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Invariants of the gas dynamics inside the mushroom clouds

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ABSTRACT

The material conservation laws are derived for the axisymmetric flows of the inviscid barotropic gas inside the mushroom clouds. The invariant functions $\Psi(\mu)$ and $\Psi_G(\mu)$ of an independent variable μ are constructed for any pure poloidal compressible gas flow.

I. INTRODUCTION

The known data[1,2] about the atomic and thermonuclear explosions in atmosphere show that the dynamics of gas in the mushroom clouds is approximately axially symmetric with initially vanishing swirl. In the cylindrical coordinates r, z, φ, the gas velocity $\mathbf{V}(r, z, t)$ has the following form:

$$\mathbf{V}(r,z,t) = u(r,z,t)\hat{\mathbf{e}}_r + v(r,z,t)\hat{\mathbf{e}}_z + w(r,z,t)\hat{\mathbf{e}}_\varphi, \tag{1}$$

where $\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_z, \hat{\mathbf{e}}_\varphi$ are the vectors of unit length in directions of the coordinates r, z, φ. In Sec. II, we show that if at the initial moment t_0 the swirl $w(r, z, t_0) = 0$, then for all times t, the swirl $w(r, z, t) \equiv 0$. This follows from the existence of the material conservation law,

$$M(r,z,t) = [\mathbf{x} \times \mathbf{V}(\mathbf{x},t)] \cdot \hat{\mathbf{e}}_z = rw(r,z,t). \tag{2}$$

The conservation law $M(r, z, t)$ for the z-axisymmetric compressible gas dynamics with an arbitrary equation of state was derived in our paper.[3] Note that function $M(r, z, t)$ is different from the z-projection of the density of gas angular momentum $\mathcal{P}(\mathbf{x},t) = \rho(\mathbf{x},t)[\mathbf{x} \times \mathbf{V}(\mathbf{x},t)] \cdot \hat{\mathbf{e}}_z$ that is not conserved along the gas streaklines.

For the z-axisymmetric flows of ideal incompressible fluid with constant density ρ, the material conservation law $rw(r, z, t)$ was first established by Hicks in Ref. 4, p. 97, and later by Batchelor in Ref. 5, p.544. Kelbin, Cheviakov, and Oberlack presented in Ref. 6 the equation $d(ru^\varphi)/dt = 0$, which is equivalent to the conservation of ru^φ along the incompressible fluid streaklines and derived from it the material conservation laws $F[rw(r, z, t)]$ for fluid flows with constant density ρ (where $F[x]$ is an arbitrary differentiable function of x). In Ref. 3, we proved that for the z-axisymmetric flows of incompressible fluid with variable density $\rho(r, z, t)$, the functions $G[\rho(r, z, t), rw(r, z, t)]$ are material conservation laws (here, $G[x, y]$ are the arbitrary differentiable functions of x, y).

In this paper, we study the Euler equations for the inviscid barotropic compressible gas dynamics,[5,7]

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = -\frac{1}{\rho}\nabla p + \nabla\Phi, \tag{3}$$

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{V}) = 0, \tag{4}$$

where $\mathbf{V}(\mathbf{x}, t)$ is the gas velocity of class C^2, $p(\mathbf{x}, t)$ is the pressure, and $\Phi(\mathbf{x}, t)$ is the Newtonian gravitational potential. The gas pressure $p(\mathbf{x}, t)$ is connected with the gas density of mass $\rho(\mathbf{x}, t)$ by a barotropic equation of state $p = f(\rho)$, where $f(\rho)$ is a differentiable function with $df(\rho)/d\rho > 0$.

On the zero level of the material conservation law $M(r, z, t)$ (2), the gas dynamics is pure poloidal as is observed in the mushroom clouds.[1,2] In Sec. II, we prove that the gas dynamics inside the mushroom clouds possesses a material conservation law $\mathcal{H}(r,z,t)$ that is constant along the gas streaklines.

The problem of invariants of the compressible gas dynamics inside the mushroom clouds after the atomic and thermonuclear explosions in atmosphere is studied in this paper for the first time in the literature. The derived material conservation law $\mathcal{H}(r,z,t)$ is new and was not known before.

In Sec. III, we construct another invariant of the axisymmetric gas dynamics without swirl. The invariants are functions $\Psi(\mu)$ and $\Psi_G(\mu)$ of one variable μ. The functions do not depend on time t and are closely related with the geometry of surfaces $\mathcal{H}(r,z,t) = \text{const.}$ In Sec. III, we demonstrate that functions $\Psi(\mu)$ and $\Psi_G(\mu)$ are linked by certain differential equations.

