up in business requires the same smarts, and courage, as does the beautifully formulated, but opaque, problems towards the back of a second year calculus exam. Biased statistics yes, but we were still reminded of a striking phrase of Alfred North Whitehead's that John Coleman is fond of quoting: *"The paradox is now fully established that the utmost abstractions are the true weapons with which to control our thought of concrete fact"*. Most everyone who has taken a course from John Coleman has certainly heard him quote from Whitehead.

The 1998 survey showed us more than simply that our graduates ascend to the top ranks of industry, with ease. The survey showed there were a sufficient number of teachers of mathematics out there, researchers and academics, to satisfy our sense of mission.

We want to know more about how you, our alumni are doing, so keep in touch. We know you've been successful – because you showed so much success as undergrads. Let us know about the role your Queen's mathematics degree has played in your lives – let us know about your interesting careers.

Launch of the Coleman Fellows Program

We have been able to launch the campaign for the Coleman Fellows Program through a generous gift from Denys Calvin, a 1980 Honours Math graduate. This is an innovative program to establish a new class of position in the Department. Denys established the endowment for the very first Coleman Fellow.

The Coleman Fellows will form a new layer of teachers and researchers between the faculty and undergrads. This new class of appointment will be for three years, immediately following doctoral studies; these appointments will be attractive to mathematicians emerging from doctoral programs since they need time to consolidate their research programs and learn how to teach. At Queen's the Fellows will hone their speaking and problem solving skills by teaching and doing research – they will bring new vibrancy to our undergraduate programs and fresh ideas to our research seminars. The Fellows Program will function as a launching pad for the next generation of leaders in mathematics. Our goal, in the long run, is to build sufficient endowment for eight Coleman Fellows.

To understand why the Coleman Fellows Program is so crucial you need to know about the rapid changes at Queen's, and in the Department. First, with reduced funding from the Province our faculty complement has shrunk, class sizes have increased, and we are hard-pressed to maintain the essential dialogue between faculty and students that has always been a part of a Queen's education. One of the important roles of the Fellows will be as teacher, helping to maintain the intimate contact with our students. Teaching is a job that requires enthusiasm, continual tinkering with everyday details, and the confidence of an expert – just right for a Coleman Fellow. Second, the Department's research program has become much more ambitious, and has expanded. Our seminar system forms the backbone of our research program, where our graduate students are trained, and where new research directions are formulated. The Coleman Fellows will bring fresh new perspectives, enthusiasm, and confidence to our seminar system.

The Coleman Fellows Program has enormous potential to improve teaching and research in the Department, and responds to the message that Shirley Tilghman brought, that we attract the brightest and ablest of our undergraduates into careers in scientific research and into our university faculties.

Fellowships for students

There are two other directions where we are trying to build endowment. We want to develop additional opportunities for both our graduate and undergraduate students, and I would like to report here on what has been accomplished in these directions.

Gail Drummond and Robert Dorrance established the Dorrance Fellowship in Mathematics and Statistics through a generous donation, making use of the Provincial OGSST program, which provides 2 for 1 matching funds. Robert Dorrance is a 1974 Honours Mathematics graduate. Using a similar strategy, Scotiabank established the Scotiabank Fellowship in Mathematics and Statistics.

Doctoral studies lie at the very centre of the research program of our Department, and play a dominating role in developing scientific infrastructure in Canada. The Department would like to expand the size of the graduate program from the current 50 to 65, and this requires additional funding. This will require sources of funding such as that provided by Gail Drummond and Robert Dorrance, or the Scotiabank.

A very generous donation by **Graham and Stevie Keyser** has provided us the opportunity to develop undergraduate research. The Keyser Fund provides stipends for undergrads to join a summer research program in the Department, and also provides stipends for graduate students to help supervise fourth year undergrad theses. Last spring, at the Mathematics and Engineering Conference the first Keyser Prizes were announced for "best talks"; Graham Keyser was on hand the present the awards himself. The Mathematics and Engineering Conference is where our students report on their thesis research, conducted throughout their fourth year. Graham Keyser graduated in Honours Mathematics and Physics in 1946, and obtained a M.Sc. in physics two years later in 1948. His wife Stevie graduated from Queen's in 1946, in the very first nursing class.

Oswald Hall established the Norman Miller Assistantships in Mathematics Education. Oswald Hall graduated with a Bachelor of Arts in 1935, and spent most of his career at McGill as a Professor of Sociology. As a Queen's undergraduate he was very much influenced by Norman Miller, a Queen's Professor of Mathematics. Norman Miller was concerned about mathematics education in the high schools, and worked closely with local high school teachers.

Transitions

Over the past three years we have hired seven new colleagues:

David Thomson was appointed as a Canada Research Chair in Statistics and Signal Processing. He had an illustrious career at Bell Labs before joining the Department, where he was the first to give a statistical proof of global warming in the early nineties. Immediately upon arriving in January 2002 David launched an ambitious program to study the sun's radiation to determine how it is linked to the dropped call problem for cell phones; this will involve gathering data using a radio telescope mounted on Jeffery Hall.

Shawn Kraut did his PhD in statistical signal processing at the University of Colorado, and was then a postdoctoral fellow at Duke University. Shortly after he was appointed to the Mathematics and Engineering Faculty, in September 2002, he won an IEEE Young Author Best Paper Award.

Troy Day was appointed as a Canada Research Chair in Mathematical Biology, in July of 2002. Shortly after the defence of his doctoral thesis at Queen's, in 1999, he won an NSERC doctoral Prize, the Canadian Industrial and Applied Mathematics Societies' Doctoral Prize, and a Premier's Research Excellence Award. He came to Queen's from the University of Toronto.

Michael Roth graduated from Queen's in 1993, and then pursued doctoral studies at Harvard under the direction of Joe Harris. In the period 1998 – 2002 Michael was an assistant professor at Michigan. He was appointed to the Algebra Group in the Department in September 2002.

