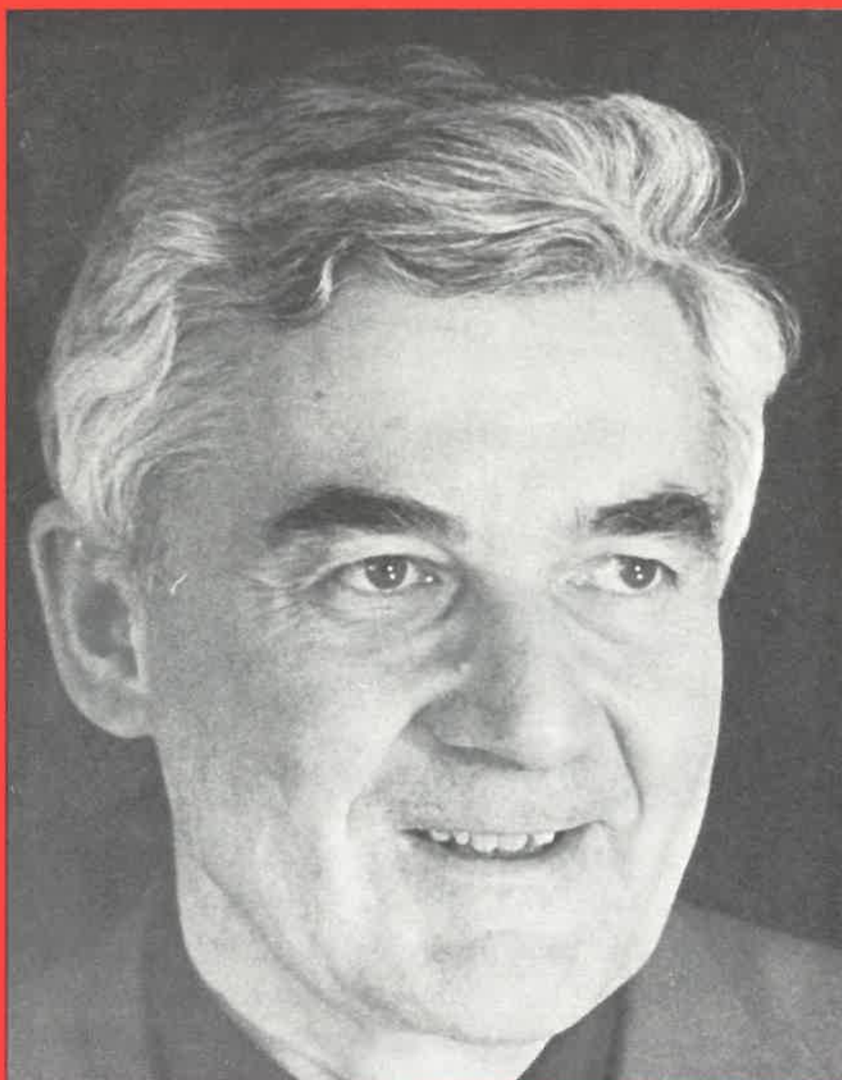
QUEEN'S MATHEMATICAL COMMUNICATOR

April 1980



An aperiodical issued at Kingston, Ontario by the
Department of Mathematics and Statistics, Queen's University

JOHN COLEMAN

John Coleman has announced his resignation after 20 years as Head of the Department as of July 1, 1980, and a successor has been appointed.

He and his wife Marie-Jeanne will be on sabbatical leave next year, probably at the University of British Columbia.

John was born in Toronto in 1918, went to high school there and graduated from the University of Toronto in mathematics in 1939. He obtained his M.A. from Princeton in 1940 and his Ph.D. from the University of Toronto in 1943 with a thesis on relativistic quantum mechanics. He was a member of the first team to win the Putnam prize in 1938.

From 1940-43 he was the Secretary of the Student Christian Movement at the University of Toronto. In 1943 he came to Queen's as an assistant professor.

From 1945-49 he was the Universities Secretary of the World's Student Christian Federation; while in that post he travelled to over 20 countries including Scandinavia, Australia, Czechoslovakia and New Zealand.

He was an assistant and later an associate professor of mathematics at the University of Toronto from 1949-60.

He came to Queen's in 1960 as Head of the Department. While here he has been very active in both the mathematical and non-mathematical community. He has been the director of the Queen's International Centre, Chairman of the Ontario Mathematics Commission, Chairman of the Exchange Commission of the International Mathematical Union, a member of the Science Council of Canada and author of the Science Council's background Study No. 37 "Mathematical Sciences in Canada". He has been active in visiting Millhaven Prison and has been Chairman of the Citizen's Advisory Committee to the prison.

He and Marie-Jeanne have two sons and two grandchildren.