II. A NEW MATERIAL CONSERVATION LAW FOR THE GAS DYNAMICS INSIDE THE MUSHROOM CLOUDS

A material derivative of a function $F(\mathbf{x},t)$ with respect to the gas flow with velocity $\mathbf{V}(\mathbf{x},t)$ is defined by the following formula:

$$\frac{\mathrm{D}F}{\mathrm{D}t} = \frac{\mathrm{d}F}{\mathrm{d}t} + \mathbf{V}\cdot\nabla F. \tag{5}$$

A function $F(\mathbf{x},t)$ is called a material conservation law for the system [(3) and (4)] if its material derivative (5) vanishes: $\mathrm{D}F/\mathrm{D}t \equiv 0$. In this case, function $F(\mathbf{x}(t),t)$ is constant along any gas streaklines defined by the system of time-dependent equations,

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{V}(\mathbf{x}(t),t). \tag{6}$$

Assume that in a z-axisymmetric mushroom cloud at an initial moment of time t_0, the gas flow has zero swirl $w(r, z, t_0) = 0$. Then, at any time t, the swirl is identically zero: $w(r, z, t) \equiv 0$.

Proof. In Ref. 3, we proved that for the z-axisymmetric compressible gas dynamics [with an arbitrary equation of state $p(r, z, t) = f[\rho(r, z, t), s(r, z, t)]$, where $s(r, z, t)$ is the density of gas entropy], the function $M(r, z, t)$ (2) satisfies equation $\mathrm{D}M(r, z, t)/\mathrm{D}t = 0$ and, therefore, is conserved along the gas streaklines. Therefore, if at an initial moment of time t_0 the gas swirl $w(r, z, t_0)$ vanishes, then $M(r, z, t) \equiv 0$ for all times t. Hence from Eq. (2), we get $w(r, z, t) \equiv 0$ for any time t. □

The gas velocity $\mathbf{V}(\mathbf{x}, t)$ for the z-axisymmetric gas flows has the form (1) in the cylindrical coordinates r, z, φ. The corresponding vorticity vector field $\nabla\times\mathbf{V}$ is[5]

$$\nabla\times\mathbf{V} = -\frac{(rw)_z}{r}\hat{\mathbf{e}}_r + \frac{(rw)_r}{r}\hat{\mathbf{e}}_z + (u_z - v_r)\hat{\mathbf{e}}_\varphi. \tag{7}$$

Let $\mathbf{A}(\mathbf{x},t) = A_1(\mathbf{x},t)\hat{\mathbf{e}}_r + A_2(\mathbf{x},t)\hat{\mathbf{e}}_\varphi + A_3(\mathbf{x},t)\hat{\mathbf{e}}_z$ be any vector field in the cylindrical coordinates r, φ, z. Then, the identity

$$(\mathbf{A}(\mathbf{x},t)\cdot\nabla)\hat{\mathbf{e}}_\varphi = \sum_{i=1}^{3}\tilde{A}_i(\mathbf{x},t)\frac{\partial}{\partial x_i}\hat{\mathbf{e}}_\varphi = -\frac{(\mathbf{A}(\mathbf{x},t)\cdot\hat{\mathbf{e}}_\varphi)}{r}\hat{\mathbf{e}}_r \tag{8}$$

holds, where $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, and $x_3 = z$ are the Cartesian coordinates.

Proof. In the Cartesian coordinates, we have

$$\hat{\mathbf{e}}_r = r^{-1}(x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2), \quad \hat{\mathbf{e}}_\varphi = r^{-1}(-x_2\hat{\mathbf{e}}_1 + x_1\hat{\mathbf{e}}_2), \tag{9}$$

where $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ are the unit vector fields in the directions of coordinates x_1, x_2, x_3. Due to formulas (9), vector field $\mathbf{A}(\mathbf{x},t)$ has the following form:

$$\begin{aligned}\mathbf{A}(\mathbf{x},t) &= \tilde{A}_1\hat{\mathbf{e}}_1 + \tilde{A}_2\hat{\mathbf{e}}_2 + \tilde{A}_3\hat{\mathbf{e}}_3 \\ &= \frac{A_1x_1 - A_2x_2}{r}\hat{\mathbf{e}}_1 + \frac{A_1x_2 + A_2x_1}{r}\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3.\end{aligned}$$