Ivan Dimitrov defended his doctoral dissertation on infinite-dimensional Lie algebras at the University of California, Riverside, in 1998. Following doctoral studies, Ivan was a Hedrick Assistant Professor at the University of California, Los Angeles, and an AMS Centennial Fellow at the Max-Planck Institute, Germany, Yale University, and the Mathematical Sciences Research Institute, Berkeley. Ivan joined the Algebra Group at Queen's in September 2003.

Navin Kashyap was appointed to our Mathematics and Engineering Faculty in January of this year. Navin did his doctoral studies at the University of Michigan, during which time he earned a master's in mathematics; both degrees were awarded in 2001. Navin then was a post-doctoral fellow at the University of California, San Diego. He works in the area of information theory and source and channel coding.

Greg Smith graduated from Queen's in 1994, and then did a doctoral degree under the direction of David Eisenbud, first at Brandeis, then at Berkeley. Greg defended his thesis in 2001, and then was an Assistant Professor at Columbia for three years. Greg will be joining the Department in July of this year.

In 2003, **Ron Hirschorn** stepped down as Chair of Mathematics and Engineering, a position he held for 12 years. Replacing him are **Fady Alajaji**, Chair, and **Tamás Linder**, Curriculum Chair. Also in 2003, **Leo Jonker** handed on his position as Co-ordinator of Graduate Studies (1999-2003) to **Roland Speicher**.

Starting on July 1, 2004, **Peter Taylor** will step in as the new Head of Mathematics and Statistics. My predecessor, **Eddy Campbell**, who is currently an Associate Dean of Arts and Science, has accepted a job at Memorial University. Perhaps quoting from Eddy's letter to the Department announcing his decision to accept the position at Memorial is appropriate; his note expresses the confidence we all have in the future of the department:

"My work as an administrator has convinced me that Queen's is an outstanding institution, complete with wonderful students, smart faculty, dedicated and thoughtful staff: a little slice of academic paradise. The place is fully deserving of its reputation and national rankings. The Department of Mathematics and Statistics is poised to move to the top when measured by our research and we have been at the top in terms of our dedication to and accomplishments in teaching for decades. ... Queen's is a wonderful place at which to work and do mathematics."

Over the past three years, ten faculty members have retired:

Jon Davis was Mr. Math and Engineering, and one of the first in the Department to win a Golden Apple, in 1972, the second year the award was offered; **David Gregory** brought the art of delivering lectures in MATH 111 to heights that will possibly never be seen again; he could write down the eigenvalues of a matrix after a brief inspection; **Malcolm Griffin** started Mod-Sat, the famous introductory statistics course that many have taken; he led the stats group for many years; **Ian Hughes** was devoted to his students, giving freely of his time to them; **Hans Kummer** was as devoted to his students as they were to him; he got a Golden Apple teaching Award in 1975, helping to establish our reputation as a good teaching Department early on; **Dan Norman**, an outstanding teacher, remembered most of the Mathematics and Engineering students we have had, from the beginning; **Kirti Oberai** was one of our best teachers, and much appreciated by our students; **Norm Rice**, another well-know teacher, won the distinguished Teaching Award of the Seaway Section of the Mathematical Association of America, a couple years ago; **Terry Smith**, the other founder of Mod-Stat,, introduced consulting as a required course for graduate students in statistics; **Joan Geramita** was co-inventor of Math 121 the team of Grace Orzech, Leo Jonker and Joan Geramita won the Frank Knox Award for this superb effort.

Significant Events in the Department of Mathematics and Statistics

The section "Significant Events" in our strategic plan is updated yearly – in it we record the important events in the Department. Since the *Communicator* has not appeared for a couple of years we are including the updates for the past few years. The student awards, described in a separate article, have been removed.

One of **Leo Jonker's** first year Calculus students said this about his lectures: "*It is like painting an entire picture for us rather than just drawing one object in the middle of the canvas. It helps us understand the concepts behind the method we are using and the very nature of the problem itself. I walk out of his lectures thinking to myself – Wow! I understand this!*" Leo received the Alumni Teaching Award at spring convocation in May 2000; the Alumni Award is the highest honour for teaching given by Queen's. Many alumni will remember Leo and the courses they took from him – he makes complicated things simple, and does this at all levels – in enrichment classes for seventh and eighth graders at a local elementary school, all the way to research seminars for grads in our doctoral program. Also, in May of 2000, Leo was awarded an Ontario Council of University Faculty Associations Teaching Prize, *or OCUFA Prize*. Each year there are only ten such awards, across all disciplines, in Ontario.

Jim Whitley is as famous a teacher on Campus now, ten years into retirement, as he was 30 years ago. He was the 2001 winner of the Alumni Teaching Award. The *J-Force Commander*, as he's affectionately know to his admirers, goes that *extra mile* for his students. If any first year engineer fails first term Calculus they are reassigned to J-Section where they come under Jim's tutelage. He helps them upgrade their marks and get back on track. Most of these students sport the popular *J-Force* patch on the sleeves of their engineering jackets, a meaningful tribute; underneath the J-force insignia is another patch reading *In Jim We Trust*.

In October 2001 **Fady Alajaji** received a **Premier's Research Excellence Award**. This prestigious award is given by the Governor of Ontario, and reserved for young faculty members in the first eight years of their academic careers. Fady is an information theorist and received his PREA for his work on source and channel coding.

In November of 2001 **Oleg Bogoyavlenskij** won a Humbolt Research Award given by the Alexander von Humbolt Society of Germany; the Award was accompanied by a stipend of DM 100,000. In March of 2002 Oleg won a Killam Fellowship. Killam Fellowships are one of the top distinctions a Canadian academic can win; each year about 15 are awarded, in all areas. Oleg had a good year – two prestigious awards in a single academic year. Oleg got these awards for his new theory of invariants for partial and ordinary differential equation, which supersedes previous work in this area. Using his theory of invariants he was able to find analytical solutions to the equations of magneto-hydrodynamics that govern fundamental processes in stars and galaxies; he was also able to find analytical solutions to the Navier Stokes Equation that govern fundamental processes in fluid flow. The only solutions of these equations known earlier had trivial physical content and were uninteresting to experimentalists; Oleg's work represents a significant break-through.

