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Exact force-free plasma equilibria with axial and with translational symmetries

https://doi.org/10.1515/zna-2020-0103
Received April 12, 2020; accepted June 12, 2020; published online August 06, 2020

Abstract: Exact force-free plasma equilibria satisfying the nonlinear Beltrami equation are derived. The construction is based on a nonlinear transformation that allows to get from any solution to the linear Beltrami equation a one-parametric family of exact solutions to the nonlinear one. Exact force-free plasma equilibria connected with the Sine-Gordon equation are presented.

Keywords: force-free plasma equilibria; Grad–Shafranov equation; non-linear Beltrami equation; Sine–Gordon equation.

1 Introduction

Equations of the ideal plasma equilibria have the form [1–3]:

$$(\nabla\times\mathbf{B}(\mathbf{x}))\times\mathbf{B}(\mathbf{x})=\mu\nabla P(\mathbf{x}),\ \ \nabla\cdot\mathbf{B}(\mathbf{x})=0. \qquad (1.1)$$

Magnetic field $\mathbf{B}(\mathbf{x})$ for a force-free plasma equilibrium with pressure $P(\mathbf{x})$=const satisfies the Beltrami equation

$$\nabla\times\mathbf{B}(\mathbf{x})=\alpha(\mathbf{x})\mathbf{B}(\mathbf{x}), \qquad (1.2)$$

where $\nabla\times\mathbf{B}(\mathbf{x})=\mathbf{J}(\mathbf{x})$ is the electric current and $\alpha(\mathbf{x})$ is a differentiable function of the coordinate vector $\mathbf{x}$. Equation (1.2) with $\alpha(\mathbf{x})\neq$ const is nonlinear because function $\alpha(\mathbf{x})$ depends on the vector field $\mathbf{B}(\mathbf{x})$. Indeed, the well-known equation $\nabla\alpha(\mathbf{x})\cdot\mathbf{B}(\mathbf{x})=0$ follows from Equation (1.2) and means that function $\alpha(\mathbf{x})$ is constant along the magnetic field $\mathbf{B}(\mathbf{x})$ lines, see Chapter 1 of monograph [3]. Hence for a general case function $\alpha(\mathbf{x})$ is constant on magnetic surfaces $\psi(\mathbf{x})$ = const ($\psi(\mathbf{x})$ is the magnetic function).

Another relation connecting $\alpha(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ is

$$\nabla\cdot[\nabla\alpha(\mathbf{x})\times\mathbf{B}(\mathbf{x})]=0. \qquad (1.3)$$

Equation (1.3) means that vector field $\nabla\alpha(\mathbf{x})\times\mathbf{B}(\mathbf{x})$ (that is also tangent to the magnetic surfaces $\alpha(\mathbf{x})=C=$ const) is divergence free. Equation (1.3) follows from Equations (1.2), $\nabla\cdot\mathbf{B}(\mathbf{x})=0$ and identity

$$\nabla\cdot(\mathbf{X}\times\mathbf{Y})=(\nabla\times\mathbf{X})\cdot\mathbf{Y}-\mathbf{X}\cdot(\nabla\times\mathbf{Y}),$$

where $\mathbf{X}$ and $\mathbf{Y}$ are arbitrary smooth vector fields.

There are well-known exact solutions to Equation (1.2) with $\alpha(\mathbf{x})=\alpha$=const, for example the spheromak magnetic field [1, 2]. The general solutions to the linear Beltrami equation with $\alpha(\mathbf{x})$=const were presented in terms of Bessel and Legendre functions in papers [1, 2, 4] and later analysed in detail in Refs. [3, 5–8]. We proved in Refs. [9–11] that any solution to the linear Beltrami Equation (1.2) with $\alpha(\mathbf{x})=\alpha=$ const has the form

$$\mathbf{B}(\mathbf{x})=\int_{S^2}[\sin(\alpha\mathbf{k}\cdot\mathbf{x})\mathbf{T}(\mathbf{k})+\cos(\alpha\mathbf{k}\cdot\mathbf{x})\mathbf{k}\times\mathbf{T}(\mathbf{k})]d\sigma, \qquad (1.4)$$

where $\mathbf{T}(\mathbf{k})$ is an arbitrary smooth vector field tangent to the unit sphere S^2: $\mathbf{k}\cdot\mathbf{k}=1$ and $d\sigma$ is an arbitrary measure on S^2. We presented in papers [9–11] the unsteady generalizations of exact solutions Equation (1.4) as exact solutions to the Navier–Stokes equations and to the viscous magnetohydrodynamics equations.

The term "spheromak" was first introduced in Ref. [12], see review [13]. Moduli spaces of vortex knots for the spheromak Beltrami field in different invariant domains were presented in Ref. [14], and for another Beltrami field in Ref. [15]. The vortex knots for non-Beltrami fluid flows were studied in Refs. [16–18].

In section 2 we introduce a transformations T_β of the axially symmetric plasma equilibria and show that transformations T_β satisfy equation $T_\gamma\cdot T_\beta=T_{\beta+\gamma}$ and hence for $\beta\geq0$ form a Lie semi-group. In section 3 we construct using transformations T_β an abundance of exact force-free plasma equilibria satisfying the nonlinear Beltrami Equation (1.2) with $\alpha(\mathbf{x})\neq$const.

For all previously known force-free plasma equilibria (Equation 1.2) with $\alpha(\mathbf{x})=\alpha=$ const the relation $\mathbf{J}(\mathbf{x})=\alpha\mathbf{B}(\mathbf{x})$ holds. Therefore the magnetic field $\mathbf{B}(\mathbf{x})$ and electric current $\mathbf{J}(\mathbf{x})$ vanish at the same isolated points. For the equilibria constructed in this paper magnetic field $\mathbf{B}(\mathbf{x})$ also vanishes at some isolated points $\mathbf{x}$ but the electric field $\mathbf{J}(\mathbf{x})$ vanishes additionally on the entire magnetic surfaces defined by equation $\overline{\psi}(\mathbf{x})=0$ where $\overline{\psi}(\mathbf{x})$ is the magnetic

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function. This follows from the fact that the new force-free plasma equilibria satisfy equation

$$\mathbf{J}(\mathbf{x}) = \frac{\pm\alpha^2\overline{\psi}(\mathbf{x})}{\sqrt{\beta + \alpha^2\overline{\psi}^2(\mathbf{x})}}\mathbf{B}(\mathbf{x}), \tag{1.5}$$

where α and $\beta>0$ are arbitrary constants. Equation (1.5) yields that for the constructed equilibria electric current $\mathbf{J}(\mathbf{x})$ changes its direction to the opposite when point $\mathbf{x}$ crosses the magnetic surface $\overline{\psi}(\mathbf{x}) = 0$. We present in section 3 two examples of axially symmetric force-free plasma equilibria where magnetic surface $\overline{\psi}(\mathbf{x}) = 0$ has infinitely many components in $\mathbb{R}^3$ and therefore the switching of direction of electric current $\mathbf{J}(\mathbf{x})$ occurs on infinitely many surfaces.