After retiring as Head he will remain a member of the Department and plans to emulate his predecessor, Ralph Jeffery, who continued to teach three courses until he was 85 years old.

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JOHN COLEMAN DAY

The Department is planning a day on which all of John's former students, colleagues and associates may gather in order to honour him on his retirement as Head of the Department.

Our tentative plans call for a session of technical lectures followed by a dinner and entertainment. The proposed date is

Saturday, November 8, 1980

If you are interested in receiving further information about this event as it becomes available please fill out the following form and return it.

- -

John Coleman Day

TO: Mrs. E.M. Wight, Dept. of Mathematics and Statistics,
Queen's University, Kingston, Ontario, K7L 3N6

Please send further information.

Name and Address:

Degree from Queen's and year: (if applicable)

Mathematics and Statistics at Queen's, 1960-1980

- John Coleman

It was on July 1, 1960 that I assumed responsibility as Head of the Department of Mathematics at Queen's. If all goes according to plan, on July 1 of this year, after twenty years, I shall step down and Lorne Campbell will take on the task of guiding the Department of Mathematics and Statistics. Nowadays, in most universities a twenty-year tenure as Head of a Department would seem so extraordinary as to be noteworthy. Not so at Queen's; in fact, I had only four predecessors (Williamson 1842-80, Dupuis 1880-1911, Matheson 1911-43, Jeffery 1943-60) whose tenure averaged 29-1/2 years!

Since this is my last chance to write for the Communicator while I am still Head, it may be appropriate to discuss briefly the significance of the period 1960-1980 in the history of this Department and of the mathematical disciplines in general.

In size, the Department expanded rapidly in response to the needs of the times and the needs of the University - from a staff of one secretary and nine regular academic staff in 1960, to eight support staff and forty-eight regular staff members plus eleven visitors this year who are with us from three to twelve months. Most of the academics devote about half their time to lecturing and attendant preparation, and half their time to research and scholarly activities. Few work as little as forty hours/week. Some work seventy.

Ralph Jeffery, whom I succeeded, believed that devotion to research is the only way a mathematician can keep abreast of our subject. He, therefore, pioneered the establishment of the Summer Research Institute of the Canadian Mathematical Congress to encourage young mathematicians to devote their summers to research.

Since when I began as an Assistant Professor at Queen's in 1943, my salary was $1900/annum and later, at the University of Toronto in 1952, the princely sum of $3000, moon-lighting in the summer was very attractive. The SRI paid $800 for twelve weeks research. This small incentive gave a remarkable fillip to mathematics in Canada.

However, Jeffery was equally concerned for good undergraduate teaching and convinced of the importance of the relations of the Department to the Faculty of Applied Science and of its outreach into Medicine, Business, etc.

Since I was and am completely sympathetic with these three concerns of Ralph Jeffery, my task at Queen's was relatively simple - to build on the excellent foundation which he had established. For several years the growth rate of mathematics teaching was double the rate of increase of students at Queen's. This was largely due to the upsurge of popular appreciation of the role of mathematics in our society caused by Sputnik and the emergence of the electronic computer as an instrument for the practical application of large reaches of pure mathematics which had previously been merely of abstract theoretical interest.

When I arrived in 1960, the then Vice-Principal, Dr. J.J. Deutsch, conducted me through the newly renovated Carruthers Hall and predicted that this would be the permanent home of mathematics for all time. By 1967 we were bursting at the seams with 32 people housed in four different locations. With the opening of Jeffery Hall in the spring of 1969 we were reunited under one roof.

Though in its general aspect, Jeffery Hall projects the air of chaste austerity which most people associate with mathematics, a delightful and outstanding feature of the building is the sunken courts. These were a happy inspiration of the architects in response to our suggestion that the Library should be displayed as the central focus of the building. Our excellent collection of monographs and journals and their pleasant physical setting has brought many satisfying expressions of admiration and envy from distinguished visiting mathematicians.

In 1968, the Department had the good fortune of being awarded a Negotiated Development Grant by the NRC. There is no doubt that this enabled Queen's to make a considerable leap forward as was documented in my Report to the NRC in 1972.

From the academic year 1967/68 to 1971/72 our establishment rose from 33 to 43 positions, or about 30 percent. The number of graduate students rose from 40 to 48. The total budget of the Department rose from approximately $550,000 in 1967-68 to slightly over one million dollars for 1972-73.

The number of subscriptions to current periodicals in the Mathematics Library had increased from 122 in 1967 to 240 in 1971. The rate of accession of monographs had quadrupled in the period between 1967-68 and 1971-72.

During 1971, my colleagues and I have prepared sixty papers which have appeared in the series of Queen's Preprints and forty-three which did not appear in that form. All of these have been submitted to regular journals and most of them will appear in due course. In addition, four volumes were added to the continuing and financially self-supporting series of Queen's Papers on Pure and Applied Mathematics.