In March 2002 **Tamás Linder** won a **Premier's Research Excellence Award** his application of artificial intelligence methods to data compression algorithms. Tamás was also tenured this spring.

Dan Norman was awarded a **Distinguished Service Award** by the Queen's University Council at its meeting on May 10, for the large number of contributions he made throughout his career at Queen's. (About 4 or 5 of these are awarded each year to people in the Queen's community; recent recipients include all retiring Queen's Principals and Chancellors, as well as some Deans, Faculty Members and Support Staff.) The citation mentioned Dan's 14 years as Chair of the Queen's Pension Board (later Committee), his work in marshalling Convocations over many years, as well as leadership on other committees.

At the Arts and Science Convocation this spring **Ole Nielsen** was awarded a **Frank Knox Teaching Award** for his outstanding work in MATH 237.

In January 2003 **Tamás Linder** won a **Chancellor's Award** for his application of artificial intelligence methods to data compression algorithms. This prestigious award is for outstanding contributions to research by new faculty members at Queen's in the first eight years of their academic careers.

Shawn Kraut has been awarded the **Young Author Best Paper Award** by the Institute of Electrical and Electronics Engineers, for his paper *Adaptive Subspace Detectors*, co-authored with Louis L. Scharf and L. Todd McWhorter. "This award honours the author or authors of an especially meritorious paper dealing with a subject related to the IEEE's technical scope and appearing in one of the IEEE's publications. The primary author must be 30-years-old or younger upon the date of the paper's submission."

In this paper adaptive statistics for radar detection are developed that both simplify earlier work and are optimal within a class. In May 2003 **Shawn** was awarded a grant from the US Office of Naval Research of US$121,000 for his research program in sonar signal processing. This grant is to support his research on detection of a moving object in shallow water when there are surface ships causing interference. This problem is difficult because there are multiple modes of propagation caused by reflections off the surface and bottom, and because the moving object sends a weak signal. This part of Shawn's research is joint with Jeffrey Krolik at Duke University.

More generally, Shawn's research focus is statistical signal processing and sensor array processing. He has made contributions in the areas of adaptive radar, sonar, and optics.

Norman Rice's book *Experimental Methods in Kinetic Studies* was published by Elsevier; this book was co-authored with B. Wojciechowski.

Ram Murty received the **2003 Jeffery-Williams Prize** at the Summer Meeting of the Canadian Mathematical Society held at the University of Alberta in June 2003. The Jeffery-Williams Prize recognizes leadership and outstanding research contributions in the field of mathematics and is awarded annually by the Canadian Mathematical Society. Ram has made systematic, significant and extensive contributions to number theory. His contributions have been described as having great depth and beauty, and have been of interest to a broad range of mathematicians.

Earlier, Ram won the Canadian Mathematics Society's Coxeter-James Prize, which is awarded annually to an outstanding young researcher for achieving a break-through in mathematics. In addition, Ram won a prestigious Steacie Fellowship in 1993, and a Killam Fellowship in 1998. The Jeffery-Williams, the Coxeter-James, the Steacie and a Killam are the four most prestigious awards that a Canadian mathematician can aspire to; Ram is the only Canadian mathematician who has won all four.

Ram Murty was one of two winners of the **2003 Queen's Research Prize** for his outstanding research contributions. In 1984, he made the first major breakthrough in the solution of Artin's conjecture, a problem that had been open for nearly a century. In 1991, Ram and his brother Kumar Murty at Toronto resolved Kolyvagin's conjecture and opened fresh ground in the theory of elliptic curves. For this work they were awarded the Balaguer prize in 1996. Ram just recently has launched a campaign to explore the interface between analytic number theory and combinatorial graph theory, which he has titled *Ramanujan graphs, zeta functions and applications.* Ram received his prize at the fall convocation.

Troy Day was awarded a *Poste Rouge Fellowship* from the *Centre National de la Recherche Scientifique* of France to work this past summer with researchers in Montpellier developing a theory on the effects of vaccination on pathogen evolution. This is a very competitive international award directed at bringing leading foreign researchers into France to foster collaboration. In August Troy was invited as a visiting scientist to the Centre for Population Biology at Imperial College, U. K.

Along with several other Canadian and US researchers, **Troy** obtained pilot funding through **MITACS** to study the epidemiological dynamics of SARS and the effectiveness of various control measures. Troy is also a CO-PI on a grant through the *Marsden Fund of the Royal Society of New Zealand.* Troy's role is to help develop mathematical models for the evolution of recombination. The value of the Marsden Fund Grant is $600K over research conducted over the next three years.

Eddie Campbell was elected **President of the Canadian Mathematical Society**. He will serve as President-Elect beginning July 1, 2003, then serve for two years as President, followed by a year as Past President, ending his term in office June 30, 2007. The Society was founded in 1945 and its goal to promote and advance the discovery, learning and application of mathematics.

Jamie Mingo stepped down as **Ontario Vice President** of the **Canadian Mathematics Society**. Earlier he served as chairman of the publications committee for four years.

Noriko Yui and her co-author **James Lewis** published *Calabi-Yau varieties and mirror symmetry*; their book appeared as Volume 38 in the Fields Communications Series. **Noriko** was appointed as the 2004 **Kloosterman Chair** at the Mathematical Institute at Leiden University. This visiting professorship in mathematics was established in 1986 in honour of Hendrik Douwe Kloosterman, a distinguished number theorist who was a professor of mathematics at the University of Leiden, Copenhagen, from 1930 to 1968. Some of the earlier Kloosterman Chairs were M. Artin, H. W. Lenstra, A. A. Borovkov, A. Granville and Y. Eliashberg.

On September 8, 2003, President Ferenc Madl of Hungary presented **András György** with a Golden Ring that is inscribed *Promotio sub auspiciis praesidentis Rei Publicae* – Awarded by the President of the Republic. This ceremony took place at the Budapest University of Technology and Economics. The *President's Academic Prize* that András received is over 100 years old, and is awarded to a small number of distinguished students each year. The terms of this award are severe, requiring that all grades must be fives, the highest possible, over the four years of high school, the five years of university, and the three years of doctoral studies. The President's Academic Prize has been awarded only 13 times to doctoral students from this famous engineering school.