Translationally invariant force-free plasma equilibria are constructed in section 4. We present equilibria based on exact solutions to the nonlinear equation

$$\nabla^2\psi(x, y) = -m^2e^{\psi(x,y)} \tag{1.6}$$

and on exact solutions to the Helmholtz equation $\nabla^2\psi(x, y) = -\alpha^2\psi(x, y)$.

Exact force-free plasma equilibria connected with the elliptic Sine–Gordon equation are constructed in section 5. The solutions are smooth and bounded in the whole space $\mathbb{R}^3$.

As known, equilibria of an ideal incompressible fluid with a constant density ρ obey the equations equivalent to Equations (1.1). Therefore the presented constructions are equally applicable to the exact axisymmetric ideal fluid equilibria and to the translationally invariant ones.

2 A transformation of the axisymmetric plasma equilibria

I. A steady z-axisymmetric magnetic field $\mathbf{B}(r, z)$ satisfying equation $\nabla\cdot\mathbf{B} = 0$ has the form

$$\mathbf{B}(r, z) = -\frac{1}{r}\frac{\partial\overline{\psi}}{\partial z}\widehat{e}_r + \frac{1}{r}\frac{\partial\overline{\psi}}{\partial r}\widehat{e}_z + \frac{1}{r}G(\overline{\psi})\widehat{e}_\varphi, \tag{2.1}$$

where $\overline{\psi}(r, z)$ is the magnetic (or flux) function and $\widehat{e}_r$, $\widehat{e}_z$, $\widehat{e}_\varphi$ are unit vector fields in the directions of the cylindrical coordinates r, z, φ. The corresponding electric current $\mathbf{J}(r, z) = \nabla\times\mathbf{B}(r, z)$ is

$$\mathbf{J}(r, z) = \frac{\mathrm{d}G(\overline{\psi})}{\mathrm{d}\overline{\psi}}\left(-\frac{1}{r}\frac{\partial\overline{\psi}}{\partial z}\widehat{e}_r + \frac{1}{r}\frac{\partial\overline{\psi}}{\partial r}\widehat{e}_z\right) - \frac{1}{r}\left(\overline{\psi}_{rr} - \frac{1}{r}\overline{\psi}_r + \overline{\psi}_{zz}\right)\widehat{e}_\varphi. \tag{2.2}$$

The magnetic field $\mathbf{B}(r, z)$ Equation (2.1) and electric current $\mathbf{J}(r, z)$ Equation (2.2) are tangent to the magnetic surfaces $\overline{\psi}(r, z) = \text{const}$.

The plasma equilibrium Equations (1.1) for the z-axisymmetric magnetic field $\mathbf{B}(r, z)$ Equation (2.1) were reduced in 1958 to the Grad–Shafranov equation [19, 20]:

$$\overline{\psi}_{rr} - \frac{1}{r}\overline{\psi}_r + \overline{\psi}_{zz} = -\mu r^2\frac{\mathrm{d}P(\overline{\psi})}{\mathrm{d}\overline{\psi}} - G(\overline{\psi})\frac{\mathrm{d}G(\overline{\psi})}{\mathrm{d}\overline{\psi}}, \tag{2.3}$$

where $P(\overline{\psi})$ is the plasma pressure.

II. Substituting into Equation (2.2) the expression of $\overline{\psi}_{rr} - \frac{1}{r}\overline{\psi}_r + \overline{\psi}_{zz}$ from the Grad–Shafranov Equation (2.3) we get:

$$\mathbf{J}(r, z) = \frac{\mathrm{d}G(\overline{\psi})}{\mathrm{d}\overline{\psi}}\left(-\frac{1}{r}\frac{\partial\overline{\psi}}{\partial z}\widehat{e}_r + \frac{1}{r}\frac{\partial\overline{\psi}}{\partial r}\widehat{e}_z + \frac{1}{r}G(\overline{\psi})\widehat{e}_\varphi\right) + \mu r\frac{\mathrm{d}P(\overline{\psi})}{\mathrm{d}\overline{\psi}}\widehat{e}_\varphi. \tag{2.4}$$

Using here Equation (2.1) we arrive at equation

$$\nabla\times\mathbf{B}(r, z) = \frac{\mathrm{d}G(\overline{\psi})}{\mathrm{d}\overline{\psi}}\mathbf{B}(r, z) + \mu r\frac{\mathrm{d}P(\overline{\psi})}{\mathrm{d}\overline{\psi}}\widehat{e}_\varphi. \tag{2.5}$$

Equation (2.5) yields that for $P(\overline{\psi}) = \text{const}$ the magnetic field $\mathbf{B}(r, z)$ satisfies the Beltrami equation

$$\nabla\times\mathbf{B}(r, z) = \frac{\mathrm{d}G(\overline{\psi})}{\mathrm{d}\overline{\psi}}\mathbf{B}(r, z) \tag{2.6}$$

that has the form of Equation (1.2) with $\alpha(\mathbf{x}) = \mathrm{d}G(\overline{\psi})/\mathrm{d}\overline{\psi}$.

III. The last term in Equation (2.3) equals to $-\frac{1}{2}\mathrm{d}G^2(\overline{\psi})/\mathrm{d}\overline{\psi}$. It is evidently unchanged after the simple nonlinear transformation

$$T_\beta: \quad G(\overline{\psi}) \to G_\beta(\overline{\psi}), \quad G_\beta(\overline{\psi}) = \pm\sqrt{\beta + G^2(\overline{\psi})}. \tag{2.7}$$

Therefore the same magnetic function $\overline{\psi}(r, z)$ satisfies also Equation (2.3) with $G_\beta(\overline{\psi})$ instead of $G(\overline{\psi})$. Substituting $G_\beta(\overline{\psi}) = \pm\sqrt{\beta + G^2(\overline{\psi})}$ into Equations (2.1) we get a new magnetic field

$$\mathbf{B}_\beta(r, z) = -\frac{1}{r}\frac{\partial\overline{\psi}}{\partial z}\widehat{e}_r + \frac{1}{r}\frac{\partial\overline{\psi}}{\partial r}\widehat{e}_z \pm \frac{1}{r}\sqrt{\beta + G^2(\overline{\psi})}\widehat{e}_\varphi \tag{2.8}$$

for that Equation (2.5) takes the form

$$\nabla\times\mathbf{B}_\beta(r, z) = \pm\frac{G(\overline{\psi})}{\sqrt{\beta + G^2(\overline{\psi})}}\frac{\mathrm{d}G(\overline{\psi})}{\mathrm{d}\overline{\psi}}\mathbf{B}_\beta(r, z) + \mu r\frac{\mathrm{d}P(\overline{\psi})}{\mathrm{d}\overline{\psi}}\widehat{e}_\varphi. \tag{2.9}$$