Excellent reasearch is not assured by any piece of machinery but only by firt-rate <u>people</u> - helped by luck. Essentially, all the money that came to Queen's has been put into the salary of first-rate people. Effectively, we have used it as a supplement to the regular university budget to make possible a research effort on the part of the whole Department, which would otherwise have been impossible. In so doing, (i) the average competence of this Department has been markedly raised; (ii) many research papers of good quality have been produced which would not have been possible without the Grant; and, (iii) insofar as the University has found it possible to fulfill its original commitment, we will carry the significant increase of mathematical power into the future. Naturally, I used the opportunity of reporting to attempt to inculcate in the Officers of the N.R.C. the philosophy which has dominated the strategy of this Department since 1943: "Mathematics is the basic tool for all the Physical, Engineering, and increasingly, of the Social Sciences, so that Canada can hope to maintain its present economic and technological position only if the work of other sciences is strongly undergirded by a large, lively, research-minded body of mathematicians. Mathematics is a subject which is especially suitable for active development in Canada in the present phase of our country, because it does not require the huge capital expenditures that are necessary to achieve pre-eminence in many other fields - such as experimental nuclear studies or interplanetary research. The interaction between pure and applied mathematicians is very complex and very intimate. It would be a great mistake to follow the line of the Report of the Senate Committee on Science Policy so single-mindedly as to conclude that we should support only applied mathematics. The simple fact is that it is quite impossible to make a clear distinction between pure and applied mathematics. There is only one subject - that is, mathematics, which in the course of history is applied to a variety of practical and theoretical ends for the good of man. The creative pure mathematician is the essential driving force of the whole mathematical enterprise and we very much need in Canada some situations where the atmosphere for the existence and activity of really first-rate mathematicians is more propitious than presently obtains anywhere.

However, I am personally strongly critical of the very widespread ethos in North America which encourages the pure mathematicians to exist independently of, and with little contact with, applied mathematicians. Though he was pre-eminent as a pure mathematician, my predecessor - Professor Jeffery - worked extremely hard to maintain happy relations between the Mathematics Department and the Engineering Faculty of Queen's. He succeeded in this and it has been a main plank in my policy to foster the same good relations. It is only in such a context that one can expect to have a proper understanding of the role of mathematics in the total republic of the sciences and make a "useful" contribution to the Canadian economy viewed in the largest possible sense.

Even though we have made significant progress at Queen's during the past four years, and even though there are some people outside the country who would rate us as the best all-around department, I would be the last to suggest that we have, in fact, achieved anything close to the level of quality which the country desperately needs in four to six centres and at which we still aim."

Heartening evidence that the commitment of the Department has paid off was provided in the report on the Ontario mathematics departments by the ACAP assessors in 1975, who concluded that our Department was unusually strong in Pure and Applied Mathematics and in Statistics. Our programmes in Control Theory and Statistics were held up as models for other Ontario universities. We were the only Department for which the ACAP Committee suggested that the number of Graduate Students might advantageously increase. Colleagues in other Ontario Universities, and in other Queen's Departments, did not hide their envy.

Although valuable publicity for our status as a research-minded Department is provided by the opinions expressed by visitors to the Department, and by the impression other mathematicians have of our published papers, the Queen's Papers in Pure and Applied Mathematics are a not insignificant factor. Edited by Professors Ribenboim and Coleman, and published by the Campus Bookstore, this series has been a financial success and now consists of 52 titles.

In an article headed "You cannot be a twentieth century man without maths", the Economist of October 27, 1979 devoted five pages (107-114) of its section on Business to an informative article about modern mathematics and its role in the contemporary world. The author asserts that "The great divide between people with some knowledge of advanced mathematics and those without it is calculus", goes on to rhapsodize on the great intellectual leap achieved by Newton and Leibniz, and concludes "It is a sobering thought that most people's mathematical education stops short of the calculus. You might think that it would at least make its way into the history syllabus. Its importance to the industrial and technological revolution is far greater than that of, say, the invention of the steam engine".

Few humanists would understand, or concur, with the last statement. However, its truth was obvious to mathematicians by 1900, to engineers by 1940 and to economists by 1960. A.N. Whitehead clearly perceived and vividly articulated the role which mathematics had played, was playing and will play in human affairs long before 1925 when he wrote Chapter 2 of Science and the Modern World in which he alleged:

> "The paradox is now fully established that the utmost abstractions are the true weapons with which to control our thought of concrete fact."