Using this formula, we find after differentiation and cancellation of similar terms the following:

$$\begin{aligned}\sum_{i=1}^{3}\tilde{A}_i(\mathbf{x})\frac{\partial\hat{\mathbf{e}}_\varphi}{\partial x_i} &= \left[\frac{A_1x_1 - A_2x_2}{r}\frac{\partial}{\partial x_1} + \frac{A_1x_2 + A_2x_1}{r}\frac{\partial}{\partial x_2}\right] \\ &\quad\times\left(-\frac{x_2}{r}\hat{\mathbf{e}}_1 + \frac{x_1}{r}\hat{\mathbf{e}}_2\right) \\ &= -\frac{A_2}{r^2}(x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2) = -\frac{A_2}{r}\hat{\mathbf{e}}_r = -\frac{(\mathbf{A}(\mathbf{x},t)\cdot\hat{\mathbf{e}}_\varphi)}{r}\hat{\mathbf{e}}_r,\end{aligned}$$

which is equivalent to the identity (8). □

As known, Euler's equations (3) and (4) for the barotropic compressible gas flows imply that the vector field

$$\mathbf{\Omega}(\mathbf{x},t) = \frac{\nabla\times\mathbf{V}(\mathbf{x},t)}{\rho(\mathbf{x},t)} \tag{10}$$

satisfies the following equation:[7–9]

$$\frac{\partial\mathbf{\Omega}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{\Omega} = (\mathbf{\Omega}\cdot\nabla)\mathbf{V}. \tag{11}$$

Let us introduce the function

$$\mathcal{H}(\mathbf{x},t) = r^{-1}\mathbf{\Omega}(\mathbf{x},t)\cdot\hat{\mathbf{e}}_\varphi = \frac{[\nabla\times\mathbf{V}(\mathbf{x},t)]\cdot\hat{\mathbf{e}}_\varphi}{r\rho(\mathbf{x},t)} = \frac{u_z - v_r}{r\rho(\mathbf{x},t)}. \tag{12}$$

Material derivative D/DT of function $\mathcal{H}(\mathbf{x},t)$ has the following form:

$$\frac{\mathrm{D}\mathcal{H}(\mathbf{x},t)}{\mathrm{D}t} = \frac{1}{r^4\rho(\mathbf{x},t)}\frac{\partial[(rw)^2]}{\partial z}. \tag{13}$$

Proof. The operator of material derivative $\mathrm{D}/\mathrm{D}t$ is defined as

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla. \tag{14}$$

From Eqs. (1), (11), and (14), we get

$$\frac{\mathrm{D}\mathbf{\Omega}}{\mathrm{D}t} = (\mathbf{\Omega}\cdot\nabla)\mathbf{V}, \qquad \frac{\mathrm{D}r^{-1}}{\mathrm{D}t} = -\frac{u}{r^2}. \tag{15}$$

Applying formula (8) to vector field $\mathbf{A}(\mathbf{x}, t) = \mathbf{V}(\mathbf{x}, t)$ (1), we find

$$\frac{\mathrm{D}\hat{\mathbf{e}}_\varphi}{\mathrm{D}t} = \left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right)\hat{\mathbf{e}}_\varphi = -\frac{(\mathbf{V}(\mathbf{x},t)\cdot\hat{\mathbf{e}}_\varphi)}{r}\hat{\mathbf{e}}_r = -\frac{w}{r}\hat{\mathbf{e}}_r. \tag{16}$$

Using the Leibnitz formula and Eqs. (15) and (16), we derive

$$\frac{\mathrm{D}\mathcal{H}}{\mathrm{D}t} = \frac{\mathrm{D}[r^{-1}\mathbf{\Omega}\cdot\hat{\mathbf{e}}_\varphi]}{\mathrm{D}t} = -\frac{u(\mathbf{\Omega}\cdot\hat{\mathbf{e}}_\varphi)}{r^2} + \frac{1}{r}[(\mathbf{\Omega}\cdot\nabla)\mathbf{V}]\cdot\hat{\mathbf{e}}_\varphi + \frac{1}{r}\mathbf{\Omega}\cdot\left(-\frac{w}{r}\hat{\mathbf{e}}_r\right). \tag{17}$$

The second term here has the equivalent form

$$\frac{1}{r}[(\mathbf{\Omega}\cdot\nabla)\mathbf{V}]\cdot\hat{\mathbf{e}}_\varphi = \frac{1}{r}(\mathbf{\Omega}\cdot\nabla)[\mathbf{V}\cdot\hat{\mathbf{e}}_\varphi] - \frac{1}{r}\mathbf{V}\cdot[(\mathbf{\Omega}\cdot\nabla)\hat{\mathbf{e}}_\varphi]. \tag{18}$$