Equation (2.9) for $P(\overline{\psi})$ = const becomes the Beltrami equation

$$\nabla \times \mathbf{B}_\beta(\mathbf{x}) = \alpha_\beta(\mathbf{x})\mathbf{B}_\beta(\mathbf{x}), \quad (2.10)$$

where

$$\alpha_\beta(\mathbf{x}) = \frac{\mathrm{d}G_\beta(\overline{\psi})}{\mathrm{d}\overline{\psi}} = \pm\frac{G(\overline{\psi})}{\sqrt{\beta + G^2(\overline{\psi})}}\frac{\mathrm{d}G(\overline{\psi})}{\mathrm{d}\overline{\psi}} = \pm\frac{G(\overline{\psi})}{\sqrt{\beta + G^2(\overline{\psi})}}\alpha(\mathbf{x}). \quad (2.11)$$

Hence transformation T_β (Equation (27)) produces from any axisymmetric solution to the Beltrami Equation (2.6) a new solution $\mathbf{B}_\beta(r, z)$ (Equation 2.8) to the Beltrami Equation (2.10) with another function $\alpha_\beta(\mathbf{x})$ Equation (2.11), $\beta > 0$. Equation (2.11) yields

$$(\alpha_\beta(\mathbf{x}))^2 = \frac{G^2(\overline{\psi}(\mathbf{x}))}{\beta + G^2(\overline{\psi}(\mathbf{x}))}(\alpha(\mathbf{x}))^2.$$

Therefore function $\alpha_\beta(\mathbf{x})$ is changing in the range $-|\alpha(\mathbf{x})|<\alpha_\beta(\mathbf{x})<|\alpha(\mathbf{x})|$.

Remark 1: Transformations T_β Equation (2.7) with $\beta > 0$ define for any solution $\overline{\psi}(r, z)$ to the general Grad - Shafranov Equation (2.3) a one-parametric family of different magnetic fields $\mathbf{B}_\beta(r, z)$ Equation (2.1) and electric currents $\mathbf{J}_\beta(r, z)$ Equation (2.2) which correspond to different functions $G_\beta(\overline{\psi})$ but have the same magnetic function $\overline{\psi}(r, z)$. A useful application of transformations T_β (Equation (2.7)) is the construction of new exact force-free plasma equilibria satisfying the Beltrami equation $\nabla\times\mathbf{B}_\beta(\mathbf{x}) = \alpha_\beta(\mathbf{x})\mathbf{B}_\beta(\mathbf{x})$ with a non-constant function $\alpha_\beta(\mathbf{x})$, see Equations (3.4) and Equation (3.5) below.

Remark 2: Transformations T_β (Equation 2.4) with sign + satisfy the relation

$$T_\gamma(T_\beta(G)) = T_{\gamma+\beta}(G). \quad (2.12)$$

Indeed, Equation (2.7) yields

$$T_\gamma(T_\beta(G)) = \sqrt{\gamma + \left[\sqrt{\beta + G^2(\overline{\psi})}\right]^2} = \sqrt{\gamma + \beta + G^2(\overline{\psi})} = T_{\gamma+\beta}(G).$$

Hence Equation (2.12) holds. For $\beta < 0$ transforms (Equation 2.7) are defined only in the domain $G^2(\overline{\psi}(\mathbf{x}))\geq|\beta|$. For $0 \leq \beta < \infty$ transforms (Equation 2.7) are defined everywhere in $\mathbb{R}^3$. Evidently $T_0(G)$ is the identity transformation. Therefore Equation (2.12) yields that transformations T_β (Equation (2.7)) with sign + and $0 \leq \beta < \infty$ form a one-dimensional Lie semi-group.

3 Exact axisymmetric force-free plasma equilibria

I. Consider the Grad–-Shafranov Equation (2.3) with $P(\overline{\psi})$ = const and $G(\overline{\psi}) = G_\zeta(\overline{\psi}) = \pm\sqrt{\zeta + \alpha^2\overline{\psi}^2}$ with an arbitrary constant α and $\zeta \geq 0$. Evidently we have $\mathrm{d}G_\zeta(\overline{\psi})/\mathrm{d}\overline{\psi} = \alpha^2\overline{\psi}/G_\zeta(\overline{\psi})$. Therefore $G_\zeta(\overline{\psi})\mathrm{d}G_\zeta(\overline{\psi})/\mathrm{d}\overline{\psi} = \alpha^2\overline{\psi}$ and hence Equation (2.3) takes the linear form

$$\overline{\psi}_{rr} - \frac{1}{r}\overline{\psi}_r + \overline{\psi}_{zz} = -\alpha^2\overline{\psi}. \quad (3.1)$$

Let $\mathbf{B}_\zeta(r,z)$ be the corresponding magnetic field (Equation 2.1):

$$\mathbf{B}_\zeta(r, z) = -\frac{1}{r}\frac{\partial\overline{\psi}}{\partial z}\hat{e}_r + \frac{1}{r}\frac{\partial\overline{\psi}}{\partial r}\hat{e}_z + \frac{1}{r}G_\zeta(\overline{\psi})\hat{e}_\varphi. \quad (3.2)$$

Substituting the Grad–Shafranov Equation (2.3) with $P(\overline{\psi})$ = const and $G_\zeta(\overline{\psi}) = \pm\sqrt{\zeta + \alpha^2\overline{\psi}^2}$ into Equation (2.2) we get the electric current

$$\mathbf{J}_\zeta(r, z) = \nabla \times \mathbf{B}_\zeta(\mathbf{x}) = \frac{\mathrm{d}G_\zeta(\overline{\psi})}{\mathrm{d}\overline{\psi}}\left(-\frac{1}{r}\frac{\partial\overline{\psi}}{\partial z}\hat{e}_r + \frac{1}{r}\frac{\partial\overline{\psi}}{\partial r}\hat{e}_z + \frac{1}{r}G_\zeta(\overline{\psi})\hat{e}_\varphi\right). \quad (3.3)$$