Whitehead went on to assert

> "We are entering upon an age of reconstruction in religion, in science, and in political thought. Such ages

if they are to avoid mere ignorant oscillation between extremes must seek truth in its ultimate depths. There can be no vision of this depth apart from a philosophy which takes full account of those ultimate abstractions, whose inter-connections it is the business of mathematics to explore."

It was my extraordinary good fortune and high responsibility to preside over the growth of the Queen's Department of Mathematics and Statistics during the historical epoch in which the popular appreciation of the key role of mathematics in civilization became widespread.

I am grateful to my students and colleagues and to the Officers of Queen's, who have encouraged and supported me. Because of current financial stringencies in university financing the next couple of years may be somewhat difficult at Queen's. However, because of the increasing need of our society for mathematics, because of the ability of the current staff of the Department, and because of their commitment to good teaching and sound research, I am certain that under the guidance of Lorne Campbell the Department will flourish in the 80's and 90's, and continue to make a significant contribution to the mathematical sciences in Canada.

THE NEW HEAD

The Principal formed a search committee in October, 1979 to advise him on a successor to Professor Coleman as head of the Department. Included on the committee were three members of the Department: Professors Caradus, Davis and Wasan.

In February the Principal announced that Lorne Campbell will be the new Head for a five year term beginning July 1, 1980.

Dr. Campbell, who has been a member of Queen's University since 1963, received his B.Sc. from the University of Manitoba, his M.S. from Iowa State and his Ph.D. from the University of Toronto. From 1954 to 1958 he served with the Defence Research Telecommunications Establishment of the Defence Research Board in Ottawa, and subsequently taught in the Department of Mathematics at the University of Windsor from 1958 to 1963.

He has served on the Admissions Committee of the Faculty of Arts and Science, on the Academic Progress Committee of the Faculty of Applied Science and on the Advisory Research Committee of the School of Graduate Studies and Research. He is currently a member of the Canadian Mathematical Society, Statistical Society of Canada, American Mathematical Society, Society for Industrial and Applied Mathematics and the Institute of Electrical and Electronics Engineers.

His major research interests are in the areas of information theory and statistical communication theory. In these and other areas he has published a total of 35 papers.

THE PRINCIPLE OF INCLUSION AND EXCLUSION

by

Dave Mason

(David Mason obtained his B.Sc. in Mathematics in 1972 and his M.Sc. in 1973, both at Queen's, his supervisor being Norm Pullman. Since then he has worked in the field of Operational Research with the Department of National Defence.)

Operational Research is a loosely defined term, but generally means "the application of scientific knowledge and techniques to the solution of operational problems". In my case these problems may include almost any type having to do with operations of our Armed Forces. The defence environment can provide a rich variety of problems suitable to analysis using many diverse mathematical methods. I will describe a recent problem I encountered involving the application of a branch of mathematics called Combinatorial Theory (the study of Permutations and Combinations), in particular of a concept called The Principle of Inclusion and Exclusion.

This principle is basically used to count the number of objects that have a specific list of properties. For instance suppose we have a class of 25 students. Each student may have one or more (or none) of the following three properties:

A) has blue eyes
B) can roller skate
C) knows math teacher's middle name.

A survey of the class shows that the actual number of students with various combinations of these properties is:

$$N(A) = 11 \qquad N(A,B) = 6$$
$$N(B) = 15 \qquad N(B,C) = 4$$
$$N(C) = 6 \qquad N(A,C) = 3$$
$$N(A,B,C) = 1$$

where $N(A,B)$, for example, denotes the number of students who have blue eyes and can roller skate. The counts generated consider only those properties listed. That means the 11 students counted with property A) are not excluded from having properties B) or C). Now the blue-eyed, roller-skating math teacher has determined that those who will fail math this year are those students which have none of the three properties. If we use the expression A' to mean "not A" then we are after

$$\text{Number of failures} = N(A',B',C') = ?$$

The expression

$$25 - N(A)-N(B)-N(C)$$

is a step in the right direction, but isn't correct. Those with both properties A and B for instance have been subtracted twice - once in the $N(A)$ count and once in the $N(B)$ count. Let us then add one back for each pair of properties to adjust the count. The result is

$$25-N(A)-N(B)-N(C)+N(A,B)+N(A,C)+N(B,C) \quad .$$

But this expression has over-compensated the other way. Those students with all three properties were subtracted three times in the $N(A)$, $N(B)$, and $N(C)$ counts and then added back in three times in the $N(A,B)$, $N(A,C)$ and $N(B,C)$ counts. Hence, they must be subtracted out again, giving the final answer as

$$
\begin{aligned}
N(A',B',C') &= 25 \\
&- N(A)-N(B)-N(C) \\
&+ N(A,B)+N(A,C)+N(B,C) \\
&- N(A,B,C) \\
&= 5
\end{aligned}
$$

You may find it easier to visualize the reasoning used by drawing a Venn Diagram and placing the appropriate number of objects in each area of intersection.