András is currently a NATO Post-Doctoral Fellow in the Department, working with Tamás Linder. As a master's student at Queen's, he won the 2001 Thesis Prize of the Science Division of Graduate School for his master's thesis *Optimal entropy constrained scalar quantization,* supervised by Tamás Linder; this event was distinguished because the competition is open to both doctoral and master's students. Following his master's studies András returned to Hungary for doctoral studies, defending his dissertation *Entropy constrained quantization and related problems,* co-supervised by Tamás Linder and Laszlo Gyorfi of Budapest, this past January.

In October 2003 **Roland Speicher** won an **NSERC Leadership Award** for his proposal *Free Probability and the Universality Conjecture for Random Matrices*; he will receive $40,000 per year for the next four years to support visiting scientists and post-doctoral students. In the recent reallocation exercise at NSERC the Mathematics Steering Committee proposed the Leadership Support Initiative to support group-based research and its leaders. Roland is one of seven winners of this new prestigious award.

Roland proposed a new approach to the Universality Conjecture based on his fundamental contributions to free probability theory, namely the theory of free stochastic analysis and the theory of free cumulants. The tools he has developed by-pass many of the difficulties encountered by other researchers, and are tailor-made for treating the Universality Conjecture in its most general formulation.

The Universality Conjecture of Random Matrix Theory has a status comparable to that of the Ergodic Hypothesis in Statistical Mechanics. According to Freeman Dyson random matrix theory is "a new kind of statistical mechanics in which we renounce exact knowledge not of the state, but of the system itself". The Universality Conjecture states that eigenvalue correlations on the scale of the average level spacing do not depend on the actual chosen ensemble of random matrices, but are universal.

In February 2004 it was announced that **Troy Day** won a **Chancellor's Research Prize** for his work on how to model the evolutionary and epidemiological dynamics of infectious diseases such as influenza and SARS. This is a prize for Queen's researchers in the first eight years of their academic careers.

In February 2004 it was announced that **Leo Jonker** won the first **Canadian Mathematical Society's Excellence in Teaching Award** for his remarkable success in teaching engineers and elementary school teachers. One of the students in his new Fundamental Concepts in Elementary Mathematics for Teachers had this to say about his teaching: *"He essentially changed us from a bunch of non-math minded students who lacked confidence in our abilities to teach it effectively, to a group of people who were excited and eager to go into our schools every week and teach math to our students."*

Academic Prizes Won by Mathematics and Statistics Students

The number of academic medals won by our students in recent years has been astounding – our undergraduates and graduates have frequently walked off with the top prizes in the Faculties of Arts and Science, Applied Science, and the School of Graduate Studies. These top students have helped push the quality of our programs steadily upwards, and then gone on to wonderful careers. Here is a look at the recent prize winners.

PRINCE OF WALES PRIZES

Established by the Prince of Wales in the 1860's, these prizes have remained the most prestigious for undergraduate science and humanities students. Each year at spring convocation four prizes are awarded for academic excellence – the Prince of Wales Science Prize and Runner-Up, and the Prince of Wales Humanities Prize and Runner-Up.

This past spring Mark Colarusso was awarded the Science Runner-Up Prize, and 15 years earlier in 1989 Robert McCann won the same prize. In the 15-year period spanning these two Prizes the Department there were 30 Prince of Wales Science Prizes awarded, and the Department won 15. There were only two years when we did not win either the Prince of Wales or Runner-Up Prize, and there were three years when walked off with both.

1989	Robert McCann	Runner-Up
1991	Alex Sands	Prince of Wales
1992	Imin Chen (Math-CISC)	Prince of Wales
1993	Michael Roth (Hons. Math)	Runner-Up
1994	Laura Scull (Hons. Math)	Prince of Wales
1995	Jonathan Burns (Math-Chem)	Prince of Wales
	Christine Tong (Math-BioChem)	Runner-Up
1996	Sean May (Hons. Math)	Prince of Wales
1997	Andrew Toms (Hons. Math)	Prince of Wales
	Paula Dow (Hons. Math)	Runner-Up
1998	Joanna Karczmarek (Math-Phys)	Prince of Wales
1999	Michael Levi (Math-Phys)	Prince of Wales
2001	John Neary (Hons. Math)	Prince of Wales
	Leigh Jansen (Hons. Math)	Runner-Up
2002	Fok-Shuen Leung (Hons. Math)	Runner-Up
2003	Mark Colarusso (Hons. Math)	Runner-Up

THE STIRLING GOLD AND PROFESSIONAL ENGINEER'S GOLD

In the Faculty of Applied Science the two most distinguished awards are the J. B. Stirling Gold Medal and the Professional Engineers' Gold Medal. The Stirling Gold goes to the student with the highest cumulative academic standing for their four Queen's years, and was first awarded in 1989; the Professional Engineers' Gold goes to the student with the highest academic standing in fourth year, and has a longer history.

1987	Mark Green (Math & Eng)	Prof. Eng Gold Medal
1988	Virginia de Sa (Math & Eng)	Prof. Eng Gold Medal
1999	Daniel Veiner (Math. & Eng.)	Stirling Gold Medal
2003	Dawn Van Weelden (Math. & Eng.)	Prof. Eng Gold Medal

GOVERNOR GENERAL'S MEDALS

Lord Dufferin, Canada's third Governor General after Confederation, created the Academic Medals in 1873 to encourage academic excellence across the nation. Pierre Trudeau, Tommy Douglas, and Robert Bourassa all received Governor General's Medals; they are awarded to students graduating with the highest average from high school, colleges and universities. They have become the most prestigious awards students can receive.

Current practice is that the Medals are awarded at four distinct levels: Bronze at the secondary school level; Collegiate Bronze at the post-secondary diploma level; Silver at the undergraduate level, and Gold at the graduate level. The practice at Queen's has varied over the years. Before 1988 there was a Gold Medal awarded to the top graduating undergraduate student, but starting in 1988 it was the Silver Medal for undergraduates, and the Gold for the top graduating student from the School of Graduate Studies. In 2003 practice again changed when two Gold Medals were awarded at the graduate level.