Inserting Equation (3.2) into Equation (3.3) we find

$$\nabla \times \mathbf{B}_\zeta(\mathbf{x}) = \frac{\mathrm{d}G_\zeta(\overline{\psi})}{\mathrm{d}\overline{\psi}}\mathbf{B}_\zeta(r, z). \quad (3.4)$$

Equation (3.4) is the Beltrami Equation (1.2) with the non-constant function

$$\alpha(\mathbf{x}) = \alpha_\zeta(r, z) = \frac{\mathrm{d}G_\zeta(\overline{\psi})}{\mathrm{d}\overline{\psi}} = \pm\frac{\alpha^2\overline{\psi}(r, z)}{\sqrt{\zeta + \alpha^2\overline{\psi}^2(r, z)}}. \quad (3.5)$$

Therefore for any solution $\overline{\psi}(r, z)$ to the linear Equation (3.1) we constructed the force-free magnetic fields $\mathbf{B}_\zeta(r, z)$ (3.2) and electric currents $\mathbf{J}_\zeta(r, z)$ (3.3) satisfying the Beltrami Equation (1.2) with non-constant function $\alpha(\mathbf{x})$ (Equation 3.5). Only for ζ=0 function $\mathrm{d}G_0(\overline{\psi})/\mathrm{d}\overline{\psi} = \pm\alpha$ becomes constant.

For function $\alpha_\zeta(r, z)$ (Equation 3.5) we have (for $\zeta > 0$)

$$(\alpha_\zeta(r, z))^2 = \alpha^2\frac{\alpha^2\overline{\psi}^2}{\zeta + \alpha^2\overline{\psi}^2} < \alpha^2.$$

Hence function $\alpha_\zeta\ (r,z)$ (Equation 3.5) satisfies inequalities $-|\alpha|<\alpha_\zeta(r,z)<|\alpha|$ and $\alpha_\zeta\ (r,z)=0$ at the points (r,z) where $\overline{\psi}(r,z)=0$.

Example 1: The magnetic function $\overline{\psi}(r,z)$ for the spheromak plasma equilibrium [1, 2] satisfies the linear Equation (3.1) and has the form

$$\psi_2(r,z) = -r^2 G_2(\alpha R) = -\frac{r^2}{\alpha^2 R^2}\left[\cos(\alpha R) - \frac{\sin(\alpha R)}{\alpha R}\right]. \quad (3.6)$$

Here $R=\sqrt{r^2+z^2}$ is the spherical radius in $\mathbb{R}^3$. The corresponding to the solution (Equation 3.6) magnetic fields (Equation 3.2)

$$\mathbf{B}_{\zeta.2}(r,z) = -\frac{1}{r}\frac{\partial\psi_2}{\partial z}\widehat{e}_r + \frac{1}{r}\frac{\partial\psi_2}{\partial r}\widehat{e}_z \pm \frac{1}{r}\sqrt{\zeta + \alpha^2\psi_2^2}\,\widehat{e}_\varphi \quad (3.7)$$

with any $0 \le \zeta < \infty$ satisfy the nonlinear Beltrami Equation (1.2) with the non-constant functions $\alpha(r,z)$:

$$\nabla\times\mathbf{B}_{\zeta.2}(r,z) = \frac{\pm\alpha^2\psi_2(r,z)}{\sqrt{\zeta+\alpha^2\psi_2^2(r,z)}}\mathbf{B}_{\zeta.2}(r,z). \quad (3.8)$$

Equation (3.8) yields that for the new force-free plasma equilibria (Equation 3.7) the electric current

$$\mathbf{J}_{\zeta.2}(r,z) = \frac{\pm\alpha^2\psi_2(r,z)}{\sqrt{\zeta+\alpha^2\psi_2^2(r,z)}}\left(\frac{1}{r}\frac{\partial\psi_2}{\partial z}\widehat{e}_r + \frac{1}{r}\frac{\partial\psi_2}{\partial r}\widehat{e}_z\right) + \frac{1}{r}\alpha^2\psi_2(r,z)\widehat{e}_\varphi \quad (3.9)$$

vanishes on the magnetic surface $\psi_2\ (r,z)=0$. Equation (3.6) implies that the latter has infinitely many components that all are spheres $S_k^2 : R = R_k$ where R_k are the roots of equation

$$\tan(\alpha R) = \alpha R. \quad (3.10)$$

The first four numerical solutions to Equation (3.10) are

$$|\alpha|R_1 \approx 4.4934,\quad |\alpha|R_2 \approx 7.7253,\quad |\alpha|R_3 \approx 10.9041,\quad |\alpha|R_4 \approx 14.0662.$$

At $k\to\infty$ the roots R_k have asymptotics $|\alpha|R_k \approx \left(k+\frac{1}{2}\right)\pi$. The electric current $\mathbf{J}_{\zeta.2}\ (r,z)$ (Equation 3.9) switches its direction to the opposite at the infinitely many spheres S_k^2. Equation (3.9) yields that electric current $\mathbf{J}_{\zeta.2}(r,z)$ is smooth everywhere in $\mathbb{R}^3$ and has zero current density on the axis z $(r=0)$.

Function $G_2(u)$ in Equation (3.6), $G_2(u) = u^{-2}(\cos u - u^{-1}\sin u)$ where $u=\alpha R$, is connected with the Bessel function $J_{3/2}(u)$ [21] of order 3/2 by the relation

$$G_2(u) = -\frac{\sqrt{\pi/2}}{u^{3/2}}J_{3/2}(u).$$

Example 2: Equation (3.1) is evidently invariant with respect to the differentiations $(\partial/\partial z)^n$ of arbitrary order n. Hence the flux functions $\psi_{2+n}\ (r,z) = \partial^n\psi_2/\partial z^n$ also are solutions to Equation (3.1). For example we have

$$\psi_3(r,z) = \frac{\partial\psi_2(r,z)}{\partial z} = -r^2\frac{dG_2(u)}{du}\frac{du}{dz} = -\alpha^2 z r^2 G_3(u), \quad (3.11)$$

$$\psi_4(r,z) = \frac{\partial^2\psi_2(r,z)}{\partial z^2} = -\alpha^2 r^2 G_3(u) - \alpha^4 z^2 r^2 G_4(u), \quad (3.12)$$

where $u=\alpha R$ and

$$G_3(u) = \frac{1}{u}\frac{dG_2(u)}{du} = \frac{1}{u^4}\left((3-u^2)\frac{\sin u}{u} - 3\cos u\right), \quad (3.13)$$

$$G_4(u) = \frac{1}{u}\frac{dG_3(u)}{du} = \frac{1}{u^6}\left((6u^2-15)\frac{\sin u}{u} - (u^2-15)\cos u\right). \quad (3.14)$$

The functions $G_k(u)$ are analytic for all u and have the following values at $u=0$: $G_2(0)=-1/3$, $G_3(0)=1/15$, $G_4(0)=-1/105$ [15].