The method I've just described may appear to be rather clumsy - progressively overshooting the target and then compensating back until it is finally hit. However it forms a useful general theory, which I will state in general form for those interested.

PRINCIPLE OF INCLUSION AND EXCLUSION

Consider a set of N objects. Let $A_1, A_2, \ldots, A_M$ be a list of properties that these objects may have. Let $N(A_1)$ denote the number of objects with property A_1 and $N(A_1')$ denote the number of objects that do not have property A_1 (etc.). The identity is

$$N(A_1', A_2', \ldots, A_M') = N$$

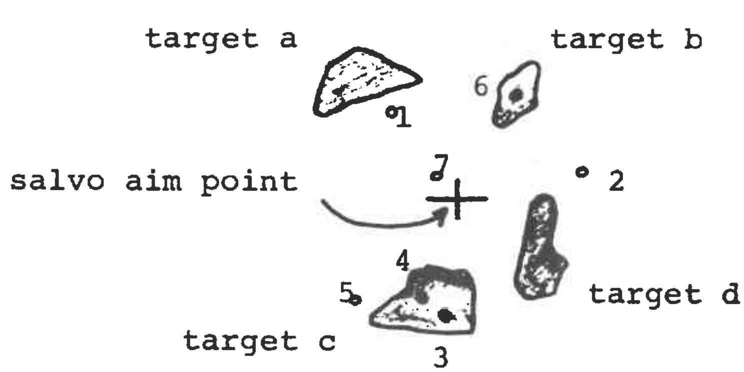
$$
\begin{aligned}
&- (\text{sum of all } N(A_i) \text{ terms}) \\
&+ (\text{sum over all distinct combinations of 2 properties of the } N(A_i,A_j) \text{ terms}) \\
&- (\text{sum over all distinct combinations of 3 properties of the } N(A_i,A_j,A_k) \text{ terms}) \\
&+ \ldots + (-1)^M \times N(A_1,A_2,\ldots,A_M)
\end{aligned}
$$

The last term in the expression represents the one and only combination of all M properties, and its sign is positive or negative depending on whether M is even or odd respectively. The term $(-1)^M$ handled this situation cleverly.

Proof of the Principle can be done by induction on M , the number of properties. Note that since "not (not A)" is precisely "A" , we could rewrite the above identity with the A_1' on the right and the A_i on the left and it would be equivalent.

Problems typically suited to the application of the Principle of Inclusion and Exclusion are those where the number of objects <u>with</u> certain properties is difficult to determine, but the number of objects <u>without</u> these properties is easily calculated (or vice-versa). One would simply apply the Principle which uses expressions involving the latter to find the former.

Let me now describe the defence problem that was considered. A fighter aircraft is loaded with salvo of 7 rockets, which it carries in a cannister under the wing. The fighter approaches a collection of ground targets and fires the rockets in rapid succession at them. The objective is to hit as many targets as possible with at least one rocket each. You might think that the rockets should all follow the same path and impact at nearly the same point. However even slight imperfections on the exterior surface of each rocket will allow aerodynamic forces to push it noticeably off the intended course. The result is that some rockets may hit one target, some hit another and some miss altogether. For instance suppose there were 4 targets on the ground.



For each rocket there are 5 possibilities: it either hits one of the targets ('a','b','c' or 'd') or it misses all of them. That means there are 5^7 or 78125 possible outcomes of one rocket salvo launch (we assume the rockets are distinguishable).

To apply our general theory in this case, we take our set of "objects" to be the set of all possible outcomes. The value of N is 78125. We are interested in four "properties" that an individual outcome can have:

A) that target "a" is hit at least once.
B) that target "b" is hit at least once.
C) that target "c" is hit at least once.
D) that target "d" is hit at least once.

We wish to know what fraction of all possible outcomes involve having all four targets hit - i.e. we are searching for $N(A,B,C,D)$.

The expression of $N(A,B,C,D)$ is not obvious, but the expressions for the number of outcomes involving A', B', C' or D' are easily calculated. In order to <u>not</u> hit target 'a', each rocket must encounter 1 of the 4 other possibilities (hits target 'b', 'c', 'd', or none at all). Hence