Applying formula (8) to vector field $\mathbf{A}(\mathbf{x}, t) = \mathbf{\Omega}(\mathbf{x}, t)$ (10), we get

$$(\mathbf{\Omega}\cdot\nabla)\hat{\mathbf{e}}_\varphi = -\frac{(\mathbf{\Omega}\cdot\hat{\mathbf{e}}_\varphi)}{r}\hat{\mathbf{e}}_r. \tag{19}$$

Inserting formula (19) into Eq. (18) and using formula (1), we find

$$\frac{1}{r}[(\mathbf{\Omega}\cdot\nabla)\mathbf{V}]\cdot\hat{\mathbf{e}}_\varphi = \frac{1}{r}(\mathbf{\Omega}\cdot\nabla)w + \frac{u}{r^2}(\mathbf{\Omega}\cdot\hat{\mathbf{e}}_\varphi). \tag{20}$$

Substituting expression (20) into Eq. (17) and canceling similar terms, we get

$$\frac{\mathrm{D}\mathcal{H}(\mathbf{x},t)}{\mathrm{D}t} = -\frac{u}{r^2}(\boldsymbol{\Omega}\cdot\hat{\mathbf{e}}_\varphi) + \frac{1}{r}(\boldsymbol{\Omega}\cdot\nabla)w + \frac{u}{r^2}(\boldsymbol{\Omega}\cdot\hat{\mathbf{e}}_\varphi) - \frac{w}{r^2}(\boldsymbol{\Omega}\cdot\hat{\mathbf{e}}_r)$$
$$= \frac{1}{r}(\boldsymbol{\Omega}\cdot\nabla)w - \frac{w}{r^2}(\boldsymbol{\Omega}\cdot\hat{\mathbf{e}}_r).$$

Using Eqs. (10) and (7) here, we derive

$$\frac{\mathrm{D}\mathcal{H}(\mathbf{x},t)}{\mathrm{D}t} = \frac{1}{r^2\rho(\mathbf{x},t)}[-(rw)_z w_r + (rw)_r w_z] - \frac{w}{r^3\rho(\mathbf{x},t)}(-rw)_z$$
$$= \frac{2rw(rw)_z}{r^4\rho(\mathbf{x},t)} = \frac{1}{r^4\rho(\mathbf{x},t)}\frac{\partial\left[(rw)^2\right]}{\partial z}.$$

Thus, we obtain Eq. (13) for the axisymmetric barotropic gas flows. □

For arbitrary axisymmetric flows of barotropic gas with initially vanishing swirl [$w(\mathbf{x},0)=0$], *the function*

$$\mathcal{H}(\mathbf{x},t) = r^{-1}\boldsymbol{\Omega}(\mathbf{x},t)\cdot\hat{\mathbf{e}}_\varphi = \frac{u_z - v_r}{r\rho(\mathbf{x},t)} \tag{21}$$

is a material conservation law. For arbitrary differentiable functions $G(x)$, *the functions* $G(\mathcal{H}(\mathbf{x},t))$ *also are material conservation laws.*

Proof. We proved in Ref. 3 that for any z-axisymmetric gas flows, the function $M(\mathbf{x},t) = (\mathbf{x}\times\mathbf{V}(\mathbf{x},t))\cdot\hat{\mathbf{e}}_z = rw(r,z,t)$ (2) is a material conservation law that means it satisfies equation $DM(\mathbf{x},t)/Dt = 0$ and, hence, is constant along any gas streaklines. Formula (13) takes the following form:

$$\frac{\mathrm{D}\mathcal{H}(\mathbf{x},t)}{\mathrm{D}t} = \frac{1}{r^4\rho(\mathbf{x},t)}\frac{\partial\left[M^2(\mathbf{x},t)\right]}{\partial z}. \tag{22}$$