Magnetic field $\mathbf{B}_{\zeta.3}(r,z)$ has the form

$$\mathbf{B}_{\zeta.3}(r,z) = -\frac{1}{r}\frac{\partial\psi_3}{\partial z}\widehat{e}_r + \frac{1}{r}\frac{\partial\psi_3}{\partial r}\widehat{e}_z \pm \frac{1}{r}\sqrt{\zeta+\alpha^2\psi_3^2}\,\widehat{e}_\varphi$$

and satisfies the nonlinear Beltrami equation

$$\nabla\times\mathbf{B}_{\zeta.3}(r,z) = \frac{\pm\alpha^2\psi_3(r,z)}{\sqrt{\zeta+\alpha^2\psi_3^2(r,z)}}\mathbf{B}_{\zeta.3}(r,z).$$

The electric current

$$\mathbf{J}_{\zeta.3}(r,z) = \frac{\pm\alpha^2\psi_3(r,z)}{\sqrt{\zeta+\alpha^2\psi_3^2(r,z)}}\left(\frac{1}{r}\frac{\partial\psi_3}{\partial z}\widehat{e}_r + \frac{1}{r}\frac{\partial\psi_3}{\partial r}\widehat{e}_z\right) + \frac{1}{r}\alpha^2\psi_3(r,z)\widehat{e}_\varphi \quad (3.15)$$

vanishes on the magnetic surface $\psi_3(r,z)=0$ that according to Equation (3.11) contains the plane $z=0$ and infinitely many spheres $S_m^2 : R = R_m$ where R_m satisfy equation $G_3(\alpha R)=0$:

$$\tan(\alpha R) = \frac{3\alpha R}{3-(\alpha R)^2}\,. \tag{3.16}$$

The electric current $\mathbf{J}_{\zeta.3}\,(r,z)$ (Equation 3.15) switches its direction to the opposite at the plane z = 0 and at infinitely many spheres S_m^2. Equation (3.15) demonstrates that electric current $\mathbf{J}_{\zeta.3}\,(r,z)$ is smooth everywhere in $\mathbb{R}^3$ and has zero current density on the axis z ($r = 0$).

The first four numerical solutions to Equation (3.16) are

$$|\alpha|R_1 \approx 5.7635,\quad |\alpha|R_2 \approx 9.0950,\quad |\alpha|R_3 \approx 12.3229,$$
$$|\alpha|R_4 \approx 15.5146\,.$$

At $m\to\infty$ the roots R_m have asymptotics $|\alpha|R_k \approx (m+1)\pi$.

II. Analogous construction exists for the magnetic function ψ_4 (r,z) (Equation 3.12). The corresponding electric current $\mathbf{J}_{\zeta.4}(r,z) = \nabla\times\mathbf{B}_{\zeta.4}(r,z)$ vanishes on the magnetic surface $\psi_4\,(r,z) = 0$ that according to Equations (3.12) and (3.14) satisfies equation

$$G_3(\alpha R) + \alpha^2 z^2 G_4(\alpha R) = 0\,. \tag{3.17}$$

Equation (3.17) yields that the surface intersects the plane $z = 0$ at infinitely many circles $\mathbb{S}_m^1 : z = 0,\ R = R_m$ where R_m are roots of equation $G_3(\alpha R) = 0$ (Equation 3.16). Therefore the magnetic surface $\psi_4\,(r,z) = 0$ (Equation 3.17) has infinitely many components that are not spheres but are z-axially symmetric.

The linearity of Equation (3.1) yields that any linear combination

$$\psi_N(r,z) = a_0\psi_2(r,z) + a_1\frac{\partial\psi_2(r,z)}{\partial z} + \cdots + a_N\frac{\partial^N\psi_2(r,z)}{\partial z^N}$$

obeys Equation (3.1). Let us consider the corresponding magnetic fields (Equation 3.2):

$$\mathbf{B}_{\zeta.N}(r,z) = -\frac{1}{r}\frac{\partial\psi_N}{\partial z}\widehat{e}_r + \frac{1}{r}\frac{\partial\psi_N}{\partial r}\widehat{e}_z \pm \frac{1}{r}\sqrt{\zeta + \alpha^2\psi_N^2}\,\widehat{e}_\varphi \tag{3.18}$$

with $0 \le \zeta < \infty$. Equation (3.4) yields that the magnetic fields $\mathbf{B}_{\zeta.N}\,(r,z)$ (Equation 3.18) satisfy the nonlinear Beltrami Equation (1.2):

$$\nabla\times\mathbf{B}_{\zeta.N}(r,z) = \frac{\pm\alpha^2\psi_N(r,z)}{\sqrt{\zeta + \alpha^2\psi_N^2(r,z)}}\mathbf{B}_{\zeta.N}(r,z)\,.$$

The electric currents

$$\mathbf{J}_{\zeta.N}(r,z) = \frac{\pm\alpha^2\psi_N(r,z)}{\sqrt{\zeta + \alpha^2\psi_N^2(r,z)}}\left(-\frac{1}{r}\frac{\partial\psi_N}{\partial z}\widehat{e}_r + \frac{1}{r}\frac{\partial\psi_N}{\partial r}\widehat{e}_z\right) + \frac{1}{r}\alpha^2\psi_N(r,z)\widehat{e}_\varphi$$

vanish on magnetic surfaces $\psi_N\,(r,z) = 0$ that have infinitely many non-spherical axisymmetric components. The current density vanishes on the axis z ($r = 0$).

4 Exact translationally invariant force-free plasma equilibria

I. In the Cartesian coordinates x, y, z, the z-independent magnetic fields $\mathbf{B}(x,y)$ satisfying the equilibrium Equations (1.1) have the form

$$\mathbf{B}(x,y) = -\psi_y\widehat{e}_x + \psi_x\widehat{e}_y + G(\psi)\widehat{e}_z\,, \tag{4.1}$$

where $\psi = \psi(x,y)$ is the magnetic function, $G(\psi)$ is an arbitrary differentiable function of ψ and $\widehat{e}_x$, $\widehat{e}_y$, $\widehat{e}_z$ are unit vectors in directions of coordinates x, y, z.