$$N(A') = 4^7 = N(B') = N(C') = N(D') .$$

Similarly

$$N(A',B') = N(A',C') = N(A',D') = N(B'C') = N(B',D') = N(C',D') = 3^7$$

$$N(A',B',C') = N(A',B',D') = N(A',C',D') = N(B',C',D') = 2^7$$

$$N(A',B',C',D') = 1 .$$

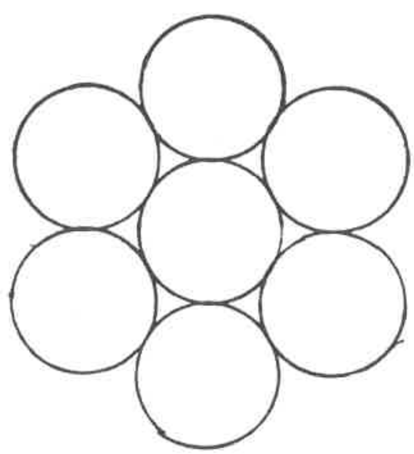
Applying the Principle then gives us

$$N(A,B,C,D) = 5^7 - 4(4^7) + 6(3^7) - 4(2^7) + 1 = 25,200 .$$

The answer to our question then is that 25,200/78,125 or 32.3% of all possible outcomes involve hitting all four targets.

The actual analysis done on this problem was considerably broader and included other features such as: more rockets; the introduction of probabilities of hitting each target (as a function of target size and its distance from the aim point); accounting for the influence of launch position of the fighter with respect to the target and; pilot aiming ability. However this application of the Principle of Inclusion and Exclusion lay at the centre of the analysis.

As an additional little problem on the geometrical side, lets look at the construction of the rocket launcher canisters themselves. The cylindrical canister that contains the seven rockets consists of one central tube with six tubes attached around it.



Rocket Launcher Cross-Section

From the drawing it appears that six circles fit exactly around one in the middle. In fact it's true, but do you know the reason? How many rockets could be carried in a launcher with an additional row of tubes around the outside of the seven? (Ans. 19). Do they also fit exactly around the perimeter? (Ans. yes). How about for one more row? (Ans. 37). Try it with a bunch of pennies and the symmetry will become visible. The pennies lie in straight lines with adjacent rows half-shifted to fit into place. The pennies are what is called hexagonally packed.

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Mathematics and Engineering students Andrew Long and Rick White competed in the Ontario Engineering Design Competition held March 1st at Queen's. Their entry "Two-wire transmission from a synchronous full-duplex modem" placed Third in a field of 12 entries from Queen's and other universities from all over Ontario.

PEOPLE IN THE NEWS

John Coleman, Liberal candidate, narrowly lost the recent federal election to Flora MacDonald. He reduced the Conservative lead from 20% in 1972 to a mere 2.5% on February 18, and, indeed, both the CBC and CTV networks predicted his victory based on early poll results. He won in the city of Kingston itself but was surpassed in the surrounding Kingston township.

Tony Geramita has been invited to speak at the American Mathematical Society special session on commutative algebra in Philadelphia in April, 1980. After that he is visiting the Università di Catania for a month to give a series of lectures on algebra and geometry. Tony has recently been elected vice-president of King Cole Homes, a non-profit community housing project in Kingston.

Paulo Ribenboim attended the Ottawa meeting of the Canadian Mathematics Society in December and the San Antonio meeting of the A.M.S. in January. He has recently given invited lectures at the University of Illinois in Urbana and at Carleton University.

Dan Norman spoke recently to a meeting of the Hastings County Mathematics teachers at Loyalist College, Belleville, on the subject of preparing students for university mathematics. His principal advice was that students must be encouraged and challenged to use ideas and to think and not simply to memorize recipes.

Hans Kummer has been invited to spend the month of June at the Nicolas Copernicus University in Toruń, Poland.

Tom Stroud is undertaking a data analysis for Educational Testing Service in connection with a study of the effectiveness of coaching programs which are designed to improve students' scores on the Scholastic Aptitude test.

Peter Taylor was invited to speak at the Potomac Regional Conference in Maryland last November. He gave one talk on the Bacon-Shakespeare controversy and another on the teaching of mathematics with the use of exploratory problems.

Ole A. Nielsen attended a three-day conference on Non-commutative Harmonic Analysis at the University of Colorado in Boulder, Colorado from March 27-29.

NEWS FROM GRADUATES

1973

Selma Tennenhouse is now working for Computech Consulting Canada Limited in Vancouver. After graduation (from the Mathematics and Engineering program) she worked for Bell Canada in Ottawa and Systemhouse in Vancouver before taking her present job.

1979

Les Gulko has been working for C.G.E. since graduation. He has been on their year-long training program which consists of several three-month assignments in various areas of engineering; he has done work in vibration and stress analysis, in electro-magnetics and in heat transfer. At present he is in Montreal working on mathematical models for ore grinding processes. He writes that his study at Queen's in the Mathematics and Engineering program was an excellent preparation for the work he is doing and that his knowledge of mathematics is highly appreciated.

QUEEN'S STUDENTS WIN NATIONAL AWARDS

This spring's graduating students at Queen's University have achieved outstanding results in competition for prestigious national awards in the sciences, social sciences and humanities.