For the gas flows with initially vanishing swirl $w(\mathbf{x},0)=0$ [that means for the flows with $M(\mathbf{x},0) = rw(\mathbf{x},0) = 0$], the material conservation law (2) implies that for all moments of time t, one has $M(\mathbf{x},t) = rw(\mathbf{x},t) = 0$. Hence, Eq. (22) yields

$$\frac{\mathrm{D}\mathcal{H}(\mathbf{x},t)}{\mathrm{D}t} = 0. \tag{23}$$

Therefore, function $\mathcal{H}(\mathbf{x},t)$ [(12) and (21)] is a material conservation law for the z-axisymmetric barotropic gas flows with initially vanishing swirl $w(\mathbf{x},0)=0$. For any composed function $G(\mathcal{H}(\mathbf{x},t))$, Eq. (23) implies that

$$\frac{\mathrm{D}G(\mathcal{H}(\mathbf{x},t))}{\mathrm{D}t} = \frac{\mathrm{d}G(\mathcal{H})}{\mathrm{d}\mathcal{H}}\frac{\mathrm{D}\mathcal{H}(\mathbf{x},t)}{\mathrm{D}t} = 0. \tag{24}$$

Hence, all functions $G(\mathcal{H}(\mathbf{x},t))$ are material conservation laws. □

Remark 1. Equations (23) and (24) are true also for the barotropic gas flows for which the material conservation law $M(\mathbf{x},t)$ is independent of z: $\partial M(\mathbf{x},t)/\partial z = 0$. For example, this is true for the rotationally and z-translationally symmetric gas dynamics.

Remark 2. Equation (22) demonstrates that for arbitrary z-axisymmetric gas flows with $\partial M(\mathbf{x},t)/\partial z \neq 0$ function, $\mathcal{H}(\mathbf{x},t)$ (12) is not a material conservation law.

Remark 3. Equation (23) implies that the diffeomorphisms defined by the axisymmetric pure poloidal gas flows preserve the values of function $\mathcal{H}(\mathbf{x},t)$ (12) [because for such flows, $M(\mathbf{x},t)\equiv 0$]. Hence, all associated geometric objects such as the range of function $\mathcal{H}(\mathbf{x},t)$, its level sets $\mathcal{H}(\mathbf{x},t) = \mu$ for any constant μ, its points of local maxima, local minima, and saddles together with the values of function $\mathcal{H}(\mathbf{x},t)$ at them are frozen into the axisymmetric gas flows.

Remark 4. The known data[1,2] about the atomic and thermonuclear explosions under water indicate that the dynamics of water after the explosion is approximately z-axisymmetric with initially vanishing swirl. Equation (2) and equation $DM(r,z,t)/Dt = 0$[3] are also applicable for the z-axisymmetric incompressible fluid flows. Therefore, as above, we get that if $w(r,z,t_0) = 0$ at an initial moment of time t_0, then the swirl $w(r,z,t)$ vanishes at any time t. As known,[5,7,8] for an ideal incompressible fluid with constant density ρ, the vorticity vector field $\nabla\times\mathbf{V}(\mathbf{x},t)$ satisfies the equation analogous to (11) with $\boldsymbol{\Omega}(\mathbf{x},t) = \nabla\times\mathbf{V}(\mathbf{x},t)$. Therefore, putting in the above proof $\rho(\mathbf{x},t) = 1$, we get that for the incompressible fluid flows without swirl the function,

$$\mathcal{H}(\mathbf{x},t) = \frac{\nabla\times\mathbf{V}(\mathbf{x},t)}{r}\cdot\hat{\mathbf{e}}_\varphi = \frac{u_z - v_r}{r} \tag{25}$$

satisfies Eq. (23). Hence, function (25) is a material conservation law for the ideal incompressible fluid flows with $\rho = \text{const}$ and with the initially vanishing swirl.

Equation (13) is also valid for the ideal incompressible fluid with $\rho(\mathbf{x},t) = 1$. Therefore, for the general axisymmetric flows with non-vanishing swirl, we have $\mathrm{D}\mathcal{H}(\mathbf{x},t)/\mathrm{D}t \neq 0$.

III. INVARIANT FUNCTIONS $\Psi(\mu)$ AND $\Psi_G(\mu)$ FOR THE GAS DYNAMICS INSIDE THE MUSHROOM CLOUDS

The level sets in $\mathbb{R}^3$ of the z-axisymmetric function $|\mathcal{H}(\mathbf{x},t)|$ are the surfaces,

$$K^2_\mu(t):\qquad |\mathcal{H}(\mathbf{x},t)| = \mu = \text{const}, \tag{26}$$

that also are z-axisymmetric. The surface $K^2_\mu(t)$ can have several disjoint connected components. If a connected component of a level set $K^2_\mu(t)$ is a smooth manifold, then due to its axial symmetry, it is topologically either a torus T^2 or a sphere S^2 or a circle S^1.