The electric current $\mathbf{J} = \nabla\times\mathbf{B}$ takes the form

$$\mathbf{J} = \frac{\mathrm{d}G(\psi)}{\mathrm{d}\psi}\psi_y\widehat{e}_x - \frac{\mathrm{d}G(\psi)}{\mathrm{d}\psi}\psi_x\widehat{e}_y + (\nabla^2\psi)\widehat{e}_z\,, \tag{4.2}$$

where $\nabla^2\psi = \psi_{xx} + \psi_{yy}$. Hence we get

$$\begin{aligned}\mathbf{J}\times\mathbf{B} = &-\left[\nabla^2\psi + G(\psi)\frac{\mathrm{d}G(\psi)}{\mathrm{d}\psi}\right]\psi_x\widehat{e}_x\\ &-\left[\nabla^2\psi + G(\psi)\frac{\mathrm{d}G(\psi)}{\mathrm{d}\psi}\right]\psi_y\widehat{e}_y\,.\end{aligned} \tag{4.3}$$

Therefore the translationally invariant plasma equilibrium Equations (1.1) take the form

$$\nabla^2\psi = -\mu\frac{\mathrm{d}P(\psi)}{\mathrm{d}\psi} - G(\psi)\frac{\mathrm{d}G(\psi)}{\mathrm{d}\psi}\,, \tag{4.4}$$

where pressure $P = P(\psi)$ is an arbitrary differentiable function of ψ.

Substituting Equation (4.4) into Equation (4.2) we find

$$\begin{aligned}\mathbf{J} = \nabla\times\mathbf{B} = &\frac{\mathrm{d}G(\psi)}{\mathrm{d}\psi}\left[\psi_y\widehat{e}_x - \psi_x\widehat{e}_y - G(\psi)\widehat{e}_z\right]\\ &-\mu\frac{\mathrm{d}P(\psi)}{\mathrm{d}\psi}\widehat{e}_z\,.\end{aligned} \tag{4.5}$$

From Equations (4.1) and (4.5) we derive

$$\nabla\times\mathbf{B} = -\frac{\mathrm{d}G(\psi)}{\mathrm{d}\psi}\mathbf{B} - \mu\frac{\mathrm{d}P(\psi)}{\mathrm{d}\psi}\widehat{e}_z\,. \tag{4.6}$$

Hence for the z-independent force-free plasma equilibria with $P(\psi)$ = const magnetic field $\mathbf{B}$ satisfies Beltrami equation: $\nabla\times\mathbf{B} = \alpha(\mathbf{x})\mathbf{B}$ with function $\alpha(\mathbf{x}) = -\mathrm{d}G(\psi)/\mathrm{d}\psi$.

Remark 3: Transformations T_β (Equation 2.7) with $\beta > 0$ turn any z-independent plasma equilibria with $P(\psi)$ = const which satisfy Equation (4.6): $\nabla\times\mathbf{B} = \alpha(\mathbf{x})\mathbf{B}$ with $\alpha(\mathbf{x}) = -\mathrm{d}G(\psi)/\mathrm{d}\psi$ into another solutions $\mathbf{B}_\beta(\mathbf{x})$ to the Beltrami equation $\nabla\times\mathbf{B}_\beta = \alpha_\beta(\mathbf{x})\mathbf{B}_\beta$ where

$$\alpha_\beta(\mathbf{x}) = -\frac{\mathrm{d}G_\beta(\psi)}{\mathrm{d}\psi} = \mp\frac{G(\psi)}{\sqrt{\beta + G^2(\psi)}}\alpha(\mathbf{x}).$$

II. Consider magnetic field

$$\mathbf{B}(x, y) = -\psi_y \hat{e}_x + \psi_x \hat{e}_y + \sqrt{2} m e^{\psi/2} \hat{e}_z, \qquad (4.7)$$

with exponential function $G(\psi) = \sqrt{2} m e^{\psi/2}$. Applying transformation T_β (Equation 2.7) we get magnetic fields

$$\mathbf{B}_\beta(x, y) = -\psi_y \hat{e}_x + \psi_x \hat{e}_y \pm \sqrt{\beta + 2m^2 e^{\psi}}\hat{e}_z \qquad (4.8)$$

with function $G_\beta(\psi) = \pm\sqrt{\beta + 2m^2 e^{\psi}}$. Beltrami Equation (4.6) with $P(\psi)$ = const for the field (Equation 4.8) becomes

$$\nabla \times \mathbf{B}_\beta = \mp \frac{m^2 e^{\psi}}{\sqrt{\beta + 2m^2 e^{\psi}}} \mathbf{B}_\beta. \qquad (4.9)$$

For the both magnetic fields Equations (4.7) and (4.8) Equation (4.4) with $P(\psi)$ = const has the form

$$\nabla^2 \psi = -m^2 e^{\psi}. \qquad (4.10)$$

III. Exact solutions to the nonlinear Equation (4.10) were first derived by Vekua [22]. Vekua's method consists of the following. Let $x + iy$ be a complex variable and $f(x + iy) = u(x, y) + iv(x, y)$ be any analytic function of $x + iy$. Then the Cauchy–Riemann equations $u_x = v_y$, $u_y = -v_x$ hold. Let function $\psi(x, y)$ has the form

$$\psi(x, y) = \log\left[\frac{8}{m^2}\left(u_x^2 + u_y^2\right)\right] - 2\log\left(1 + u^2 + v^2\right) = \log\left[\frac{8|f'|^2}{m^2\left(1 + |f|^2\right)^2}\right]. \qquad (4.11)$$

Since $f' = \mathrm{d}f(x + iy)/\mathrm{d}(x + iy) = u_x + iv_x = u_x - iu_y$ we get $\log|f'|^2 = \log(u_x^2 + u_y^2)$. Since f' also is an analytic function we have $\nabla^2 \log|f'|^2 = 2\nabla^2 \log|f'| = 0$. Hence $\nabla^2 \log(u_x^2 + u_y^2) = 0$. Therefore we get

$$\begin{aligned} -\nabla^2 \psi &= 2\nabla^2 \log\left(1 + u^2 + v^2\right) \\ &= 4\frac{\partial}{\partial x}\left[\frac{uu_x + vv_x}{1 + u^2 + v^2}\right] + 4\frac{\partial}{\partial y}\left[\frac{uu_y + vv_y}{1 + u^2 + v^2}\right] \\ &= -8\frac{\left(uu_x + vv_x\right)^2 + \left(uu_y + vv_y\right)^2}{\left(1 + u^2 + v^2\right)^2} \\ &\quad + 4\frac{u\nabla^2 u + v\nabla^2 v + u_x^2 + u_y^2 + v_x^2 + v_y^2}{1 + u^2 + v^2}. \end{aligned}$$

Using here equations $\nabla^2 u = 0$, $\nabla^2 v = 0$ and the Cauchy–Riemann equations we find

$$\nabla^2 \psi = -\frac{8\left(u_x^2 + u_y^2\right)}{\left(1 + u^2 + v^2\right)^2}. \qquad (4.12)$$

Equation (4.11) yields

$$e^{\psi} = \frac{8\left(u_x^2 + u_y^2\right)}{m^2\left(1 + u^2 + v^2\right)^2}. \qquad (4.13)$$

Equations (4.12) and (4.13) imply that for arbitrary analytic functions $f(x + iy)$ functions $\psi(x,y)$ (Equation 4.11) satisfy Equation (4.10).