In recent times our students have walked off with about a third of all the Academic Medals awarded at Queen's, twelve of thirty-seven since 1985. On two separate occasions, in 1991 and 1999, our students won both the Silver and the Gold.

UNDERGRADUATE ACADEMIC MEDALS

1985	Stephan Norman (Math & Eng)	Gold Medal
1987	Mark Green (Math & Eng)	Gold Medal
1988	Virginia de Sa (Math & Eng)	Silver Medal
1991	Alex Grossman (Hons. Math)	Silver Medal
1992	Imin Chen (Math-CISC)	Silver Medal
1997	Andrew Toms (Hons. Math)	Silver Medal
1998	Joanna Karczmarek (Math-Phys)	Silver Medal
1999	Michael Levi (Math-Phys)	Silver Medal
2001	John Neary (Hons. Math)	Silver Medal

GOVERNOR GENERAL'S GOLD MEDAL FOR GRADS

1991	Wojciech Jaworski (PhD, Mathematics)	Gold Medal
1999	Jian Shen (PhD, Mathematics)	Gold Medal
2003	Lousindi Sabourin (PhD, Mathematics)	Gold Medal

SCIENCE DOCTORAL PRIZE AT QUEEN'S

Starting in 1999 each division of the Graduate School started awarding Doctoral Prizes for the best theses in division. For the Science Division (Division IV) our students have won the Thesis Prize four of the five times the Prize has been awarded.

1999	Jian Shen (PhD, Mathematics)	Science Prize
2000	Konstantin Rybnikov (PhD, Mathematics)	Science Prize
2001	András György (MSc, Math & Eng)	Science Prize
2003	Lousindi Sabourin (PhD, Mathematics)	Science Prize

The case of András is particularly distinguished because he won as a master's student, and the competition is open to both doctoral and master's students.

NATIONAL AWARDS FOR GRADUATE STUDENTS

Starting in 1999 national awards started coming regularly to our graduate students. In this period our graduating doctor students won the thesis prizes of the three major Canadian societies for the mathematical sciences – the Canadian Mathematics Society, the Canadian Applied and Industrial Mathematics Society, and the Statistical Society of Canada. In addition, the prestigious NSERC Doctoral Prize went to the Department once; the NSERC Doctoral Prize is awarded to two science students each year, and two engineering students.

1999	Troy Day (PhD, Mathematics)	NSERC Doctoral Prize, Canadian Applied and Industrial Mathematics Doctoral Prize, American Naturalist's Young Investigators Prize
1999	Jian Shen (PhD, Mathematics)	Canadian Mathematics Society Doctoral Prize
2001	Tim Ramsey (PhD, Statistics)	Statistical Society of Canada Pierre Robbillard Prize
2002	Alina Cojocaru (PhD, Mathematics)	Canadian Mathematics Society Doctoral Prize
2003	András György (post-doctoral)	Hungarian President's Academic Prize

The *American Naturalists Award* is open to scientists in the first eight years of their career; it is somewhat unusual for an emerging doctoral student to win it.

The *Hungarian President's Academic Prize* that András received is over 100 years old – it is awarded to a small number of distinguished students each year. The terms of this award require that all grades must be fives, the highest possible, over the four years of high school, the five years of university, and the three years of doctoral studies. András was the 13th doctoral student at the Budapest University of Technology and Economics to receive this award in the past 100 years.

The Tyranny of Reality

Peter Taylor

This article is based on a talk given at the 2003 summer meeting of the Canadian Mathematics Society.

They started from opposite ends. Pisano is a master of form and is striving towards reality; Julia aches to throw off the tyranny of reality and reach the essential that lies somewhere underneath.

The Dream of Scipio, Iain Pears, p. 279.

It is a powerful weapon, *yet it's aesthetically superb.*

Tom Cruise in awe of the art that has gone into the making of his samurai sword.
Kingston Whig Standard, Sept. 5, 2003, p.28.

My premise is that we mathematicians are sitting on a gold mine. In terms of structural beauty, stunning insights, unexpected power, all from simple, accessible, ingredients, very little can compare with our wonderful subject. But we do a terrible job at communicating that to most of our students. In the classroom we blow it, and we thereby alienate just about the entire population. And we've no one to blame but ourselves.

Well hang on here. Surely a lot of that has been done over the past 20 years (reform calculus, for example) and indeed it remains today an active field of development. Yes, that's true. So why haven't we seen more of an impact?

I believe the problem is that there are *two* things we have to do. One of these is to find a new way to teach and that's what most of our reform efforts have been working on. [Indeed for me this is mostly about curriculum, at least I consider teaching methodology as being based in or driven by curriculum.] But the other is to let go of the old way, and I have a feeling that's a lot more difficult, or at least it poses a much more subtle problem.

I see this when I share with colleagues some neat exploratory problems that might work well in their courses. They unfailingly like the problems, but in their wrinkled foreheads I can see a calculation of the time it will take. "What can I afford to leave out?" That's "old way" thinking, and if we don't let go of that, we will never successfully embrace the new.

Letting go is hard. To succeed I believe that we need a different model about what it is that we are doing. I believe that we need a new metaphor.

Picasso's Guernica 1937. In the afternoon of April 26, 1937, German bombers, flying for Franco, annihilated the defenseless Spanish town of Guernica, the centre of the Basque cultural tradition. For over three hours, a powerful fleet of bombers and fighters circled and wheeled over the town, dropping thousands of bombs, and setting everything on fire. The fighters then dropped low to spatter with machine gun fire those who had fled to the fields.

Over the next few days, the news of the massacre at Guernica spread to a shocked and outraged world. It was not the first of Franco's atrocities, but it was the one which galvanized Picasso into action. He had already accepted a commission for a mural at the Spanish pavilion at the Paris World fair, but he had so far produced nothing. In the six weeks following Guernica, he worked at a feverish pitch to produce a memorial to the innocent dead and a manifesto against the brutality of modern war. (See front cover).