Since function $|\mathcal{H}(\mathbf{x},t)|$ is non-negative, it has a minimum value inside the mushroom cloud, which we denote as μ_1.

Let $\mathcal{O}^3_\mu(t)$ be the domain inside the mushroom cloud defined by the following conditions:

$$\mathcal{O}^3_\mu(t):\qquad \mu_1 \leq |\mathcal{H}(\mathbf{x},t)| \leq \mu, \qquad \mu > \mu_1. \tag{27}$$

The boundary of domain $\mathcal{O}^3_\mu(t)$ is the surface $K^2_\mu(t)$ (26). The conservation of the function $\mathcal{H}(\mathbf{x},t)$ along the compressible gas streaklines implies that the surface $K^2_\mu(t_1)$ of a constant level of function $|\mathcal{H}(\mathbf{x},t_1)| = \mu$ is transported by the gas flow diffeomorphisms at any time $t > t_1$ into the surface $K^2_\mu(t)$ (26) defined by the same constant value μ of function $|\mathcal{H}(\mathbf{x},t)|$. Therefore, the domain $\mathcal{O}^3_\mu(t_1)$ (27) is transported along the gas streaklines at any time $t > t_1$ into the domain $\mathcal{O}^3_\mu(t)$. Hence, the domains $\mathcal{O}^3_\mu(t)$ and their boundary surface $K^2_\mu(t)$ are frozen into the flow, in the standard terminology. Hence, the total mass of gas inside the domain $\mathcal{O}^3_\mu(t)$,

$$\int_{\mathcal{O}^3_\mu(t)} \rho(\mathbf{x},t)\mathrm{d}x, \tag{28}$$

is conserved in time t. Here, $\mathrm{d}x$ is the element of volume in $\mathbb{R}^3$. Conservation of mass (28) is true inside any domain $\mathcal{D}(t)$ that is frozen into the gas flow; see Refs. 7 and 8. We study below its dependence on the parameter μ for the concrete domains $\mathcal{O}^3_\mu(t)$.

Any z-axisymmetric flow of inviscid compressible gas in the mushroom cloud has an invariant function,

$$\Psi(\mu) = \int_{\mathcal{O}^3_\mu(t)} \rho(\mathbf{x},t)\mathrm{d}x, \tag{29}$$

that is the total mass of gas frozen into the gas flow domain $\mathcal{O}^3_\mu(t)$. The function $\Psi(\mu)$ is continuous and piece-wise differentiable with respect to the independent variable μ. Function $\Psi(\mu)$ is monotonously increasing with μ, and its derivative $\mathrm{d}\Psi(\mu)/\mathrm{d}\mu > 0$, $\Psi(\mu_1) = 0$.

Proof. Equations (4) and (5) yield that the material derivative of the gas density $\rho(\mathbf{x},t)$ is

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial\rho}{\partial t} + \mathbf{V}\cdot\nabla\rho = -\rho(\nabla\cdot\mathbf{V}). \tag{30}$$

Differentiating the integral (28) with respect to time t and using the frozenness of the domain $\mathcal{O}^3_\mu(t)$ into the gas flow and the material derivative (30) and equation $\mathrm{D}(\mathrm{d}x)/\mathrm{D}t = (\nabla\cdot\mathbf{V})\mathrm{d}x$,[10] we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}^3_\mu(t)} \rho(\mathbf{x},t)\mathrm{d}x = \int_{\mathcal{O}^3_\mu(t)}\left[\frac{\mathrm{D}\rho}{\mathrm{D}t}\mathrm{d}x + \rho\frac{\mathrm{D}(\mathrm{d}x)}{\mathrm{D}t}\right]$$
$$= \int_{\mathcal{O}^3_\mu(t)}[-\rho(\nabla\cdot\mathbf{V}) + \rho(\nabla\cdot\mathbf{V})]\mathrm{d}x = 0.$$

Therefore, the integral (28) is conserved in time t and, hence, depends only on the variable μ. We denote it as a function $\Psi(\mu)$ (29).