IV. Consider analytic function $f(x + iy) = a(x + iy)^k$ where a = const and $k \geq 1$ is an integer. Then $f' = ka(x + iy)^{k-1}$ and function (Equation 4.11) becomes

$$\begin{aligned} \psi_k(x, y) = &\log\left[\frac{8|a|^2 k^2}{m^2}\left(x^2 + y^2\right)^{k-1}\right] \\ &- 2\log\left(1 + |a|^2\left(x^2 + y^2\right)^k\right), \end{aligned} \qquad (4.14)$$

For $k \geq 2$ function $\psi(x, y)$ (Equation 4.14) has singularity (tends to $-\infty$) at $x = 0$, $y = 0$. For $k = 1$ we get

$$\psi_1(x, y) = \log\left[\frac{8|a|^2}{m^2}\right] - 2\log\left(1 + |a|^2\left(x^2 + y^2\right)\right),$$

$$e^{\psi_1} = \frac{8|a|^2}{m^2\left(1 + |a|^2\left(x^2 + y^2\right)\right)^2}.$$

Function $\psi_1(x, y)$ is smooth everywhere. The corresponding force-free magnetic field $\mathbf{B}_{\beta.1}(x, y)$ (Equation 4.8) is a generalization of the Bennett pinch solution [23, 24]. It satisfies Equation (4.9) that takes the form

$$\nabla \times \mathbf{B}_{\beta.1} = \mp \frac{8|a|^2}{\left(1 + |a|^2\left(x^2 + y^2\right)\right)\sqrt{\beta\left(1 + |a|^2\left(x^2 + y^2\right)\right) + 16|a|^2}} \mathbf{B}_{\beta.1}.$$

V. Consider analytic function $f(x+iy) = e^{\alpha(x+iy)}$ where α is real. Hence $\left|f(x + iy)\right| = e^{\alpha x}$ and function $\psi(x, y)$ (Equation 4.11) becomes

$$\psi(x, y) = \log\left[\frac{8\alpha^2}{m^2} e^{2\alpha x}\right] - 2\log\left(1 + e^{2\alpha x}\right),$$

$$e^{\psi} = \frac{8\alpha^2 e^{2\alpha x}}{m^2\left(1 + e^{2\alpha x}\right)^2}.$$

Hence Equation (4.9) for the corresponding force-free magnetic field $\mathbf{B}_{\beta.\alpha}(x, y)$ (Equation 4.8) takes the form

$$\nabla \times \mathbf{B}_{\beta.\alpha} = \mp \frac{8\alpha^2 e^{2\alpha x}}{\left(1 + e^{2\alpha x}\right)\sqrt{\beta\left(1 + e^{2\alpha x}\right)^2 + 16\alpha^2 e^{2\alpha x}}} \mathbf{B}_{\beta.\alpha}.$$

VI. Let us consider functions

$$G_\zeta(\psi) = \pm\sqrt{\zeta + \alpha^2 \psi^2}, \qquad (4.15)$$

where α and $\zeta > 0$ are arbitrary parameters. We have $G_\zeta(\psi)\,\mathrm{d}G_\zeta(\psi)/\mathrm{d}\psi = \alpha^2\psi$. Hence Equation (4.4) with $P(\psi)$ = const takes the linear form

$$\psi_{xx} + \psi_{yy} = -\alpha^2\psi. \qquad (4.16)$$

The two-dimensional Helmholtz Equation (4.16) evidently has exact solutions

$$\psi(x, y, \theta) = f(\theta)\sin(\alpha x\cos\theta + \alpha y\sin\theta), \qquad (4.17)$$

where $0 \le \theta < 2\pi$ and $f(\theta)$ is any piece-wise continuous function of angle θ.

Any finite sum of functions (Equation 4.17) is an exact solution to the linear Equation (4.16):

$$\psi_N(x, y) = \sum_{k=1}^{N} C_k \sin(\alpha x\cos\theta_k + \alpha y\sin\theta_k),$$

where C_k, θ_k are arbitrary constants. For the corresponding magnetic fields Equation (4.1), Equation (4.15) with $\zeta > 0$

$$\mathbf{B}_{\zeta.N}(x, y) = -(\psi_N)_y\hat{e}_x + (\psi_N)_x\hat{e}_y \pm \sqrt{\zeta + \alpha^2\psi_N^2}\,\hat{e}_z,$$

Equation (4.6) with $P(\psi)$ = const takes the form

$$\nabla\times\mathbf{B}_{\zeta.N}(x, y) = \mp\frac{\alpha^2\psi_N(x, y)}{\sqrt{\zeta + \alpha^2\psi_N^2(x, y)}}\mathbf{B}_{\zeta.N}(x, y). \qquad (4.18)$$

Equation (4.18) is the Beltrami Equation (1.2) with the non-constant function

$$\alpha(\mathbf{x}) = \alpha_{\zeta.N}(x, y) = \mp\frac{\alpha^2\psi_N(x, y)}{\sqrt{\zeta + \alpha^2\psi_N^2(x, y)}}.$$

VII. Integrating functions $\psi(x,y,\theta)$ (Equation 4.17) with respect to the angle θ and using the linearity of Equation (4.16) we derive the general exact solution to the Helmholtz Equation (4.16):

$$\hat{\psi}(x, y) = \int_0^{2\pi} f(\theta)\sin(\alpha x\cos\theta + \alpha y\sin\theta)\mathrm{d}\theta. \qquad (4.19)$$

The corresponding magnetic fields Equation (4.1), Equation (4.15) with $\zeta > 0$ have the form

$$\hat{B}_\zeta(x, y) = -\hat{\psi}_y\hat{e}_x + \hat{\psi}_x\hat{e}_y \pm \sqrt{\zeta + \alpha^2\hat{\psi}^2}\,\hat{e}_z. \qquad (4.20)$$

Equation (4.6) with $P(\hat{\psi})$ = const and $G(\psi) = G_\zeta(\hat{\psi}) = \pm\sqrt{\zeta + \alpha^2\hat{\psi}^2}$ with $\zeta > 0$ takes the form

$$\nabla\times\hat{B}_\zeta(x, y) = \mp\frac{\alpha^2\hat{\psi}(x, y)}{\sqrt{\zeta + \alpha^2\hat{\psi}^2(x, y)}}\hat{B}_\zeta(x, y). \qquad (4.21)$$

Equation (4.21) is the Beltrami Equation (1.2) with non-constant function

$$\alpha(\mathbf{x}) = \hat{\alpha}_\zeta(x, y) = \mp\frac{\alpha^2\hat{\psi}(x, y)}{\sqrt{\zeta + \alpha^2\hat{\psi}^2(x, y)}}.$$

Thus Equations (4.19), (4.20) provide an abundance of exact solutions to the nonlinear Beltrami Equation (1.2), (4.21). The exact solutions (Equations 4.19, 4.20) for $\zeta > 0$ are bounded for all x, y. It is evident that $-|\alpha| < \hat{\alpha}_\zeta(x, y) < |\alpha|$.