Dr. Maurice Yeates, dean of graduate studies and research at Queen's, announced on April 3 that Queen's has won four of the 44 "1967 Scholarships" offered by the Natural Sciences and Engineering Research Council. Queen's was the only university in Canada to receive as many as four awards, with two other universities receiving three each.

The four NSERC scholarships, valued at $11,200 a year for three years, have been won by mathematics and statistics student Ross Ethier of Calgary, Alberta, physics students John Brace of Toronto and David DiFilippo of Sudbury, and chemistry student Erica Weinberg of Willowdale. The students are all in the final year of honours BA or B.Sc. programs.

"These national awards reaffirm the quality of Queen's academic strength," said Dr. Yeates. "The results demonstrate that Queen's continues to maintain its position as a full-service institution, offering high-quality programs in the sciences, social sciences and humanities."

In addition four undergraduates from the Department have been awarded NSERC scholarships for graduate study next year; all are from the Mathematics and Engineering program and are

Graham Avis
Ross Ethier
Gillian Woodruff
and Barbara Wyslouzil

This reflects, of course, the extremely high calibre of the students entering the Mathematics and Statistics Department and the excellent training they receive while here.

PROBLEM SECTION

by

Peter Taylor

The following problem was posed in the last issue.

Problem No. 2

The Iterated Mother Problem

Let X be the set of all n-vectors whose entries are non-negative integers. Define the transformation T from X to itself as follows

$$T(x_1,x_2,\dots,x_n) = (|x_1-x_2|,|x_2-x_3|,\dots,|x_{n-1}-x_n|,|x_n-x_1|).$$

Here is an example for $n = 5$.

$$\begin{aligned} x &= 4\ \ 7\ \ 8\ \ 1\ \ 4 \\ Tx &= 3\ \ 1\ \ 7\ \ 3\ \ 0 \\ T^2x &= 2\ \ 6\ \ 4\ \ 3\ \ 3 \\ T^3x &= 4\ \ 2\ \ 1\ \ 0\ \ 1 \end{aligned}$$

What happens, in general, if T is iterated indefinitely? I am especially interested in receiving any concise proofs. I am indebted to Joel Hllel of Concordia University for this nice problem.

Solution

This problem turns out to have a great deal more to it than I thought when I originally set it down in our last issue. I had worked out the solution for 3-triples and 4-tuples and said to myself, "Ah, it ought to be a good problem, in general". Well, in a sense it was! Indeed, attempts to get at the structure of the transformation T have led us into the reaches of linear algebra, group theory, and even finite field extensions. This may explain why so few readers submitted progress reports to me, although I was aware of quite a bit of excellent exploratory activity.

Probably the way to begin thinking about a problem like this is to try to settle the low-dimensional cases. For example, for 3-tuples you quickly discover that starting with any non-constant triple, you eventually get to MOM for some non-zero M and this pattern henceforth repeats itself, up to a shift. Hence the iterated mother. On the other hand, starting with <u>any</u> 4-tuple, you eventually get to 0000 .

Further experimentation with longer sequences will convince you that you always eventually wind up with all zeros, or in a periodic orbit of n-tuples all consisting of 0's and M's for some non-zero M . In fact, this is a general result, and the most basic non-obvious result of the problem. The proof is not sophisticated, but not all that easy to find either. Only Paul-Jean Cahen of Tunisia has submitted a proof of this result. I will not reproduce his rather nice argument, but will be pleased to supply it upon request.

We can take M to be 1 , so by the above result we will have discovered everything about the asymptotic behaviour of repeated applications of T , if we know how it behaves on sequences of 0's and 1's. Let $Z_2 = \{0,1\}$ endowed with the usual binary addition and multiplication, and $V_n = (Z_2)^n$. Then V_n is a vector space over Z_2 of dimension n . Let S_n be the left shift on V_n [eg. $S_5(10010)=(00101)$] . Then it is easy to see that $T_n = S_n + I_n$ where I_n is the identity operator (remember that $-1 = 1$). In particular we see that T_n commutes with S_n , which means that T_n preserves orbits of S_n . So if we want to know how T_n moves around elements of V_n , we don't have to distinguish two sequences in the same S_n-orbit.

Now the characteristic polynomial of S_n is $p_n(\lambda) = \lambda^n + 1$. If we factor out as many copies as we can of $\lambda + 1$ (over Z_2) we get $\lambda^n+1 = (\lambda+1)^k r(\lambda)$ where $r(\lambda)$ is not divisible by $\lambda+1$, and has degree $n-k$. Then it can be shown that V_n is the direct sum of the two subspaces $A = \ker(S_n+I_n)^k$ of dimension k and $B = \ker \underline{r}(S_n)$ of dimension $n-k$. Further, T_n is nilpotent on A (some power of T_n vanishes) and is nonsingular on B. So starting with any element of V_n, if it is in A, then some power of T_n annihilates it; if it is not in A, then some power of T_n maps it into B. So we need only know how T_n acts on the S_n-orbits in B.