To prove the differentiability of function $\Psi(\mu)$ (29), we assume that $\Delta\mu$ is a sufficiently small number $|\Delta\mu| \ll 1$. The surfaces $K^2_{\mu+\Delta\mu}(t)$ and $K^2_\mu(t)$ have zero intersection and bound a domain $\mathcal{O}^3_{\mu.\Delta\mu}(t)$ between them. The distance $|\tilde{\mathbf{x}}(\mathbf{x}) - \mathbf{x}|$ between a point $\mathbf{x} \in K^2_\mu(t)$ and the point closest to it $\tilde{\mathbf{x}}(\mathbf{x}) \in K^2_{\mu+\Delta\mu}(t)$ satisfies the following equation:

$$\Delta\mu \approx |\nabla\mathcal{H}(\mathbf{x},t||\tilde{\mathbf{x}}(\mathbf{x}) - \mathbf{x}|. \tag{31}$$

The continuity of functions $\rho(\mathbf{x},t)$ and $\mathcal{H}(\mathbf{x},t)$ implies that the difference $\Psi(\mu+\Delta\mu) - \Psi(\mu)$ is equal to the integral

$$\int_{\mathcal{O}^3_{\mu.\Delta\mu}(t)} \rho(\mathbf{x},t)\mathrm{d}x \approx \int_{K^2_\mu(t)} \rho(\mathbf{x},t)|\tilde{\mathbf{x}}(\mathbf{x}) - \mathbf{x}|\mathrm{d}s, \tag{32}$$

where $\mathrm{d}s$ is the element of area on the surface $K^2_\mu(t)$. Substituting the expression of $|\tilde{\mathbf{x}}(\mathbf{x}) - \mathbf{x}|$ from Eq. (31) into (32), we get

$$\Psi(\mu+\Delta\mu) - \Psi(\mu) \approx \Delta\mu\int_{K^2_\mu(t)} \frac{\rho(\mathbf{x},t)}{|\nabla\mathcal{H}(\mathbf{x},t)|}\mathrm{d}s. \tag{33}$$

Equation (33) implies that function $\Psi(\mu)$ is continuous. In the limit $\Delta\mu \to 0$, we find from (33) the following:

$$\lim_{\Delta\mu\to 0}\frac{\Psi(\mu+\Delta\mu) - \Psi(\mu)}{\Delta\mu} = \frac{\mathrm{d}\Psi(\mu)}{\mathrm{d}\mu} = \int_{K^2_\mu(t)} \frac{\rho(\mathbf{x},t)}{|\nabla\mathcal{H}(\mathbf{x},t)|}\mathrm{d}s, \tag{34}$$

provided that the denominator $|\nabla\mathcal{H}(\mathbf{x},t)|$ in the integral (34) is non-zero. Evidently, it is zero only if $\nabla\mathcal{H}(\mathbf{x},t) = 0$. Hence, the function $\Psi(\mu)$ is differentiable at all μ for which the level set $K^2_\mu(t)$ does not contain critical points of function $\mathcal{H}(\mathbf{x},t)$.

Equation (34) yields $\mathrm{d}\Psi(\mu)/\mathrm{d}\mu > 0$. Hence, function $\Psi(\mu)$ is monotonously increasing. The definitions (27) and (29) of domain $\mathcal{O}^3_\mu$ and function $\Psi(\mu)$ imply that $\Psi(\mu_1) = 0$. □

Any z-axisymmetric inviscid gas flow has the invariants

$$\Psi_G(\mu) = \int_{\mathcal{O}^3_\mu(t)} \rho(\mathbf{x},t)G(|\mathcal{H}(\mathbf{x},t)|)\mathrm{d}x, \tag{35}$$

which, for any differentiable function $G(x)$, are continuous and piece-wise differentiable functions of the independent variable μ.

Proof. Differentiating the integral $\Psi_G(\mu)$ (35) with respect to time t, we get

$$\frac{\mathrm{d}\Psi_G(\mu)}{\mathrm{d}t} = \int_{\mathcal{O}^3_\mu(t)}\left[\left(\frac{\mathrm{D}\rho}{\mathrm{D}t}\mathrm{d}x + \rho\frac{\mathrm{D}(\mathrm{d}x)}{\mathrm{D}t}\right)G(|\mathcal{H}(\mathbf{x},t)|) + \rho(\mathbf{x},t)\right.$$
$$\left.\times\frac{\mathrm{D}G(|\mathcal{H}(\mathbf{x},t)|)}{\mathrm{D}t}\right]\mathrm{d}x.$$