5 Plasma equilibria connected with the Sine–Gordon equation

Let us consider a trigonometric function $G(\psi) = A\sin[\alpha(\psi + \gamma)]$ where A, α and γ are arbitrary constants. We get

$$G(\psi)\frac{\mathrm{d}G(\psi)}{\mathrm{d}\psi} = \alpha A^2\sin[\alpha(\psi + \gamma)]\cos[\alpha(\psi + \gamma)] = \frac{\alpha A^2}{2}\sin[2\alpha(\psi + \gamma)].$$

Therefore Equation (4.4) with $P(\psi)$ = const takes the form

$$\psi_{xx} + \psi_{yy} = -\frac{\alpha A^2}{2}\sin[2\alpha(\psi + \gamma)], \qquad (5.1)$$

that coincides with the elliptic Sine-Gordon equation. The Beltrami Equation (4.6) (with $P(\psi)$ = const) is

$$\nabla\times\mathbf{B} = -\frac{\mathrm{d}G(\psi)}{\mathrm{d}\psi}\mathbf{B} = -\alpha A\cos[\alpha(\psi + \gamma)]\mathbf{B}. \qquad (5.2)$$

Hence function $\alpha(\mathbf{x})$ in the corresponding Beltrami Equation (1.2) is $\alpha(\mathbf{x}) = -\alpha A cos[\alpha(\psi+\gamma)]$.

To construct exact solutions to the nonlinear Equation (5.1) we consider equation of first order

$$\psi_x = A\cos[\alpha(\psi + \gamma)]. \qquad (5.3)$$

Differentiating Equation (5.3) with respect to x we get $\psi_{xx} = -\alpha A^2\sin[\alpha(\psi+\gamma)]\cos[\alpha(\psi+\gamma)] = -(\alpha A^2/2)\sin[2\alpha(\psi+\gamma)]$. Hence any solution to Equation (5.3) satisfies Equation (5.1). Integrating Equation (5.3) we find its exact solutions

$$\psi_1(x) = \frac{1}{\alpha}\arcsin\{\tanh[\alpha A(x + c)]\} - \gamma, \qquad (5.4)$$

that satisfy also Equation (5.1). Solutions (Equation 5.4) lead (after rotation of variables x, y for an angle θ) to the more general solutions to Equation (5.1):

$$\psi_\lambda(x, y) = \frac{1}{\alpha}\arcsin\{\tanh[\alpha A(v + c)]\} - \gamma, \quad v = \lambda x + \sqrt{1-\lambda^2}\,y, \qquad (5.5)$$

where $\lambda = \cos\theta$, $\sqrt{1-\lambda^2} = \sin\theta$. For functions $\psi_\lambda(x, y)$

(Equation 5.5) Equation (5.2) with function $G(\psi_\lambda) = A \sin[\alpha(\psi_\lambda+\gamma)]$ takes the form

$$\nabla \times \mathbf{B} = -\alpha A \cos[\alpha(\psi_\lambda + \gamma)]\mathbf{B} = -\frac{\alpha A}{\cosh\left[\alpha A\left(\lambda x + \sqrt{1-\lambda^2}\,y + c\right)\right]}\mathbf{B}. \quad (5.6)$$

Equation (5.6) shows that electric current $\mathbf{J} = \nabla \times \mathbf{B}$ always has the same direction as vector field $-\alpha A\mathbf{B}$ and no switching of its direction occurs.

Applying transformations T_β (Equation 2.7) with $\beta > 0$: $G(\psi_\lambda) \to G_\beta(\psi_\lambda) = \pm\sqrt{\beta + G^2(\psi_\lambda)}$ we get the magnetic fields (Equation 4.1):

$$\mathbf{B}_\beta(x, y) = -(\psi_\lambda)_y \hat{e}_x + (\psi_\lambda)_x \hat{e}_y \pm \sqrt{\beta + A^2 \sin^2[\alpha(\psi_\lambda + \gamma)]}\,\hat{e}_z. \quad (5.7)$$

Magnetic fields (Equation 5.7) with the exact magnetic functions $\psi_\lambda(x, y)$ (Equation 5.5) satisfy Beltrami equation

$$\nabla \times \mathbf{B}_\beta = -\frac{\mathrm{d}G_\beta(\psi_\lambda)}{\mathrm{d}\psi_\lambda}\mathbf{B}_\beta = \mp\frac{\alpha A^2 \sin[2\alpha(\psi_\lambda + \gamma)]}{2\sqrt{\beta + A^2\sin^2[\alpha(\psi_\lambda + \gamma)]}}\mathbf{B}_\beta. \quad (5.8)$$

Using here exact solution (Equation 5.5) we obtain

$$\nabla \times \mathbf{B}_\beta = \mp\frac{\alpha A^2 \sinh[\alpha A(v + c)]}{\cosh^2[\alpha A(v + c)]\sqrt{\beta + A^2\tanh^2[\alpha A(v + c)]}}\mathbf{B}_\beta. \quad (5.9)$$

Hence electric current $\mathbf{J}_\beta = \nabla \times \mathbf{B}_\beta$ vanishes on the plane $v + c = 0$. The switching of direction of the electric current $\mathbf{J}_\beta$ occurs when point (x, y, z) crosses the plane $v + c = 0$.

It is evident that Equations (5.7)–(5.9) with $\beta > 0$ provide new everywhere bounded force-free plasma equilibria satisfying the Beltrami Equation (1.2) with non-constant function $\alpha_\lambda(\mathbf{x}) = -\mathrm{d}G_\beta(\psi_\lambda)/\mathrm{d}\psi_\lambda$.

Acknowledgement: The author is grateful to the referees for useful remarks.
Author contribution: The author has accepted responsibility for the entire content of this submitted manuscript and approved submission.
Research funding: None declared.
Conflict of interest statement: The author declares no conicts of interest regarding this article.

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