Well now this is getting rather esoteric, and it certainly isn't everyone's cup of tea, but if you recall some of your linear algebra, it gives you an idea of how far you can go in simplifying the analysis of the structure of T_n. What we have accomplished is this. If we know the number k of times that $\lambda+1$ is a factor of λ^n+1 over the field Z_2, then we know the dimension $n-k$ of the space of "periodic" vectors x of T_n (some power of T_n applied to x gives x again). Furthermore we can construct all such x from the condition that they must be annihilated by $r(S_n)$, a sum of powers of the shift. The above factorization problem is itself quite interesting. Let me indicate a couple of examples.

For the case $n=5$, $\lambda^5+1 = (\lambda+1)(\lambda^4+\lambda^3+\lambda^2+1)$ is the required factorization. The space $A = \ker(S_5+I_5)$ has dimension 1 and contains only (00000) and (11111). The subspace B has dimension 4, hence contains $2^4 = 16$ elements. To say that a vector is in B, the kernel of $S_5^4 + S_5^3 + S_5^2 + I_5$, is the same as saying it has an even number of 1's. Such a vector is $x = (11110)$. Then $T_5 x = (00011)$, $T_5^2 x = (01111)$ and $T_5^3 x = (01111)$ which is in the same S_5-orbit as x. Now the S_5-orbits of x, $T_5 x$ and $T_5^2 x$ each contain 5 elements, and we have just seen that T_5 cycles these orbits. That is 15 elements of B accounted for. The other element of B is, of course, (00000). So we have described the structure of T_5 . If we start with a constant in V_5 , we eventually wind up in a fixed cycle of period 3 - and it's always the same cycle up to shifts.

For another example, I will consider the case where n is a power of 2 , $n = 2^k$. It can be shown that, over Z_2 , $\lambda^n+1 = (\lambda+1)^n$ is the factorization in this case. This implies that $A = V_n$, and T is nilpotent: every vector is eventually transformed into zero. So the pattern we observed for $n = 4$ holds for any power of 2 . This result was also obtained by a slightly more direct route by Paul-Jean Cahen.

Comments were also received from Peter Leipa (Toronto) who observed that all T-orbits wind up in a limit cycle. The case n=4 is treated in Ross Hensberger's book, Ingenuity in Mathematics, on page 80. I am indebted to Jack Harvey for this reference. If the problem still interests you and you find the sophistication forbidding, play with a few more examples (n=6 and n=7) and try to perceive the structures I have described.

Problem No. 3

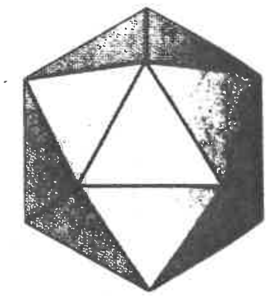
A Four Dimensional Polygon

Let K be the set of all 3×3 matrices with real entries ≥ 0 and all row and column sums equal to 1 . K can be realized as a subset of nine-dimensional space. What does it look like? Give a geometric description.

"Is he kidding" He wants me to visualize something in 9 dimensions? I have trouble with 3".

Now hang on. Don't get excited. K is really only 4-dimensional (these are 4 "degrees of freedom" in specifying such a matrix - why?) and it's a regular convex polygon with lots of symmetry. I realize you still can't quite visualize it, but to see what you might do, imagine how you would describe an icosahedron to an inhabitant of a 2-dimensional world.

You would say there are 12 vertices (0-dimensional faces) each subtending 5 edges (1-dimensional) faces of which there are 30 in all. There are 20 2-dimensional faces, all triangular. So far he can visualize everything.



Icosahedron

You can describe how the triangular faces fit together by labelling them and specifing adjacencies. Still, this doesn't give much of a picture of what the whole thing looks like. You might try the following. Take a horizontal "cutting plane" and lower it through the polygon describing how the 2-dimensional intersection changes. We start with a point, then get a regular 5-gon which grows to some maximum size. Suddenly each vertex of the 5-gon becomes a short side, and we get a 10-gon with 5 long edges and 5 short edges. The short edges grow and the long edges decrease until we get a regular 10-gon. We are now half-way down the icosohedron. The bottom half is the antisymmetric image of the top half.

What can you do for K ?

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Undergraduates in the Physics Department did outstandingly well in the recent Canadian Association of Physics Prize Examination. 95 students from 21 different universities competed and John Sloan of Queen's stood first. Out of the top ten students, five were from Queen's.