The painting is 26 feet wide and 11 feet high. The figures rage across the canvas in a rush of terror. Heads everywhere are flung high, mouths forced open in a frozen outcry. A jagged light casts its sharp illumination on the scene. A woman from the outside world leans through the window surveying the carnage with a feeble lamp, her face a mask of horror. Except for the harsh whites, everything is dark, claustrophobic, compressed in gloom. The images are stark and simple, almost childlike, a woman and a child, a peasant woman, farm animals, a single stricken household says it all. [Excerpted in part from Life 65, December 1968 pp. 86-93.]

The way of the artist. A work of art is a representation of reality, a representation subject to certain essential constraints (the canvas, the sonnet, the steps of the dance). However the *objective* of the work is not in fact to *represent* but to *transform*, to transform our perception of the reality, to allow us to see what's truly there, to open our eyes, to free and empower us. It accomplishes this by stripping away the inessential aspects of the experience, and rendering with imagination the simple lines that remain. This imaginative transformation is such that the work, if successful, *conveys the experience more sharply and truly than can reality itself.* In this way, art, which, because of its self-appointed constraints of form and structure appears to work at a disadvantage, manages to turn these constraints into a more focused, more memorable, more telling experience than the real thing. That word "telling" is a good one here because the raw experience itself is often overlaid with complexities and irrelevancies which interfere with our attention. Art, as a highly particular retelling, focuses us and allows us to listen in a new way.

An interesting example we are perhaps all familiar with is the movie *A Beautiful Mind* that attempts to provide an artistic portrayal (within a certain medium, that being the genre of big Hollywood films) of the life of John Nash. This is all the more interesting because, though it is widely regarded as having succeeded on a number of levels as a work of art, it was criticized for departing significantly from Nash's life. But the important point (well made by Keith Devlin and others) is that the movie is not a "photograph" of the life lived. If it had attempted to be that it would almost certainly not have worked in that particular artistic context. Instead it took on the (formidable) challenge of capturing the essence of that life (both personally and mathematically) in a 3-hour Hollywood-style film, and by most accounts succeeded wonderfully. For those who want more (and the movie has almost certainly inspired many to seek out more) there are always books and webs, for example, Sylvia Nasar's excellent book of the same name

As teachers of mathematics we are artists. The landscape we gaze upon, brush in hand, is a coherent body of mathematical ideas and results. It is however not our job to thrust this body of results upon our students. Rather our challenge as artists is first to "strip away the inessential aspects," and then to render imaginatively "the simple lines that remain." This stripping away is quite different from asking, "what can I leave out?" If you ask the artist what she has left out of her picture, she might regard you with puzzled amusement, and then reply, "Everything; I pitched the lot," but she might just as well reply, "Nothing; everything is there." Indeed, just as art is less than reality, so the problems and explorations we conjure up will be less than the whole mathematical theory. And just as art is so much more than reality, sharper, more focused, more particular, *so these problems can convey the true mathematical experience better than could the mathematics itself.*

Restraint is a key component of artistic integrity and here it comes down to trusting the problems to do the work they are designed to do. That's the "letting go" part and it's not easy—caught up in the complexities of the subject, we are too forcefully aware of so much that has to be said, explained, clarified, and we are seized with doubts that the few students who actually might need something that we have left out will be able to capture that on their on. But the rewards of restraint can be enormous. It gives room for the encounter to continue to work (and play!) in the mathematical lives of our students, and it encourages them to be artistic in their own efforts.

An example might help, and I choose one from my introductory linear algebra course. A central concept in the course is the notion of eigenvector, or more generally of *eigensolution* that being a special solution which has the virtue of being easy to describe, but has the vice of not being a solution to the problem at hand. But it *is* the solution to a closely related one and the idea is that with luck (and linearity) we can put these special solutions together to get the solution we are after. This strategy is so central to the subject, that I build a large canvas around it, large enough to occupy an entire third of the two-semester course. I begin by *counting trains*. This is a simple exploratory problem with lots of fine side-roads (for example massive explorations into Fibonacci numbers), which contains the essence of the idea of eigenfunction expansion.

Problem 1. Counting trains. I am constructing trains using cars that are either 1 unit long or 2 units long, where there is one type of car of length 1 but two kinds of cars of length 2, type A and type B. Let t_n be the number of trains of total length n. For example $t_3 = 5$, the 5 different 3-trains being 111, 1A, A1, 1B, B1. [Note that trains are ordered so that 1A and A1 are indeed different.] Find a formula for t_n in terms of n.

Solution. By counting, students can generate a number of terms of the sequence: 1, 3, 5, 11, 21... I suggest the possibility of recursive thinking and they eventually come up with the argument that

$$t_n = t_{n-1} + 2t_{n-2}.$$

[Count the number of n-trains conditional on the first car.] This leads to the initial value problem:

$$t_n = t_{n-1} + 2t_{n-2} \qquad t_0 = 1,\ t_1 = 1.$$

Armed with this, the students can easily generate more terms:

$$1, 3, 5, 11, 21, 43, 85, 171\ldots$$

Many students see that each term is twice the preceding term except you alternately add or subtract 1. By comparing terms with powers of 2, they are lead to the formula:

$$t_n = \frac{2}{3}2^n + \frac{1}{3}(-1)^n.$$

It's a nice formula and it fits the terms so far, but can we be sure it will work forever? [One way to prove this is with mathematical induction, but I'm after bigger game here. We will look at induction later in the course.]

Here's where I put forward our fundamental strategy: *look for alternative initial conditions that have simple solutions. Then try to use these as building blocks to* *construct other solutions*. What the students find (perhaps by trial and error, trying different initial conditions) are the geometric sequences $\{2^n\}$ and $\{(-1)^n\}$. And then we argue that sums and scalar multiples of solutions are solutions and we manage to write the solution we are after as a linear combination of the geometric solutions, and we have found a rigorous argument for our formula.

This is the problem that introduces the general notion of eigenvector. From here we go on to study a number of standard matrix recursions (age-structured population growth, systems of brine tanks, equilibrium price vectors, etc.) In each of the past two years I have restricted myself to real eigenvalues, partly because I wanted to do justice to the above (real) examples, but also because the last time I "did" complex eigenvalues, the students found it difficult and it did seem to take a long time. But again this year the question arose. Can I include complex eigenvalues? What would I have to leave out?