Substituting here Eqs. (24) and (30) and equation $\mathrm{D}(\mathrm{d}x)/\mathrm{D}t = (\nabla\cdot\mathbf{V})\mathrm{d}x$,[10] we find that

$$\frac{\mathrm{d}\Psi_G(\mu)}{\mathrm{d}t} = 0.$$

Therefore, functions $\Psi_G(\mu)$ (35) are invariants of the gas flow. □

To prove the differentiability of functions $\Psi_G(\mu)$, assume as above that $\Delta\mu$ is a small number $|\Delta\mu| \ll 1$. The continuity of functions $\rho(\mathbf{x},t)$ and $G(\mathcal{H}(\mathbf{x},t))$ implies that the difference $\Psi_G(\mu+\Delta\mu) - \Psi_G(\mu)$ is equal to the integral

$$\int_{\mathcal{O}^3_{\mu.\Delta\mu}(t)} \rho(\mathbf{x},t)G(\mathcal{H}(\mathbf{x},t))\mathrm{d}x \approx \int_{K^2_\mu(t)} \rho(\mathbf{x},t)G(|\mathcal{H}(\mathbf{x},t)|)|\tilde{\mathbf{x}}(\mathbf{x})-\mathbf{x}|\mathrm{d}s, \tag{36}$$

where $\mathrm{d}s$ is the element of area on the surface $K^2_\mu(t)$. Substituting the expression of $|\tilde{\mathbf{x}}(\mathbf{x}) - \mathbf{x}|$ from formula (31) into Eq. (36), we get

$$\Psi_G(\mu+\Delta\mu) - \Psi_G(\mu) \approx \Delta\mu\int_{K^2_\mu(t)} \frac{\rho(\mathbf{x},t)G(|\mathcal{H}(\mathbf{x},t)|)}{|\nabla\mathcal{H}(\mathbf{x},t)|}\mathrm{d}s. \tag{37}$$

Equation (37) implies that function $\Psi_G(\mu)$ is continuous. In the limit $\Delta\mu \to 0$, we find from (37) the following:

$$\lim_{\Delta\mu\to 0}\frac{\Psi_G(\mu+\Delta\mu) - \Psi_G(\mu)}{\Delta\mu} = \frac{\mathrm{d}\Psi_G(\mu)}{\mathrm{d}\mu} = \int_{K^2_\mu(t)} \frac{\rho(\mathbf{x},t)G(|\mathcal{H}(\mathbf{x},t)|)}{|\nabla\mathcal{H}(\mathbf{x},t)|}\mathrm{d}s, \tag{38}$$

provided that the denominator in the integral (38) is non-zero. As above, this is true for all μ for which the level set $K^2_\mu(t)$ does not contain critical points of function $\mathcal{H}(\mathbf{x},t)$.

Functions $\Psi_G(\mu)$ (35) are connected with function $\Psi(\mu)$ (29) by the following formula:

$$\Psi_G(\mu) = \int_{\mu_1}^{\mu} G(\xi)\frac{d\Psi(\xi)}{d\xi}d\xi. \quad (39)$$

Proof. On the surface $K_\mu^2(t)$, function $|\mathcal{H}(\mathbf{x},t)|$ has constant value $|\mathcal{H}(\mathbf{x},t)| = \mu$. Hence, on the surface $K_\mu^2(t)$, we have $G(|\mathcal{H}(\mathbf{x},t)|) = G(\mu)$. Inserting this into Eq. (38), we find that

$$\frac{d\Psi_G(\mu)}{d\mu} = G(\mu)\int_{K_\mu^2(t)} \frac{\rho(\mathbf{x},t)}{|\nabla\mathcal{H}(\mathbf{x},t)|}ds.$$

Substituting here formula (34), we arrive at the following differential equation:

$$\frac{d\Psi_G(\mu)}{d\mu} = G(\mu)\frac{d\Psi(\mu)}{d\mu}. \quad (40)$$

Integrating Eq. (40) and using the evident equality $\Psi_G(\mu_1) = 0$, we get formula (39). □

Example 1. For the function $G(|\mathcal{H}|) = |\mathcal{H}|$, Eq. (35) yields

$$\Psi_{|\mathcal{H}|}(\mu) = \int_{\mathcal{O}_\mu^3(t)} \rho(\mathbf{x},t)|\mathcal{H}(\mathbf{x},t)|dx. \quad (41)$$

Equation (39) takes the following form:

$$\Psi_{|\mathcal{H}|}(\mu) = \int_{\mu_1}^{\mu} \xi\frac{d\Psi(\xi)}{d\xi}d\xi = \mu\Psi(\mu) - \int_{\mu_1}^{\mu}\Psi(\xi)d\xi,$$

where we used the equality $\Psi(\mu_1) = 0$.

The functions $\Psi(\mu)$ (29) and $\Psi_G(\mu)$ [(35) and (41)] are new invariants of the z-axisymmetric inviscid gas flows inside the mushroom clouds.

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

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