And I suddenly realize I've fallen into the same "old ways" of thinking that I have warned others to avoid. I have been automatically assuming that to "do" complex numbers would entail a whole bag of stuff—complex arithmetic, trigonometry, and enough examples of different kinds to "cover all the angles." But why not just do one example—a well chosen work of art that convey the magic of the topic, shows off the power of our brave decision to try to push through with complex eigenvalues an idea that we previously realized with real ones. For example:

Problem 2. Solve the recursive equation $t_{n+1} = 2t_n - 2t_{n-1}$ $t_0 = 1$, $t_1 = 3$.

Solution. If we tabulate the first 12 values

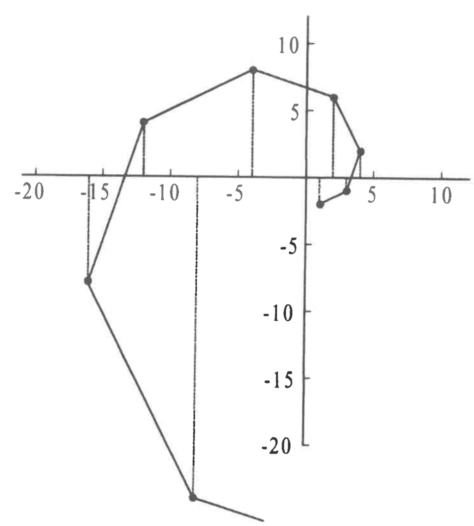
$$1, 3, 4, 2, -4, -12, -16, -8, 16, 48, 64, 32,$$

we perceive a block pattern with blocks of size 4. From here we could again use mathematical induction to show that the pattern continues, but we actually want to "see" how the pattern unfolds. The students are used to looking for "multipliers," and here they find one in −4 but the trouble is that it seems to take 4 terms to act. How might we encapsulate that? Could such a "jerky" pattern ever be described by any kind of natural construction?

Using the train technique, we look for geometric solutions $\{r^n\}$ to the equation and we find two with $r = 1 \pm i$. Now these are complex, but we push forward in spite of that. [A precocious student might be unable to resist calculating $(1+i)^4$ and getting −4. What a discovery!] We try to write our target sequence as a linear combination of the two geometric sequences and since the two terms of the sum are conjugates, we get a sum of conjugates which can be written as the real part of a sequence of complex numbers. We get:

$$t_n = \mathrm{Re}[(1-2i)(1+i)^n].$$

The sequence in the square brackets is geometric (with multiplier $1+i$), and it is therefore a spiral in the complex plane with a 45° rotation each term. The projection of this on the real axis is our desired sequence. This is a lovely example of the visual power of embracing the imaginary dimension—the spiral is seen as a *deux ex machina* that generates the sequence from above, as it were. And in displaying multiplication as rotation it showcases the fundamental contribution that complex numbers make to our understanding of arithmetic.



What of the rest of the course? How does it develop? Which ideas, which theorems, which technical results? For example, do I go farther with complex eigenvectors? Do I go on to a 2×2 matrix equation where I use the same complex plane representation to track both x and y together? Should I get into change of basis stuff (something I've actually so far done without, even with the real eigenvalues)?

Such questions as these we always struggle with, and they are very much the struggle between Pisano and Julia. In this process we are guided in our thinking and feeling the way an artist is so guided. The course evolves as does a painting grows or a dramatic work. We draw on our deep knowledge of the landscape, on the character of the work and the nature of the artistic medium. In this we must look clearly and carefully; we must strive to "see" with fresh eyes. As the work grows, so do the possibilities. But there's an essential closing down as well. Each new piece must fit the emerging whole. It's a question of integrity. [What distressed me about our recent high school curriculum revisions was a blatant disregard of this principle. Topics were stuffed in here and there with little connection to the whole.]

In an article a few years ago, William Kirwan, mathematician and President of Ohio State University, called for "a reshaping and restructuring of the curriculum with greater emphasis on active learning at all levels." The ideas put forward here are exactly that— a reshaping and restructuring. It is however a big change. It questions the very canon of the subject, at least at the introductory level. To do it right requires many creative ideas and courage as well.

William Kirwan, *Mathematics departments in the 21st century: role, relevance and responsibility* MAA Monthly 108, January 2001.

In Memoriam
Dominique de Caen (1956 – 2002)

Professor Dominique (Dom) de Caen died suddenly on June 19, 2002 after a long struggle with severe neck pain.

Born in Montreal in 1956, Dom obtained his M.Sc. at Queen's University in 1979 under the direction of Norman Pullman and his Ph.D. in 1982 at the University of Toronto under Eric Mendelsohn. He held an NSERC Postdoctoral Fellowship at the University of Waterloo from 1982 to 1983 and was an Assistant Professor at Northeastern University from 1983-1985. In 1985, he was enticed to return to Canada with the award of a prestigious NSERC University Research Fellowship (URF). From 1985 on, he was on staff at Queen's University and was a model for the success of the URF program. Through his scholarship, insightful research and generous support he became the linchpin of the discrete mathematics program at Queen's. He was promoted to Full Professor in 1997.

Dom became well known for the estimates in his doctoral thesis on Turan's extremal problem for hypergraphs. His interest in extremal graph theory continued throughout his life: in 1999 he obtained an asymptotically sharp estimate with Z. Furedi on the maximum size of a 3-uniform hypergraph not containing a Fano plane. He was also known for his expertise in other branches of discrete mathematics. He had an impressive familiarity with the theory of designs and with algebraic graph theory. His joint work with E. van Dam on association schemes later resulted in his construction of the asymptotically largest known families of equiangular lines in Euclidean space. He also made significant contributions to the theory of tournaments and to the theory of graph decompositions and is known for his lower bound on the probability of a union of events in probability space. He published over 50 papers covering a wide variety of topics in discrete mathematics. His Erdös number was one.

Dom will always be fondly remembered by many of his colleagues for fine conference talks, helpful suggestions, a love of good food, and for cryptic crosswords and many games of scrabble and backgammon. A calm and generous spirit, a respected researcher, an inspiring lecturer and Putnam coach, he is greatly missed.

David Gregory

Even or odd?

(from Fall 2000 issue)

Peter Taylor

Eeyore and Owl play the following game: they flip 10 coins, and Eeyore wins if the number of heads is even, and Owl wins if it's odd. The question is, is the game fair, or does it favour one or the other? Well that's not so hard to settle if the coins are unbiased. The game is fair and there are a number of simple arguments for that, some quite clever. But the situation here is that the coins are *biased* and each comes up heads with probability 2/3 and tails with probability 1/3. Is the game still fair?

Solution

One of the interesting things about this problem is that there are so many different ways to tackle it. Maybe that's why I received more solutions and comments than I ever have before. More about that later.

Most of the solutions I received were actually what might be called the "brute force" approach. Hey that doesn't mean they're bad—for those who like calculating it's a very satisfying approach. Essentially you work out the probability of getting exactly 1 head, 2 heads, 3, heads etc., and then do some adding. Actually a few of the folks using this approach, Garth Scott (Arts '97), John Greenhorn (Arts '75, South Grenville DHS), Allan Brett and Warren Wolfe (PhD. '75), pointed out that we don't really need to find the probability of winning; all we are asked to do is find out whether or not it's equal to 1/2. And if you just look at the various numbers you are adding together you see that the resulting sum has denominator 3^{10} so no matter what the numerator is, the result could never equal 1/2. Very nice observation. In fact, the probability that Eeyore wins turns out to be $29525/3^{10}$ which is close to 0.5000085. Close to fair but (as they say) no cigar.

There's another approach: more subtle, more elegant, more powerful—the recursive approach.

Whenever we have a family of problems indexed by the integers (for example, here we can consider the problem for 9 coins or 10 coins or 11 coins, etc.) it's natural ask whether we can move easily from one level to the next. If we happened to know the answer for the 9-coin problem, would that allow us to easily find the answer to the 10-coin problem? Such inductive or recursive ways of thinking can be very powerful.

Let P_n be the probability of getting an even number of heads with n coins. Suppose we knew P_9, the answer to the 9-coin problem. Could we find from there the answer to the 10-coin problem?

Well let's see. Suppose we flip 10 coins, 9 of them green and the other one red. We want to know when the total number of heads will be even. Now look at the green coins. Either there's an even number of heads or not, and the probabilities for each case are P_9 and $1-P_9$. In each case can we work out the probabilities for the set of 10 coins?

Yes we can. If there are an even number of green heads, then to stay even, the red coin better be tails (prob. 1/3), and if there are an odd number of green heads, then to get even, the red coin better be heads (prob. 2/3). So in a proportion P_9 of the cases, the probability of an even number of heads is 1/3, and in a proportion $1-P_9$ of the cases, the probability of an even number of heads is 2/3. This gives us:

$$P_{10} = \frac{1}{3}P_9 + \frac{2}{3}(1-P_9).$$

Simplifying:

$$P_{10} = \frac{2}{3} - \frac{1}{3}P_9.$$

The argument is quite general and gives us:

$$P_n = \frac{2}{3} - \frac{1}{3}P_{n-1}.$$

This is the "recursive" formula we were after. What we want to do now is "solve" it, that is, use it to find a formula for P_n. There are lots of ways to do that, standard and non-standard, including guessing and then using math induction, but again, all we need do is determine whether the game is fair and, as noted by Hamish Taylor (Prof. School of Business), Ross Ethier (Science '80, Prof. U of T) and Philip Nidd (Science '74) it's clear from the recursion that either all of the P_n are equal to 1/2 or none of them are. Since P_1 is 1/3, none of them are. You can also easily show that they must be alternately less than and greater than 1/2, so P_{10} will be greater than 1/2 and Eeyore has the edge.

Also solved by Tom Kerr (Welland Centennial SS and the entire (!) OAC algebra class of Jen Scriel (Arts '96, Earl Haig SS) all of whom sent me their solutions. Thanks guys!

New Problems

Peter Taylor

More coin flipping problems

Because of the success of the last problem, I include a couple more coin flipping problems.

1. Player A has 11 coins, while player B has 10 coins. Both players throw all of their coins simultaneously and observe the number that come up heads. Assuming all the coins are unbiased, what is the probability that A obtains more heads than B?

2. Amar and Belinda play the following game. A fair coin is flipped repeatedly and Amar wins if the sequence Heads-Tails occurs before the sequence Heads-Heads; otherwise Belinda wins. For example, Amar wins on the toss sequence T, T, H, T and Belinda wins on the toss sequence T, T, H, H.

(a) (easy) Show that Amar and Belinda both win with probability 1/2.

(b) Now consider a simple variation on this game. Instead of betting on the outcome of a single coin, imagine what would happen if A and B each had a coin and flipped it until the desired sequence was obtained, HT for A and HH for B. The game is now played as follows. The winner is the one who gets the right sequence in the least number of flips. [If there's a tie you play again.]

At first glance it seems like the second game might also result in each player having a probability of winning of 1/2. However this is not true—this version favours A.

I don't think it's so easy to calculate the winning probabilities, but it's certainly possible. A simpler but closely related question is to calculate the average number of flips it takes either player to obtain their desired pair. On average, how long does it take for A to get HT and for B to get HH? That's your problem.

3. *[Editor: Please omit this problem from this issue. It is simply too difficult for this audience.]* You start flipping a fair coin, and can at any time stop and claim a prize in cents equal to the fraction of flips that has come up heads. So, if you stop playing after 5 flips with 4 heads, you win 80 cents. Find an optimal strategy.

Send your solutions, or new problem suggestions, to:
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Words of wisdom...

John Coleman, now 85.7 years old, says: "The best way to keep mathematically active past 80 is to learn touch-typing and walk daily".

Did you know...? (submitted by Norm Rice)

$e^{(i\pi)} + 1 = 0$ remarkably combines the five most important numbers in mathematics.